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STABILITY INDEX OF DEPTH FUNCTIONS OF COVER IDEALS OF BIPARTITE GRAPHS

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ABSTRACT. Let $G = (V(G), E(G))$ be a bipartite graph over the vertex set
$V(G) = \{1, \ldots, r\}$ and $J(G)$ be the cover ideal of G in the polynomial ring $R = K[x_1, \ldots, x_r]$. In this paper, we give a good bound for the index of depth stability of $J(G)$.

INTRODUCTION

Throughout on this paper, let K be a field and $R = K[x_1, \ldots, x_r]$, $r \geqslant 1$ be a polynomial ring over a field K with $\mathfrak{m} := (x_1, \ldots, x_r)$ the maximal homogeneous ideal of R. The main goal of our work is focused on the important invariant in commutative algebra, namely the depth of an ideal. By means of local cohomology, we can define definitions as follows:

Let M be a finitely generated graded R-module, and $H^i_{\mathfrak{m}}(M)$, for $i \geqslant 0$, denote the i-th local cohomology module of M with respect to $\mathfrak{m}$. We define

$$\operatorname{depth}(M) := \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}.$$

Let I be a homogeneous ideal in R, the function $t \mapsto \operatorname{depth} R/I^t$ for $t \geqslant 1$ is called the *depth function* of I. It is well known result by Brodmann [3] that depth function becomes the constant for big enough t. Moreover,

$$\lim_{t\to\infty} \operatorname{depth} R/I^t \leqslant \dim R - \ell(I),$$

where $\ell(I)$ is the *analytic spread* of I. Eisenbud and Huneke [5] showed that equality holds when the associated graded ring of I is Cohen–Macaulay. The *index of depth stability* of I is defined by

$$\operatorname{dstab}(I) := \min\{t_0 \geqslant 1 \mid \operatorname{depth} R/I^t = \operatorname{depth} R/I^{t_0} \text{ for all } t \geqslant t_0\}.$$

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In [8] they investigated the depth functions of cover ideals $J(\mathcal{H})$ of unimodular hypergraphs $\mathcal{H}$. We proved the non-increasing property of depth $R/J(\mathcal{H})^t$, and we showed the $\operatorname{dstab}(J(\mathcal{H})) \leqslant \dim R$.

In fact, when we restrict the unimodular hypergraphs to graphs, we obtain the bipartite graphs. So, in this paper if G is bipartite graph with cover ideal $J(G)$, we consider the depth function of $J(G)$. We also obtain the result for the non-increasing property of depth $R/J(G)^t$. However, we improve the result in [8] and show a better bound for $\operatorname{dstab}(J(G))$. More precisely, the main result of paper is the following theorem.

Theorem 3.5 *Let $G = (V, E)$ be a bipartite graph with the vertex set $V = \{1, \dots, r\}$ and let l be the length of a longest simple path in G. Then,*

$$\operatorname{depth} R/J(G)^t = r - \nu_0(G) - 1 \textit{ for all } t \geqslant \lceil \frac{l+1}{2} \rceil.$$

In particular, $\operatorname{dstab}(J(G)) \leqslant \lceil \frac{l+1}{2} \rceil$.

Our technique is based on a formula contributed by Takayama [15]. By using this formula we lead to investigate the integer solutions of certain systems of linear inequalities. This allows us to use the theory of integer programming as the key role in this paper (see e.g. [8, 10, 11] for this approach).

The paper is organized as follows. In Section 1, we recall some basic notation and terminology for simplicial complex, the relationship between simplicial complexes and cover ideals of graphs, and give Takayama's formula for computing local cohomology modules. In Section 2, we consider the integer solutions of systems of linear inequalities with bipartite matrices. In Section 3, we investigate an upper bound for $\operatorname{dstab}(J(G))$ of any bipartite graph G.

1. Preliminary

In this section, for the convenience of the reader, we recollect notation, terminology and basic results used in the paper. We follow standard texts [2, 9, 12, 14].

1.1. **Depth function.** Throughout the paper, the main invariant of our work is the depth of graded modules and ideals over R. This invariant can be defined via either the minimal free resolutions or the local cohomology modules.

Let M be a nonzero finitely generated graded R-module and let

$$0 \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \to 0$$

be the minimal free resolution of M. The *depth* of M is given by Auslander-Buchsbaum formula $\operatorname{depth}(M) = r - p$, in which p is projective dimension of M.

On the other hands, the depth of M can also be computed in terms of the local cohomology modules of M. Let $H^i_{\mathfrak{m}}(M)$ be the i-th cohomology module of M with support in $\mathfrak{m}$. Then,

$$\operatorname{depth}(M) = \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}.$$

1.2. **Simplicial complexes and Stanley-Reisner ideals.** A *simplicial complex* on $V = \{1, \ldots, r\}$ is a collection of subsets of V, called faces, such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. A face of Δ not properly contained in another face of Δ is called a facet. The set of facets is denoted by $\mathcal{F}(\Delta)$.

