East-West J. of Mathematics: Vol. 24, No 2 (2023) pp. 78-88
https://doi.org/10.36853/ewjm0395

STRUCTURES OF POWER DIGRAPHS ASSOCIATED WITH $x^{p^k} \equiv y \pmod{n}$ WHERE p IS AN ODD PRIME

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Abstract

Let k and n be positive integers and p be an odd prime. A power digraph $G(p^k, n)$ for which the vertex set is $\{0, 1, 2, \ldots, n-1\}$ and (u, v) is a directed edge from a vertex u to a vertex v if $u^{p^k} \equiv v \pmod{n}$. We study the structures of this power digraphs. Moreover, we provide some interesting results when p is $3, 5$ or 7.

1 Introduction

Graph structures and number theory are closely related. For a positive integer k, the study of digraph associated with the congruence $x^k \equiv y \pmod{n}$ becomes interesting in the recent years. We first introduce the power digraph with some important definitions.

Let n be a positive integer and $\bar{r}$ denote the set of all integers which leave remainder r when divided by n. Then, the set $\{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{(n-1)}\}$ is the set of complete residue classes of all integers when divided by n. For simplicity, in this article we will use $\{0, 1, 2, \ldots, n-1\}$ instead. Let p be an odd prime. We define a digraph $G(p^k, n)$ over the residue classes of n where the vertex set of

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Key words: power digraph, congruence relation, fixed point
(2010) Mathematics Subject Classification: 05C20; 11A07

$G(p^k, n)$ is the set of complete residue class of all integers when divided by n, $\{0, 1, 2, \ldots, n-1\}$ and (u, v) is a directed edge of $G(p^k, n)$ from a vertex u to a vertex v if $u^{p^k} \equiv v \pmod{n}$.

C is a cycle of length c if vertices $u_1, u_2, u_3, \ldots, u_c$ satisfy the following condition

$$\begin{aligned} {u_1}^{p^k} &\equiv u_2 \pmod{n}, \\ {u_2}^{p^k} &\equiv u_3 \pmod{n}, \\ {u_3}^{p^k} &\equiv u_4 \pmod{n}, \\ &\vdots \\ {u_c}^{p^k} &\equiv u_1 \pmod{n} \end{aligned}$$

The vertex u is called a fixed point of $G(p^k, n)$ if $u^{p^k} \equiv u \pmod{n}$. In term of graphs, we can say that $G(p^k, n)$ has a loop at a vertex u. We see that the fixed points are the solution of the congruence equation $x^{p^k} \equiv x \pmod{n}$. Moreover, we see that an 1-cycle is said to be a loop or a fixed point and a cycle of length c is called a c-cycle.

Example 1.1. *Let $n = 11, p = 3$ and $k = 2$. Since*

$$\begin{array}{ccc} 0^{3^2} \equiv 0 \pmod{11} & 1^{3^2} \equiv 1 \pmod{11} & 2^{3^2} \equiv 6 \pmod{11} \\ 3^{3^2} \equiv 4 \pmod{11} & 4^{3^2} \equiv 3 \pmod{11} & 5^{3^2} \equiv 9 \pmod{11} \\ 6^{3^2} \equiv 2 \pmod{11} & 7^{3^2} \equiv 8 \pmod{11} & 8^{3^2} \equiv 7 \pmod{11} \end{array}$$
$$9^{3^2} \equiv 5 \pmod{11} \quad 10^{3^2} \equiv 10 \pmod{11},$$

the power digraph $G(3^2, 11)$ can be drawn as in Figure 1.

We see that $0, 1$ and 10 are fixed points and the other vertices are in some 2-cycles.

Much research has been done on the topic of power digraphs associated with the congruence. Bryant [1] considered quadratic digraphs and isomorphic subgroups of a finite group. In 1992, some properties of power digraphs associated with the congruence and the existence of cycles are studied by Szalay [2]. Next, Rogers [3] and Somer et al. [4] provided some results on fixed points, cycles and components in the square mapping graphs. After that, the symmetric structures of power digraphs are investigated, see [5, 6]. Many researchers proved useful results of such digraphs based on the congruence $x^k \equiv y \pmod{n}$, see [7, 8] and proposed quadratic and cubic residue graphs, see [9, 10, 11]. Then, Mateen et al. [12] generalized the power digraph on the congruence $x^p \equiv y \pmod{n}$ when p is an odd prime.