The Stanley-Reisner ideal associated to a simplicial complex Δ is the squarefree monomial ideal

$$I_\Delta := (x_\tau \mid \tau \notin \Delta) \subseteq R.$$

Note that if I is a squarefree monomial ideal, then it is a Stanley-Reisner ideal of the simplicial complex $\Delta(I) := \{\tau \subseteq V \mid \mathbf{x}_\tau \notin I\}$. If I is a monomial ideal (maybe not squarefree) we also use $\Delta(I)$ to denote the simplicial complex corresponding to the squarefree monomial ideal $\sqrt{I}$.

If $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m\rangle$. Then, I_Δ has the primary-decomposition (see [12, Theorem 1.7]):

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \notin F).$$

For $n \geqslant 1$, the n-th symbolic power of I_Δ is

$$I_\Delta^{(n)} = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \notin F)^n.$$

1.3. **Degree complexes.** Let I be a non-zero monomial ideal. Since R/I is an $\mathbb{N}^r$− graded algebra, $H^i_m(R/I)$ is an $\mathbb{Z}^r$-graded module over R/I for every i. For each degree $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$, in order to compute $\dim_K H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}$ we use a formula given by Takayama (see [15, Theorem 1]).

Set $\operatorname{supp}^-\boldsymbol{\alpha} := \{i \mid \alpha_i < 0\}$. For a subset $F \subseteq V$, we let $R_F := R[x_i^{-1} \mid i \in F]$. We define the *degree complex* $\Delta_{\boldsymbol{\alpha}}(I)$ by

$$\Delta_\alpha(I) := \left\{F \subseteq V \setminus \operatorname{supp}^-\boldsymbol{\alpha} \mid x^\alpha \notin IR_{F\cup \operatorname{supp}^-\boldsymbol{\alpha}}\right\}. \tag{1}$$

Now, we state the Takayama's formula (see [15, Theorem 1]) in the following form (see [13, Theorem 1.1]).

Lemma 1.1. $\dim_K H^i_m(R/I)_{\boldsymbol{\alpha}} = \dim_K \widetilde{H}_{i-|supp^-\boldsymbol{\alpha}|-1}(\Delta_{\boldsymbol{\alpha}}(I); K)$.

The following result of Minh and Trung is very useful for computing $\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(t)})$, which allows us to study the depth function by using the theory of convex polyhedra.

Lemma 1.2. ([13, Lemma 1.3]) *Let Δ be a simplicial complex and $\boldsymbol{\alpha} \in \mathbb{N}^r$. Then,*

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(t)})) = \left\{F \in \mathcal{F}(\Delta) \mid \sum_{i \notin F} \alpha_i \leqslant t-1\right\}.$$

1.4. **Graphs and their cover ideals.** Let G be a simple graph. We use the symbols $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Let $p\colon v_0, v_1, \ldots, v_k$ be a sequence of vertices of G. Then,

(1) p is called a *path* if $\{v_{i-1}, v_i\} \in E(G)$ for $i = 1, \ldots, k$. In this case, we say that p is a path from v_0 to v_k.
(2) p is called a *simple path* if it is a path and every vertex appears exactly once.
(3) p is called a *cycle* if $k \geqslant 3$ and p is a path with distinct vertices except for $v_0 = v_k$.

In each case, k is called the *length* of p. A simple path is longest if it is among the simple paths of largest lengths of G.

According to Constantinescu and Varbaro [4], we define an ordered matching as follows.

Definition 1.3. A matching $M = \{\{u_i, v_i\} \mid i = 1, \ldots, s\}$ in a graph G is called an *ordered matching* if:

(1) $\{u_1, \ldots, u_s\}$ is an independent set in G,
(2) $\{u_i, v_j\} \in E(G)$ implies $i \leqslant j$.

The *ordered matching number* of G, denoted by $\nu_0(G)$ is the maximum size of an ordered matching in G.

A graph is *connected* if there is a path from any point to any other point in the graph. A connected component of a graph G is a connected subgraph that is not part of any larger connected subgraph. The components of any graph partition its vertices into disjoint sets

The graph G is *bipartite* if $V(G)$ can be partitioned into two subsets X and Y so that every edge has one end in X and another end in Y; such a partition (X, Y) is called a *bipartition* of the graph. Note that G is bipartite if and only if it has no cycle of odd length (see [2, Theorem 4.7]).

A connected graph is a *tree* if it has no cycles. From [2, Theorem 2.4 and Theorem 2.2], we deduce that.

Lemma 1.4. *If G is a connected graph, then $|E(G)| \geqslant |V(G)| - 1$. The equality occurs if and only if G is a tree.*

If G is a tree, then for each pair of vertices u and v of G, there is a unique simple path from u to v according to [2, Theorem 2.1]. The length of this path is just the distance between u and v, and we denoted by $\mathrm{d}_G(u, v)$ or $\mathrm{d}(u, v)$.

In the sequence, we give the fact on bipartite graphs.

Lemma 1.5. ([7, Lemma 1.4]) *Let G be a bipartite graph with at least one edge. Assume that for each edge $\{i, j\}$ of G we have a real number a_{ij}. Then,* *the linear system:*

$$\begin{cases} x_i + x_j = a_{ij}, \\ \{i,j\} \in E(G) \end{cases}$$

has no unique solution.

A *vertex cover* of G is a subset of $V(G)$ which meets every edge of G; a vertex cover is *minimal* if none of its proper subsets is itself a cover. For a subset $\tau = \{i_1, \ldots, i_t\}$ of V, set $\mathbf{x}_\tau := x_{i_1} \cdots x_{i_t}$. Then, the *cover ideal* of G is defined by

$$J(G) := (\mathbf{x}_\tau \mid \tau \text{ is a minimal vertex cover of } G),$$

Note that $J(G)$ can be written as

$$J(G) = \bigcap_{\{u,v\} \in E(G)} (x_u, x_v), \tag{2}$$

and $J(G)$ is the Stanley-Reisner ideal corresponding to the simplicial complex

$$\Delta(J(G)) = \langle V \setminus e \mid e \in E \rangle . \tag{3}$$

For $t \geqslant 1$, the t-th symbolic power of $J(G)$ is

$$J(G)^{(t)} = \bigcap_{\{u,v\} \in E(G)} (x_u, x_v)^t. \tag{4}$$

When G is bipartite graph, the cover ideal $J(G)$ is normally torsion-free, i.e. $J(G)^{(t)} = J(G)^t$ for all $t \geqslant 1$ by [6, Corollary 2.6]. Therefore, Lemma 1.2 can be written as follows.

Lemma 1.6. *Let $G = (V, E)$ be a bipartite graph with vertex set $V = \{1, \ldots, r\}$ and edge set E. For every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ and $t \geqslant 1$, we have*

$$\Delta_{\boldsymbol{\alpha}}(J(G)^t) = \langle V \setminus \{u,v\} \mid \{u,v\} \in E \text{ and } \alpha_u + \alpha_v \leqslant t-1 \rangle .$$

2. Integer polytopes

For a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$, we set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_r$ and for a nonempty bounded closed subset S of $\mathbb{R}^r$ we set

$$\delta(S) := \max\{|\boldsymbol{\alpha}| \mid \boldsymbol{\alpha} \in S\}.$$

Let $G = (V, E)$ be a bipartite graph on the vertex set $V = \{1, \ldots, r\}$, and edge set E. Assume that

$$H^i_{\mathfrak{m}}(R/J(G)^t)_{\boldsymbol{\beta}} \neq 0, \text{ for some } i \geqslant 0,\ t \geqslant 1, \text{ and } \boldsymbol{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r .$$

By Lemma 1.1 we have

$$\dim_K \widetilde{H}_{i-1}(\Delta_{\boldsymbol{\beta}}(J(G)^t); K) = \dim_K H^i_{\mathfrak{m}}(R/J(G)^t)_{\boldsymbol{\beta}} \neq 0. \tag{5}$$

In particular, $\Delta_{\boldsymbol{\beta}}(J(G)^t)$ is not acyclic.

Let $E = \{e_1, \ldots, e_n\}$ where $n \geqslant 1$. Then, by Equation (3)

$$\Delta(J(G)) = \langle V \setminus e_1, \ldots, V \setminus e_n \rangle .$$

Since $\Delta_{\boldsymbol{\beta}}(J(G)^t)$ is not acyclic, by Lemma 1.6 we may assume that

$$\Delta_{\boldsymbol{\beta}}(J(G)^t) = \langle V \setminus e_1, \ldots, V \setminus e_k \rangle\,, \text{ in which } 1 \leqslant k \leqslant n.$$

For each integer $t \geqslant 1$, let $\mathcal{P}_t$ be the set of solutions in $\mathbb{R}^r$ of the following system:

$$\begin{cases} x_u + x_v \leqslant t-1 & \text{for } \{u,v\} \in E_1, \\ x_u + x_v \geqslant t & \text{for } \{u,v\} \in E_2, \\ x_1 \geqslant 0, \ldots, x_r \geqslant 0, \end{cases} \tag{6}$$

where $E_1 = \{e_1, \ldots, e_k\},\ E_2 = \{e_{k+1}, \ldots, e_n\},\ 1 \leqslant k \leqslant n$. Then, $\boldsymbol{\beta} \in \mathcal{P}_t$. Moreover, by Lemma 1.6 one has

$$\Delta_{\boldsymbol{\alpha}}(J(G)^s) = \langle V \setminus e_1, \ldots, V \setminus e_k \rangle = \Delta_{\boldsymbol{\beta}}(J(G)^t) \text{ whenever } \boldsymbol{\alpha} \in \mathcal{P}_s \cap \mathbb{N}^r\,.$$

In order to study the set $\mathcal{P}_t$, we consider $\mathcal{C}_t$ to be the set of solutions in $\mathbb{R}^r$ of the following system:

$$\begin{cases} x_u + x_v \leqslant t & \text{for } \{u,v\} \in E_1, \\ x_u + x_v \geqslant t & \text{for } \{u,v\} \in E_2, \\ x_1 \geqslant 0, \ldots, x_r \geqslant 0. \end{cases} \tag{7}$$

Note that if G is a bipartite graph, then G is a unimodular hypergraph by [1, Theorem 5, page 164]. So, we have both $\mathcal{P}_t$ and $\mathcal{C}_t$ are integer convex polyhedra by [14, Theorem 19.1], i.e. all their vertices have integral coordinates. Especially, we have the result about $\mathcal{C}_1$ as follows.