Motivated by [12], in Section 2, we investigate the general structures of power digraph $G(p^k, n)$ when p is an odd prime. In Section 3, we study some

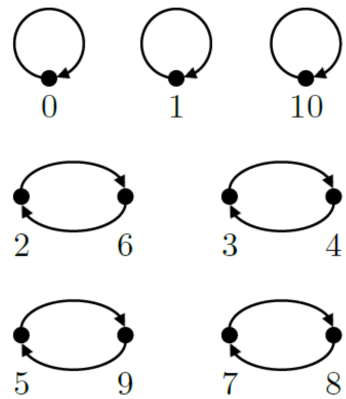

Figure 1: The power digraph $G(3^2, 11)$

results on fixed points of $G(p^k, n)$ where $p \in \{3, 5, 7\}$. In the last section, conclusions and discussions are provided.

2 When p is an odd prime

First, we consider the congruence $x^{m^k} \equiv y \pmod{n}$ when m is a positive integer and then provide the following simple and straightforward results. By the fact that $0^{m^k} \equiv 0 \pmod{n}, 1^{m^k} \equiv 1 \pmod{n}$,

$$(-1)^{m^k} = \begin{cases} 1 & \text{if } m \text{ is even} \\ -1 & \text{if } m \text{ is odd} \end{cases}$$

and

$$(-u)^{m^k} = \begin{cases} u & \text{if } m \text{ is even} \\ -u & \text{if } m \text{ is odd,} \end{cases}$$

we obtain the following lemma.

Lemma 2.1. *Let n be a positive integer. The following statements are true.*

1. *The number 0 and 1 are fixed points of $G(m^k, n)$ when m is even.*
2. *The number $0, 1$ and -1 are fixed points of $G(m^k, n)$ when m is odd.*
3. *Assume that m is even. For any vertices u and v in $G(m^k, n)$, (u, v) is an edge in $G(m^k, n)$ if and only if $(-u, v)$ is an edge in $G(m^k, n)$.*
4. *Assume that m is odd. For any vertices u and v in $G(m^k, n)$, (u, v) is an edge in $G(m^k, n)$ if and only if $(-u, -v)$ is an edge in $G(m^k, n)$.*

5. *Assume that m is odd. For any vertices u in $G(m^k, n)$, u is a fixed point of $G(m^k, n)$ if and only if $-u$ is a fixed point of $G(m^k, n)$.*

6. *Assume that m and c are odd. Then, u is a vertex in some c-cycle if and only if $-u$ is a vertex in c-cycle.*

Before showing the relationship between the congruence classes modulo m and the congruence classes modulo m^l, we give the necessary definition as follow. A digraph $G(n)$ over the set of residue classes of all integers when divided by n and (u, v) is a directed edge of $G(n)$ from a vertex u to a vertex v if $u \equiv v \pmod{n}$.

Lemma 2.2. *Let $m \geq 2$ and $k \geq 1$ be integers. If $u \equiv v \pmod{m}$, then $u^{m^k} \equiv v^{m^k} \pmod{m^l}$ for all $1 \leq l \leq k+1$. That is, if (u, b_1) and (v, b_1) are directed edges in $G(m)$, then (u^{m^k}, b_2) and (v^{m^k}, b_2) are directed edges in $G(m^k, m^l)$.*