Lemma 2.1. ([7, Lemma 2.1]) *$\mathcal{C}_1$ is a polytope with $\dim \mathcal{C}_1 = r$. Moreover, if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ is a vertex of $\mathcal{C}_1$, then $\boldsymbol{\alpha} \in \{0,1\}^r$.*

Remark 2.2. Since $\mathcal{C}_t = t\mathcal{C}_1$, we deduce $\mathcal{C}_t$ is also a polytope. Observe that $\mathcal{P}_t \subseteq \mathcal{C}_t$, so is $\mathcal{P}_t$.

Since $\mathcal{C}_1$ is a polytope of dimension r, there exists a vertex $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r)$ of $\mathcal{C}_1$ such that

$$\delta(\mathcal{C}_1) = |\boldsymbol{\gamma}| = \gamma_1 + \cdots + \gamma_r.$$

Let $\rho := |\boldsymbol{\gamma}| = \delta(\mathcal{C}_1)$, we have $\rho \geqslant 1$. Note that $t\boldsymbol{\gamma}$ is also a vertex of $\mathcal{C}_t$ and $\delta(\mathcal{C}_t) = \rho t$. Since $\mathcal{P}_t \subseteq \mathcal{C}_t$, we have $\delta(P_t) \leqslant \rho t$, so we can write

$$\delta(P_t) = \rho t - b_t \text{ for some integer } b_t \geqslant 0. \tag{8}$$

Lemma 2.3. *If $\mathcal{P}_t \cap \mathbb{N}^r \neq \emptyset$, then $\mathcal{P}_{t+1} \cap \mathbb{N}^r \neq \emptyset$ and $b_t \geqslant b_{t+1}$ for any $t \geq 1$.*

Proof. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathcal{P}_t \cap \mathbb{N}^r$ such that $\delta(\mathcal{P}_t) = |\boldsymbol{\alpha}|$. Since $\boldsymbol{\alpha}$ is a solution of the System (6), and $\boldsymbol{\gamma}$ is a solution of the System (7) with $t = 1$, by Lemma 2.1 we have $\boldsymbol{\gamma} \in \{0,1\}^r$. Let $\boldsymbol{\alpha} + \boldsymbol{\gamma} = \boldsymbol{\theta} = (\theta_1, \ldots, \theta_r)$, we imply that

$$\begin{cases} \theta_u + \theta_v = (\alpha_u + \alpha_v) + (\gamma_u + \gamma_v) \leqslant t - 1 + 1 = t & \text{for } \{u,v\} \in E_1, \\ \theta_u + \theta_v = (\alpha_u + \alpha_v) + (\gamma_u + \gamma_v) \geqslant t + 1 & \text{for } \{u,v\} \in E_2. \end{cases}$$

In other words, $\boldsymbol{\theta} \in \mathcal{P}_{t+1} \cap \mathbb{N}^r$. Therefore, $\mathcal{P}_{t+1} \cap \mathbb{N}^r \neq \emptyset$ and $\delta(P_{t+1}) \geqslant |\alpha| + |\gamma|$. Since $\delta(\mathcal{P}_{t+1}) = \rho(t+1) - b_{t+1}$ and $|\boldsymbol{\alpha}| + |\boldsymbol{\gamma}| = \rho(t+1) - b_t$, we have $b_t \geqslant b_{t+1}$. $\square$

Let l be the length of a longest simple path in G. In the following key lemma, we show that $\mathcal{P}_t$ has vertices with integral coordinates for all $t \geqslant \lceil \frac{l+1}{2} \rceil$.

Lemma 2.4. *We have $\mathcal{P}_t \cap \mathbb{N}^r \neq \emptyset$ for any $t \geq \lceil \frac{l+1}{2} \rceil$.*

Proof. Let $\rho = \delta(\mathcal{C}_1)$. For $t \geqslant 1$ with $P_t \neq \emptyset$, we represent $\delta(P_t) = \rho t - b_t$ where b_t is an integer by (8). By Lemma 2.3 we have $b_t \geqslant b_{t+1} \geqslant \cdots \geqslant 0$. It follows that there is $t_0 \geqslant 1$ such that $b_t = b_{t_0}$ for $t \geqslant t_0$. Let $b := b_{t_0}$. Then,

$$\delta(\mathcal{P}_t) = \rho t - b, \text{ for all } t \geqslant t_0.$$

By Lemma 2.3 again, we deduce that $\delta(\mathcal{P}_t) \leqslant \rho t - b$, whenever $\mathcal{P}_t \neq \emptyset$.

Let s be an integer such that $s \geqslant \max\left\{2r^2 + b, t_0\right\}$. Then, we have

$$\delta(\mathcal{P}_s) = \rho s - b.$$

Since $\mathcal{P}_s$ is a polytope, $\delta(\mathcal{P}_s) = |\boldsymbol{\alpha}|$ for some vertex $\boldsymbol{\alpha}$ of $\mathcal{P}_s$. Note that the polytope $\mathcal{P}_s$ is defined by the following system

$$\begin{cases} x_u + x_v \leqslant s - 1 & \text{for } \{u, v\} \in E_1 \\ x_u + x_v \geqslant s & \text{for } \{u, v\} \in E_2, \\ x_1 \geqslant 0, \ldots, x_r \geqslant 0. \end{cases}$$

By [14, Formula 23 in Page 104], $\boldsymbol{\alpha}$ is the unique solution of a system of linear equations of the form

$$\begin{cases} x_u + x_v = s - 1 & \text{for } \{u, v\} \in S_1, \\ x_u + x_v = s & \text{for } \{u, v\} \in S_2, \\ x_t = 0, & \text{for } t \in S_3, \end{cases} \tag{9}$$

where $S_1 \subseteq E_1,\ S_2 \subseteq E_2,\ S_3 \subseteq [r]$ such that $|S_1| + |S_2| + |S_3| = r$.

Let H be the subgraph of G with $V(H) = V(G)$ and $E(H) = S_1 \cup S_2$ and let $H_1, \ldots, H_p$ be connected components of H. We next prove following claims:

Claim 1: H_i is a tree and $|V(H_i) \cap S_3| = 1$ for each $i = 1, \ldots, p$.

Indeed, because of $H = H_1 \sqcup \ldots \sqcup H_p$, so the System (9) can be separated into p systems, which has the form

$$\begin{cases} x_u + x_v = s - 1 & \text{for } \{u, v\} \in S_1 \cap V(H_i), \\ x_u + x_v = s & \text{for } \{u, v\} \in S_2 \cap V(H_i), \\ x_t = 0, & \text{for } t \in S_3 \cap V(H_i), \end{cases} \tag{10}$$

where $i = 1, \ldots, p$, and there is no common variable between the two different systems. Since the System (9) has a unique solution, then so does System (10). In particular, the number of equations is greater than or equal to the number of variables, i.e.

$$|V(H_i)| \leqslant |E(H_i)| + |S_3 \cap V(H_i)|.$$

Therefore,

$$\sum_{i=1}^{p} |V(H_i)| \leqslant \sum_{i=1}^{p} (|E(H_i)| + |S_3 \cap V(H_i)|), \text{ or}$$

$$|V(H)| \leqslant |S_1| + |S_2| + |S_3| = r.$$

On the other hand, $|V(H)| = |V(G)| = r$ and the System (9) has the number of equations equals the number of variables, which implies

$$|V(H_i)| = |E(H_i)| + |S_3 \cap V(H_i)|, \tag{11}$$

for each $i = 1, \ldots, p$. It is equivalent to the System (10) has the number of equations equals the number of variables.

Note that $S_3 \cap V(H_i) \neq \emptyset$ by Lemma 1.5. This fact together with the Equality (11) and Lemma 1.4, implies that

$$|E(H_i)| = |V(H_i)| - 1, \; |S_3 \cap V(H_i)| = 1, \text{ and } H_i \text{ is a tree, as claimed.}$$

From Claim 1, for $i = 1, \ldots, p$, denote the unique vertex in $V(H_i) \cap S_3$ by u_i. Since H_i is a tree, for every vertex v of H_i, there is a unique simple path in H_i from v to u_i, and we assume that this path is of the form

$$u_i = v_0, v_1, \ldots, v_n = v,$$

where $n = \mathrm{d}_{H_i}(v, u_i)$ is the distance between v and u_i.

From the system (10) we have $\alpha_{v_{j-1}} + \alpha_{v_j} = s - \epsilon_j$, for $j = 1, \ldots, n$, where

$$\epsilon_j = \begin{cases} 1 & \text{if } \{v_{j-1}, v_j\} \in E(H_i) \cap S_1 \\ 0 & \text{if } \{v_{j-1}, v_j\} \in E(H_i) \cap S_2. \end{cases}$$

Let $\sigma_v = \sum\limits_{k=0}^{n} (-1)^{k+1} \epsilon_k$ where we make a convention that $\epsilon_0 = 0$.

Claim 2. For every vertex v of H_i, we have $0 \leqslant \sigma_v \leqslant \lceil \mathrm{d}(v, u_i)/2 \rceil$ and

$$\alpha_v = \begin{cases} \sigma_v & \text{if } \mathrm{d}(v, u_i) \text{ is even,} \\ s - \sigma_v & \text{if } \mathrm{d}(v, u_i) \text{ is odd.} \end{cases}$$

Indeed, for each $q = 0, \ldots, n$, put $\eta_q = \sum\limits_{k=0}^{q} (-1)^{k+1} \epsilon_k$.