Proof. Let $m \geq 2, k \geq 1$ and $l \geq 1$ be positive integers such that $l \leq k+1$. Let $s \geq 0$ and $0 \leq t \leq m-1$ be integers. Consider $(ms+t)^{m^k}$. We know that

$$(ms+t)^{m^k} = \sum_{i=0}^{m^k} \binom{m^k}{i} (ms)^{m^k - i} t^i.$$

Since $l \leq k+1$, we have

$$\sum_{i=0}^{m^k - 1} \binom{m^k}{i} (ms)^{m^k - i} t^i \equiv 0 \pmod{m^l}.$$

Then,

$$(ms+t)^{m^k} \equiv \tbinom{m^k}{m^k} t^{m^k} \equiv t^{m^k} \pmod{m^l}.$$

Assume $u \equiv v \pmod{m}$. Then, $u = ms_1 + t$ and $v = ms_2 + t$ where $s_1 \neq s_2$. Thus,

$$u^{m^k} = (ms_1 + t)^{m^k} \equiv t^{m^k} \pmod{m^l}$$

and

$$v^{m^k} = (ms_2 + t)^{m^k} \equiv t^{m^k} \pmod{m^l}.$$

Therefore, $u^{m^k} \equiv v^{m^k} \pmod{m^l}$. That is, if (u, b_1) and (v, b_1) are directed edges in $G(m)$, then (u^{m^k}, b_2) and (v^{m^k}, b_2) are directed edges in $G(m^k, m^l)$. $\square$

Next, we study the congruence $x^{p^k} \equiv y \pmod{n}$ when p is an odd prime and give some results on fixed points of $G(p^k, 2^l)$, where $l \geq 4$. The following result is the initial step for our result.

Lemma 2.3. *[12] For a prime* p *of the type* $p \equiv 3 \pmod 4$ *and* $l \geq 4$, $0, 1, 2^{l-1} \pm 1$ *and* 2^{l-1} *are fixed points of* $G(p, 2^l)$.

Theorem 2.4. *For a prime* p *of the type* $p \equiv 3 \pmod 4$ *and* $l \geq 4$, $0, 1, 2^{l-1} \pm 1$ *and* 2^{l-1} *are fixed points of* $G(p^k, 2^l)$ *for all integers* $k \geq 1$.

Proof. Let $l \geq 4$ be an integer and $p \equiv 3 \pmod 4$. We prove the theorem by mathematical induction on k.

Basis step Let $k = 1$. By Lemma 2.3, we obtain that $0, 1, 2^{l-1} \pm 1$ and 2^{l-1} are fixed points of $G(p, 2^l)$.

Induction step Assume that for $k \geq 1$, $0, 1, 2^{l-1} \pm 1$ and 2^{l-1} are fixed points of $G(p^k, 2^l)$. We claim that $0, 1, 2^{l-1} \pm 1$ and 2^{l-1} are fixed points of $G(p^(k+1), 2^l)$. By Lemma 2.1 (2), we obtain that 0 and 1 are fixed points of $G(p^(k+1), 2^l)$. By the induction hypothesis,

$$\begin{aligned}
(2^{l-1}+1)^{p^{k+1}} &\equiv (2^{l-1}+1)^p \pmod{2^l} \\
&\equiv 2^{l-1}+1 \pmod{2^l}, \\
(2^{l-1}-1)^{p^{k+1}} &\equiv (2^{l-1}-1)^p \pmod{2^l} \\
&\equiv 2^{l-1}-1 \pmod{2^l} \text{ and} \\
(2^{l-1})^{p^{k+1}} &\equiv (2^{l-1})^p \pmod{2^l} \\
&\equiv 2^{l-1} \pmod{2^l}.
\end{aligned}$$

By mathematical induction on k, we obtain that $0, 1, 2^{l-1} \pm 1$ and 2^{l-1} are fixed points of $G(p^k, 2^l)$ when $p \equiv 3 \pmod 4$ and $l \geq 4$. □

For a prime p of the type $p \equiv 5 \pmod 8$, the following result is the initial case for our result.

Lemma 2.5. *[12] For a prime* p *of the type* $p \equiv 5 \pmod 8$ *and* $l \leq 4$, $0, 1, 2^{l-1} \pm 1, 2^{l-2)} \pm 1, 2^{l-1}$ *and* $-(2^{l-2} \pm 1) + 2^l$ *are fixed points of* $G(p, 2^l)$.

Theorem 2.6. *For a prime* p *of the type* $p \equiv 5 \pmod 8$ *and* $l \geq 4$, $0, 1, 2^{l-1} \pm 1, 2^{l-2} \pm 1, 2^{l-1}$ *and* $-(2^{l-2} \pm 1) + 2^l$ *are fixed points of* $G(p^k, 2^l)$ *for all integers* $k \geq 1$.