Then, $\sigma_v = \eta_n$. In order to prove the claim, it suffices to show that

$$0 \leqslant \eta_{2p} \leqslant p, \text{ and } 0 \leqslant \eta_{2p+1} \leqslant p + 1,$$

$$\alpha_{v_{2p}} = \eta_{2p}, \quad \text{and } \alpha_{v_{2p+1}} = s - \eta_{2p+1},$$

whenever the indices do not exceed n. We proceed by induction on p.

If $p = 0$, we have $\eta_0 = 0$ and $\alpha_{v_0} = \alpha_{u_i} = 0$ since $u_i \in S_3$. Note that $\eta_1 = \epsilon_1 \in \{0, 1\}$, so that $0 \leqslant \eta_1 \leqslant 1$. On the other hand, since $\alpha_{v_1} + \alpha_{v_0} = s - \epsilon_1$, one has $\alpha_{v_1} = s - \epsilon_1 = s - \eta_1$, and the case $p = 0$ holds.

Assume that $p \geqslant 1$. By the induction hypothesis, $0 \leqslant \eta_{2p-1} \leqslant p$ and $\alpha_{2p-1} = s - \eta_{2p-1}$. From the equation $\alpha_{2p-1} + \alpha_{2p} = s - \epsilon_{2p}$, we have

$$\alpha_{2p} = s - \epsilon_{2p} - (s - \eta_{2p-1}) = \eta_{2p-1} - \epsilon_{2p} = \eta_{2p}.$$

Since $\eta_{2p-1} \leqslant p$ by the induction hypothesis, we get $\eta_{2p} \leqslant p$. On the other hand, since $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in P_s$, we have $\alpha_{2p} \geqslant 0$, and so $\eta_{2p} \geqslant 0$.

From the equation $\alpha_{2p} + \alpha_{2p+1} = s - \epsilon_{2p+1}$, we have

$$\alpha_{2p+1} = s - \epsilon_{2p+1} - \eta_{2p} = s - (\epsilon_{2p+1} + \eta_{2p}) = s - \eta_{2p+1}.$$

Note that $0 \leqslant \eta_{2p} \leqslant p$, so $0 \leqslant \eta_{2p+1} \leqslant p+1$, and the claim follows.

For each $t \geqslant (l+1)/2$, we consider the integer point $\boldsymbol{\beta}(t) = (\beta_1(t), \ldots, \beta_r(t)) \in \mathbb{Z}^r$ where

$$\beta_v(t) = \begin{cases} \sigma_v & \text{if } v \in H_i \text{ and } \mathrm{d}(v, u_i) \text{ is even,} \\ t - \sigma_v & \text{if } v \in H_i \text{ and } \mathrm{d}(v, u_i) \text{ is odd.} \end{cases}$$

Then, $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}$ by Claim 2.

Claim 3: $\boldsymbol{\beta}(t) \in P_t$ for all $t \geqslant \lceil \frac{l+1}{2} \rceil$.

Firstly, we show that $\boldsymbol{\beta}(t) \in \mathbb{N}^r$. By Claim 2, it suffices to show that $\beta_v(t) \geqslant 0$ if $v \in V(H_i)$ and $\mathrm{d}(v, u_i)$ is odd for some $i = 1, \ldots, p$. In this case, $\beta_v(t) = t - \sigma_v$. By Claim 2 again, $\sigma_v \leqslant \lceil \mathrm{d}_{H_i}(v, u_i) \rceil \leqslant (l+1)/2$, and thus $\beta_v(t) \geqslant 0$.

Secondly, we prove that $\beta_u(t) + \beta_v(t) \leqslant t - 1$ for $\{u, v\} \in E_1$. We may assume that $u \in V(H_i)$ and $v \in V(H_j)$. We now consider four possible cases:

Case 1: $\mathrm{d}(u, u_i)$ and $\mathrm{d}(v, u_j)$ are even. If $i = j$, there are two even paths from u and v to u_i, respectively. Since $\{u, v\}$ is an edge of G, we deduce that G contains an odd cycle, which contradicts the fact that G is bipartite. Thus, $i \neq j$. In this case $\beta_u(t) = \sigma_u$ and $\beta_v(t) = \sigma_v$, so that

$$\beta_u(t) + \beta_v(t) = \sigma_u + \sigma_v \leqslant \frac{\mathrm{d}_{H_i}(u, u_i)}{2} + \frac{\mathrm{d}_{H_j}(v, u_j)}{2} = \frac{\mathrm{d}_{H_i}(u, u_i) + \mathrm{d}_{H_j}(v, u_j)}{2}.$$

If we have a simple path, say p_1 in H_i from u_i to u, and a simple path, say p_2, in H_j from v to u_j, then we have a simple path

$$p_1, u, v, p_2$$

from u_i to u_j in G. This implies that $\mathrm{d}_{H_i}(u, u_i) + \mathrm{d}_{H_j}(v, u_j) \leqslant l - 1$. Together with the inequality above, it gives

$$\beta_u(t) + \beta_v(t) \leqslant \frac{l-1}{2} \leqslant t - 1.$$

Case 2: $\mathrm{d}(u, u_i)$ is even and $\mathrm{d}(v, u_j)$ is odd. In this case, by Claim 2, one has

$$\alpha_u + \alpha_v = s + \sigma_u - \sigma_v \leqslant s - 1,$$

hence $a_u - a_v \leqslant -1$. It follows that

$$\beta_u(t) + \beta_v(t) = t + \sigma_u - \sigma_v \leqslant t - 1.$$

Case 3: $\mathrm{d}(u,u_i)$ is odd and $\mathrm{d}(v,u_j)$ is even. In this case the proof is similar to the previous case.

Case 4: $\mathrm{d}(u,u_i)$ and $\mathrm{d}(v,u_j)$ are odd. In this case, by Claim 2 we have

$$\alpha_u + \alpha_v = 2s - \sigma_u - \sigma_v \leqslant s - 1.$$

But this is not true, since $s \geqslant 2r$ and $\sigma_u \leqslant r-1$ and $\sigma_v \leqslant r-1$.

Therefore, we have proven that $\beta_u(t) + \beta_v(t) \leqslant t-1$ for $\{u,v\} \in E_1$. Similarly, we can verify $\beta_u(t) + \beta_v(t) \geqslant t$ for $\{u,v\} \in E_2$.

In summary, we have $\boldsymbol{\beta}(t) \in \mathcal{P}_t$ for $t \geqslant (l+1)/2$, and the claim follows. □

3. The index of depth stability

In this section, besides studying the non-increasing property of the depth functions, we establish the good upper bound for $\operatorname{dstab}(J(G))$ of cover ideals of bipartile graphs. Note that, we may assume that $E \neq \emptyset$ and thus $J(G) \neq 0$.

We recall that the analytic spread of a homogeneous ideal I of R is defined by

$$\ell(I) := \dim \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I), \text{ where } \mathcal{R}(I) = \bigoplus_{s=0}^{\infty} I^s \text{ is the Rees ring of } I.$$

For the behaviour of depth functions of cover ideals of bipartile graphs, by [8, Theorem 2.3] we have the first main result as follows:

Theorem 3.1. *Let G be a bipartile graph. Then,*

$$\operatorname{depth} R/J(G)^t \geq \operatorname{depth} R/J(G)^{t+1}, \text{ for all } t \geq 1.$$

In other words, $J(G)$ has non-inceasing depth function.

In order to investigate the index of depth stability of $J(G)$, we need the following two lemmas to do induction on the number of variables.

Lemma 3.2. [10, Lemma 1.3] *Let I be a monomial ideal of R and $F \subseteq \{1,\ldots,r\}$ such that $IR_F \neq R_F$. Let $S = K[x_i \mid i \notin F]$ and $J = IR_F \cap S$. Then,* $\operatorname{depth} R/I \leqslant |F| + \operatorname{depth} S/J$.

Lemma 3.3. [10, Lemma 1.4] *Let I be a monomial ideal of R with* $\operatorname{depth} R/I = d$. *Assume that $H^d_m(R/I)_{\boldsymbol{\alpha}} \neq \mathbf{0}$ for some $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_r) \in \mathbb{Z}^r$. Let $F = \operatorname{supp}^-\boldsymbol{\alpha}$, $S = K[x_i \mid i \notin F]$ and $J = IR_F \cap S$. Then,* $\operatorname{depth} R/I = \operatorname{depth} S/J + |F|$.

Next, we establish an upper bound for $\operatorname{dstab}(J(G))$. In comparison with the result in [8], the bound that we get in this paper is more optimal

Lemma 3.4. *Let G be a bipartite graph. Then,*

$$\lim_{t\to\infty} \operatorname{depth} R/J(G)^t = \dim R - \ell(J(G)).$$

Proof. Since G is bipartite, $J(G)$ is totally torsion-free by [6, Corollary 2.6]. The lemma can be deduced from [9, Proposition 10.3.2 and Theorem 10.3.13]. $\square$

Now, we are in position to prove the main result of our paper.

Theorem 3.5. *Let $G = (V, E)$ be a bipartite graph with the vertex set $V = \{1, \dots, r\}$ and let l be the length of a longest simple path in G. Then,*

$$\operatorname{depth} R/J(G)^t = r - \nu_0(G) - 1 \textit{ for all } t \geqslant \lceil \frac{l+1}{2} \rceil.$$

In particular, $\operatorname{dstab}(J(G)) \leqslant \lceil \frac{l+1}{2} \rceil$.

Proof. Since G is bipartite, by Theorem 3.1 and Lemma 3.4 we have

$$\operatorname{dstab}(J(G) = \min\{t \geqslant 1 \mid \operatorname{depth} R/J(G)^t = \dim R - \ell(J(G)\}. \tag{12}$$

Thus, it remains to show that $\operatorname{dstab}(J(G)) \leqslant \lceil \frac{l+1}{2} \rceil$.

We prove the assertion by induction on r.