Proof. Let $l \geq 4$ be an integer and $p \equiv 5 \pmod 8$. We prove the theorem by mathematical induction on k.

Basis step Let $l = 1$. By Lemma 2.5, we obtain that $0, 1, 2^{l-1} \pm 1, 2^{l-2} \pm 1, 2^{l-1}$ and $-(2^{l-2} \pm 1) + 2^l$ are fixed points of $G(p, 2^l)$.

Introduction step Assume that for $k \geq 1$, $0, 1, 2^{l-1} \pm 1, 2^{l-2} \pm 1, 2^{l-1}$ and $-(2^{l-2} \pm 1) + 2^l$ are fixed points of $G(p^k, 2^l)$. We claim that $0, 1, 2^{l-1} \pm 1, 2^{l-2} \pm 1, 2^{l-1}$ and $-(2^{l-2} \pm 1) + 2^l$ are fixed points of $G(p^k, 2^l)$. By Lemma 2.1 (2), we obtain that 0 and 1 are fixed points of $G(p^{k+1}, 2^l)$. By the induction hypothesis

$$\begin{aligned}
(2^{l-1}+1)^{p^{k+1}} &\equiv (2^{l-1}+1)^p \pmod{2^l}\\
&\equiv 2^{l-1}+1 \pmod{2^l},\\
(2^{l-1}-1)^{p^{k+1}} &\equiv (2^{l-1}-1)^p \pmod{2^l}\\
&\equiv 2^{l-1}-1 \pmod{2^l},\\
(2^{l-2}+1)^{p^{k+1}} &\equiv (2^{l-2}+1)^p \pmod{2^l}\\
&\equiv 2^{l-2}+1 \pmod{2^l},\\
(2^{l-2}-1)^{p^{k+1}} &\equiv (2^{l-2}-1)^p \pmod{2^l}\\
&\equiv 2^{l-2}-1 \pmod{2^l},\\
(2^{l-1})^{p^{k+1}} &\equiv (2^{l-1})^p \pmod{2^l}\\
&\equiv 2^{l-1} \pmod{2^l},\\
(-2^{l-2}+1+2^l)^{p^{k+1}} &\equiv (-2^{l-2}+1+2^l)^p \pmod{2^l}\\
&\equiv -2^{l-2}+1+2^l \pmod{2^l} \text{ and}\\
(-2^{l-2}-1+2^l)^{p^{k+1}} &\equiv (-2^{l-2}-1+2^l)^p \pmod{2^l}\\
&\equiv -2^{l-2}-1+2^l \pmod{2^l}.
\end{aligned}$$

By mathematical induction on k, we obtain that $0, 1, 2^{l-1} \pm 1, 2^{l-2} \pm 1, 2^{l-1}$ and $-(2^{k-2} \pm 1) + 2^k$ are fixed points of $G(p^k, 2^l)$ when $p \equiv 5 \pmod 8$ and $l \geq 4$. □

3 When p is $3, 5$ or 7

We consider the structures of power digraphs for the special case when p is $3, 5$ or 7. We first introduce some important number theory background.

Definition 1. *Let m be a positive integer. Define the Euler's totient function $\phi(m)$ by*

$$\phi(m) = \left|\{r \in \mathbb{Z} : 0 \leq r \leq m \text{ and } \gcd(r, m) = 1\}\right|.$$

Note that $\phi(1) = 1$ and $\phi(m) \leq m-1$ for all $m \geq 2$. Moreover, $\phi(p) = p-1$ if and only if p is prime. In addition, if p is a prime, then $\phi(p^k) = p^k - p^{k-1}$ for every $k \in \mathbb{N}$

Definition 2. *The Carmichael λ-function is defined at $1, 2, 4, 2^k$ and p^k as follows: $\lambda(1) = 1, \lambda(2) = 1, \lambda(4) = 2, \lambda(2^k) = \frac{1}{2}(2^k); k \geq 3$ and $\lambda(p^k) = \phi(p^k); k \geq 1$, where p is an odd prime.*

Note that $\lambda({p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}\cdots{p_l}^{\alpha_l}) = \mathrm{lcm}(\lambda({p_1}^{\alpha_1}), \lambda({p_2}^{\alpha_2}), \lambda({p_3}^{\alpha_3}), \ldots, \lambda({p_l}^{\alpha_l}))$.
The followings are properties of ϕ and α due to Euler and Carmicheal.