If $r = 2$, we have

$$\operatorname{depth} R/J(G)^t = \operatorname{depth} R/J(G)$$

for all $t \geqslant 1$, and then the assertion holds.

Assume that $r \geqslant 3$. Let $t := \operatorname{dstab}(J(G))$ and $d := r - \ell(J(G))$. Then, $H^d_{\mathfrak{m}}(R/J(G)^t)_{\boldsymbol{\alpha}} \neq \mathbf{0}$ for some $\boldsymbol{\alpha} \in \mathbb{Z}^r$. For convenience, we let $F := \operatorname{supp}^- \boldsymbol{\alpha}$. We deduce two cases to consider.

Case 1: $F \neq \emptyset$. Let $V' := V \setminus F$ and $S := k[x_i \mid i \in V']$. We have $|V'| = r' < r$.

Let G' be the graph on the vertex set V' with the edge set $E' = \{e \in E \mid e \notin F\}$. Then, by Equation (2) we have $J(G') = J(G)R_F \cap S$. Since G is bipartite, G' is bipartite too.

Note that $\operatorname{depth} R/J(G)^t = \operatorname{depth} S/J(G')^t + |F|$ by Lemma 3.3. Now let $p := \operatorname{dstab}(J(G'))$. Together with Lemma 3.2 and Theorem 3.1 we have

$$\operatorname{depth} S/J(G')^p \geqslant \operatorname{depth} R/J(G)^p - |F| \geqslant \operatorname{depth} R/J(G)^t - |F| = \operatorname{depth} S/J(G')^t.$$

Together with Theorem 3.1, this fact follows that

$$\operatorname{depth} S/J(G')^t = \operatorname{depth} S/J(G')^p.$$

Hence, the inequalities above yields $\operatorname{depth} R/J(G)^p = \operatorname{depth} R/J(G)^t$. By combining this with Equation (12), we get $t \leqslant p$.

On the other hand, $p \leqslant \lceil \frac{l'+1}{2} \rceil < \lceil \frac{l+1}{2} \rceil$ by the induction hypothesis. Thus, $t < \lceil \frac{l+1}{2} \rceil$, and the assertion holds for this case.

Case 2: $F = \emptyset$, i.e. $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$. By Lemma 1.1 we have

$$\dim_k \widetilde{H}_{d-1}(\Delta_{\boldsymbol{\alpha}}(J(G)^t); k) = \dim_k H^d_{\mathfrak{m}}(R/J(G)^t)_{\boldsymbol{\alpha}},$$

so $\widetilde{H}_{d-1}(\Delta_{\boldsymbol{\alpha}}(J(G)^t); k) \neq \mathbf{0}$.

Suppose that $E = \{e_1, \ldots, e_m\}$. By (3) and Lemma 1.1 we may assume that

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(J(G)^t) = \{V \setminus e_1, \ldots, \mathcal{V} \setminus e_q\},$$

for some $1 \leqslant q \leqslant m$.

For each $s \geqslant 1$, let $\mathcal{P}_s$ the set of solotions in $\mathbb{R}^r$ of the following system of linear inequalities

$$\begin{cases} x_u + x_v \leqslant s-1 & e_j = \{u, v\} \ \text{ for } j = 1, \ldots, q, \\ x_u + x_v \geqslant s & e_j = \{u, v\} \ \text{ for } j = q+1, \ldots, v, \\ x_1 \geqslant 0, \ldots, x_p \geqslant 0. \end{cases} \tag{13}$$

In the proof of the key Lemma 2.4, we have that the system of linear inequalities (13) has integral vertex for all $s \geqslant \lceil \frac{l+1}{2} \rceil$. Let $\boldsymbol{\gamma} \in \mathcal{P}_s \cap \mathbb{N}^r$ for $s \geqslant \lceil \frac{l+1}{2} \rceil$. Then, by (13) and Lemma 1.6 we have $\Delta_{\boldsymbol{\gamma}}(J(G)^s) = \Delta_{\boldsymbol{\alpha}}(J(G)^t)$ for all $s \geqslant \lceil \frac{l+1}{2} \rceil$.

In particular,

$$\widetilde{H}_{d-1}(\Delta_{\boldsymbol{\gamma}}(J(G)^{\lceil \frac{l+1}{2} \rceil}); k) \neq \mathbf{0}.$$

Together with Lemma 1.1 we deduce that $H^d_{\mathfrak{m}}(R/J(G)^{\lceil \frac{l+1}{2} \rceil}) \neq \mathbf{0}$. Consequently, $\operatorname{depth} R/J(G)^{\lceil \frac{l+1}{2} \rceil} \leqslant d$.

Hence, $\operatorname{depth} R/J(G)^{\lceil \frac{l+1}{2} \rceil} = d$, and thus $t = \operatorname{dstab}(J(G)) \leqslant \lceil \frac{l+1}{2} \rceil$ by (12), as required.

Moreover, if we denote the ordered matching number of G is $\nu_0(G)$, then by [4, Corollary 3.9] we have $\ell(J(G)) = \nu_0(G) + 1$. Hence, the theorem is proved. $\square$

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