Theorem 3.1. *Let $a \geq 1$ and $m \geq 1$ be integers.*

1. *[13] (Euler) Assume that gcd$(a, m) = 1$. Then, we have $a^{\phi(m)} \equiv 1 \pmod{m}$.*
2. *[14] (Carmichael) $a^{\lambda(m)} \equiv 1 \pmod{m}$ if and only if gcd$(a, m) = 1$.*

Next, we study the results on fixed points of such digraphs arising from $x^{3^2} \equiv y \pmod{n}$.

Theorem 3.2. *u is a fixed point of $G(3^2, 32)$ if and only if gcd$(u, 32) = 1$.*

Proof. Consider $\lambda(32) = \frac{1}{2}\phi(2^5) = \frac{1}{2}(2^5 - 2^4) = 8$. By Theorem 3.1 (2), $u^8 = u^{\lambda(32)} \equiv 1 \pmod{32}$ if and only if $\gcd(u, 32) = 1$. That is, $u^{3^2} \equiv u \pmod{32}$ if and only if $\gcd(u, 32) = 1$. □

Theorem 3.3. *If $n \neq 1$ and $n \mid 30$, then u is a fixed point of $G(3^2, n)$ for all $u \in \{0, 1, 2, \ldots, n-1\}$.*

Proof. Let $u \in \{0, 1, 2, \ldots, n-1\}$. By Theorem 3.1 (1), we obtain that

$$u \equiv 1 \pmod{2}, u^2 \equiv 1 \pmod{3} \text{ and } u^4 \equiv 1 \pmod{5}.$$

Thus,

$$u^8 \equiv 1 \pmod{2}, u^8 \equiv 1 \pmod{3} \text{ and } u^8 \equiv 1 \pmod{5}.$$

Therefore,

$$u^9 \equiv u \pmod{2}, u^9 \equiv u \pmod{3} \text{ and } u^9 \equiv u \pmod{5}.$$

Since $2, 3$ and 5 are mutually relatively prime, we have

$$u^9 \equiv u \pmod{6}, u^9 \equiv u \pmod{10}, u^9 \equiv u \pmod{15} \text{ and } u^9 \equiv u \pmod{30}.$$

Hence, u is a fixed point of $G(3^2, n)$ for all $u \in \{0, 1, 2, \ldots, n-1\}$ when $n \neq 1$ and $n \mid 30$. □

Moreover, we construct a power digraph $G(3^2, 3^2)$. We obtain that

$$\begin{array}{ccc} 0^{3^2} \equiv 0 \pmod{3^2} & 1^{3^2} \equiv 1 \pmod{3^2} & 2^{3^2} \equiv 8 \pmod{3^2} \\ 3^{3^2} \equiv 0 \pmod{3^2} & 4^{3^2} \equiv 1 \pmod{3^2} & 5^{3^2} \equiv 8 \pmod{3^2} \\ 6^{3^2} \equiv 0 \pmod{3^2} & 7^{3^2} \equiv 1 \pmod{3^2} & 8^{3^2} \equiv 8 \pmod{3^2}. \end{array}$$

Then, we see that $G(3^2, 3^2)$ consists of 3 copies of isomorphic component as

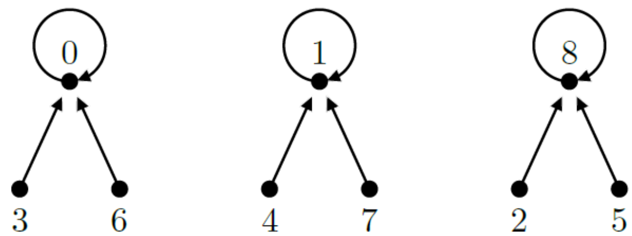

Figure 2: The power digraph $G(3^2, 3^2)$

Figure 2 motivates us to consider the structure of $G(3^k, 3^l)$ and see that such digraph is involved with a specific digraph $\mathcal{G}$ shown in Figure 3.

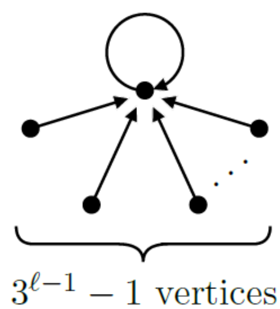

Figure 3: A specific digraph $\mathcal{G}$

Theorem 3.4. *Let l be an integer such that $1 \le l \le k+1$. Then, $G(3^k, 3^l)$ consists of only 3 copies of digraph $\mathcal{G}$ shown in Figure 3.*

Proof. By Lemma 2.1 (2), we see that $0, 1$ and -1 are fixed point of $G(3^k, 3^l)$. That is,

$$0^{3^k} \equiv 0 \pmod{3^l}, 1^{3^k} \equiv 1 \pmod{3^l} \text{ and } (-1)^{3^k} \equiv -1 \pmod{3^l}.$$

Since $3s_1 \equiv 0 \pmod 3$ for all integers $s_1 \ge 1$, by Lemma 2.2, we have $(3s_1)^{3^k} \equiv 0^{3^k} \equiv 0 \pmod{3^l}$ for all integers $1 \le l \le k+1$.

Since $3s_2 + 1 \equiv 1 \pmod 3$ for all integers $s_2 \ge 1$, by Lemma 2.2, we have $(3s_2+1)^{3^k} \equiv 1^{3^k} \equiv 1 \pmod{3^l}$ for all integers $1 \le l \le k+1$.

Since $3s_3 - 1 \equiv -1 \pmod 3$ for all integers $s_3 \ge 1$, by Lemma 2.2, we have $(3s_3-1)^{3^k} \equiv (-1)^{3^k} \equiv -1 \pmod{3^l}$ for all integers $1 \le l \le k+1$.

Then, we obtain 3 copies of the digraph $\mathcal{G}$ in $G(3^k, 3^l)$ as shown in Figure 4.

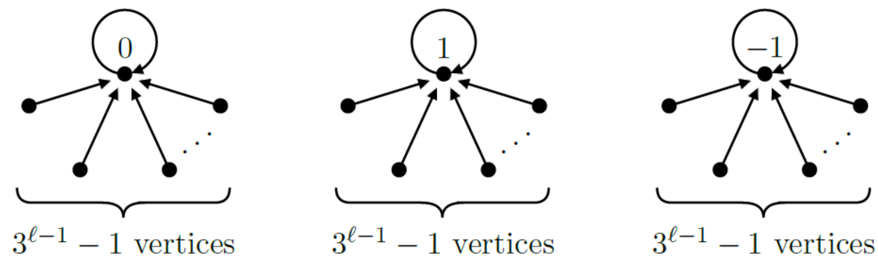

Figure 4: The power digraph $G(3^k, 3^l)$

□

After that, we investigate results of the power digraph over the considered congruence equation when $p = 5$ and $k = 2$ which are resemble to Theorem 3.2 and Theorem 3.3.

Theorem 3.5. *u is a fixed point of $G(5^2, 288)$ if and only if $gcd(u, 288) = 1$.*

Proof. Consider $\lambda(288) = \lambda(2^5 \cdot 3^2) = \mathrm{lcm}(\lambda(2^5), \lambda(3^2)) = \mathrm{lcm}(8, 6) = 24$. By Theorem 3.1 (2), $u^{24} = u^{\lambda(288)} \equiv 1 \pmod{288}$ if and only if $\gcd(u, 288) = 1$. That is, $u^{5^2} \equiv u \pmod{288}$ if and only if $\gcd(u, 288) = 1$. □

Theorem 3.6. *If $n \neq 1$ and $n \mid 2730$, then u is a fixed point of $G(5^2, n)$ for all $u \in \{0, 1, 2, \ldots, n-1\}$.*

Proof. Let $u \in \{0, 1, 2, \ldots, n-1\}$. By Theorem 3.1 (1), we obtain that

$$u \equiv 1 \pmod{2}, u^2 \equiv 1 \pmod{3}, u^4 \equiv 1 \pmod{5},$$
$$u^6 \equiv 1 \pmod{7} \text{ and } u^{12} \equiv 1 \pmod{13}.$$

Thus,

$$u^{24} \equiv 1 \pmod{2}, u^{24} \equiv 1 \pmod{3}, u^{24} \equiv 1 \pmod{5},$$
$$u^{24} \equiv 1 \pmod{7} \text{ and } u^{24} \equiv 1 \pmod{13}.$$

Therefore,

$$u^{25} \equiv u \pmod{2}, u^{25} \equiv u \pmod{3}, u^{25} \equiv u \pmod{5},$$
$$u^{25} \equiv u \pmod{5} \text{ and } u^{25} \equiv u \pmod{13}.$$

Since $2, 3, 5, 7$ and 13 are mutually relatively prime, we have that

$$u^{25} \equiv u \pmod{6}, u^{25} \equiv u \pmod{10}, u^{25} \equiv u \pmod{14}, u^{25} \equiv u \pmod{15},$$
$$u^{25} \equiv u \pmod{21}, u^{25} \equiv u \pmod{30}, u^{25} \equiv u \pmod{35}, u^{25} \equiv u \pmod{42},$$
$$\vdots$$
$$u^{25} \equiv u \pmod{910}, u^{25} \equiv u \pmod{1365} \text{ and } u^{25} \equiv u \pmod{2730}.$$

Hence, u is a fixed point of $G(5^2, n)$ for all $u \in \{0, 1, 2, \dots, n-1\}$ when $n \neq 1$ and $n \mid 2730$. $\square$

Then, we prove results on fixed points of a digraph $G(7^2, n)$ which are resemble to Theorem 3.2 and Theorem 3.3, respectively.

Theorem 3.7. *u is a fixed point of $G(7^2, 576)$ if and only if gcd$(u, 576) = 1$.*

Proof. Consider $\lambda(576) = \lambda(2^6 \cdot 3^2) = \operatorname{lcm}(\lambda(2^6), \lambda(3^2)) = \operatorname{lcm}(16, 6) = 48$. By Theorem 3.1 (2), $u^{48} = u^{\lambda(576)} \equiv 1 \pmod{576}$ if and only if $\gcd(u, 288) = 1$. That is, $u^{7^2} \equiv u \pmod{576}$ if and only if $\gcd(u, 576) = 1$. $\square$

Theorem 3.8. *If $n \neq 1$ and $n \mid 2730$, then u is a fixed point of $G(7^2, n)$ for all $u \in \{0, 1, 2, \dots, n-1\}$.*

Proof. The proof is similar to the proof of Theorem 3.6. $\square$

4 Conclusion and Discussion

In the study of structures of power digraphs over the congruence equation $x^{p^k} \equiv y \pmod{n}$, we provide fixed points of a digraph $G(p^k, 2^l)$ where $p \equiv 3 \pmod{4}; l \geq 4$ and $p \equiv 5 \pmod{8}; l \geq 4$ which generalize useful results on fixed points of a digraph $G(p, 2^l)$ under the same conditions.

According to some specific prime integers, we discuss the conditions on the number x and n enabled us to study fixed points of digraphs $G(3^2, n), G(5^2, n)$ and $G(7^2, n)$. We obtain that u is a fixed point of $G(p^2, n)$ if and only if $\gcd(u, n) = 1$ where ordered pair (p, n) is $(3, 32), (5, 288)$ or $(7, 576)$. Moreover, u is a fixed point of $G(p^2, n)$ for all $u \in \{0, 1, 2, \dots, n-1\}$ when $n \neq 1$ and $n \mid m$ where ordered pair (p, m) is $(3, 30), (5, 2730)$ or $(7, 2730)$. Besides fixed points, we consider the structure of a digraph $G(3^k, 3^l)$ where $1 \leq l \leq k+1$.

Furthermore, we show some general results based on such the congruence equation $x^{m^k} \equiv y \pmod{n}$ when m is a positive integer. As for future, we suggest proposing the results on the existence of cycles and few decompositions of components and enumerating.

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