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# THE THREE-DIMENSIONAL NON-UNIFORM AND ASYMMETRIC $n$-PLAYER GAME WITH EQUAL INITIAL FORTUNES

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**Abstract**

In this paper, we find expected duration time of the three-dimensional gambler's ruin problem with $n$ players when the players have equal initial fortunes of $d$ in each currency, $1 \le d \le n+1$. Our result is an extension of Hashemiparast and Sabzevari's work when they found the expected duration time of two-dimensional game. A limiting result when our three-dimensional game is reduced to the two-dimensional game is also discussed.

## 1 Introduction

Gambler's ruin problem is one of the classical topics in probability theory, that has brought researcher's attentions worldwild. Several aspects of the problem have been studied. For example, in the aspect of multi-player game and multi-dimensional game. In the aspect of the $n$-player game, the player $i^{th}$ ($i = 1, \ldots, n$) either wins with probability $p_i$ or loses with probability $1 - p_i$ in each round where $p_1 + p_2 + \cdots + p_n = 1$. If $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, the game is said to be **symmetric**; otherwise it is **asymmetric**. When one of the $n$ players wins, each of other players must pay one dollar to the winner, so the winner

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has a total gain of $n-1$ dollars for the round. Play continues until one of the players is ruined. In the aspect of $m$-dimensional game, the players randomly select 1 of the $m$ currencies to gambling in each round and the game is over when one of the players ruins one currency. In each round, the gamblers select the $j^{th}$ currency ($j = 1, \dots, m$) with probability $r_j$ where $r_1+r_2+\cdots+r_m = 1$. When $r_1 = r_2 = \cdots = r_m = \frac{1}{m}$, the game is said to be **uniform**; otherwise it is **non-uniform**.

The one-dimensional non-uniform and asymmetric $n$-player game for $n > 3$ with equal initial fortunes of $d$ dollars, where $1 \leq d \leq n+1$, was first examined by Rocha and Stern [9]. In 2004, they considered the non-uniform and asymmetric $n$-player gambler's ruin problem with equal initial fortunes of $n+c$ dollars for fixed $c$, $c = 0, 1, 2, \dots$, and $n \geq \max(c, 2)$ (see [10]).

The two-dimensional uniform and symmetric game was first considered by Orr and Zeilberger [8]. They derived a formula for the expected duration time of the two-player game by using generating function. In 2012, Hashemiparast and Sabzevari [6] extended Orr and Zeilberger's results to the two-dimensional non-uniform and asymmetric $n$-player game with equal initial fortunes of $d$, $1 \leq d \leq n+1$. They claimed that their work was the first combination of the $n$-player and multi-dimensional games.

In this work, we extend Hashemiparast and Sabzevari's technique used for the two-dimensional game to derive the expected duration time of the three-dimensional game when the players start with equal initial fortunes of $d$, $1 \leq d \leq n+1$. Formulae of the expected duration time of the three-dimensional game are derived in Section 2 and a limiting result when our three-dimensional game is reduced to the two-dimensional game is discussed in Section 3.

## 2 The Three-Dimensional $n$-Player Game with Equal Initial Fortunes

Throughout this paper, we assume that the players start with equal initial fortunes of $d$ dollars, $d$ euros and $d$ bahts where $1 \leq d \leq n+1$. In each round, the players randomly select one currency to gambling. The probabilities of selecting dollar, euro and baht are $p, q$ and $r$ ($p+q+r = 1$), respectively. The $i^{th}$ player wins with probability $p_i$, $i = 1, \dots, n$, where $p_1 + \cdots + p_n = 1$. Let $T$ denote the number of rounds until the first ruin. In this section, we derive formulae of the expected duration time of three-dimensional non-uniform and asymmetric $n$-player game in three different cases, where short versions of the first two cases can be found in Chaidee et al.[1]. However, for smooth presentation and better understanding in materials of audiences, those materials will be repeated.

### 2.1 The Game with Initial Fortunes of $d$, $1 \leq d \leq n-1$

In one-dimensional game, if the initial fortune of each player, $d$, is less than $n$, then the game is eventually ended at time $d$, in which we have $P(T=d)=1$ in this case. In two-dimensional game, Hashemiparast and Sabzevari [6] suggested that a ruin can occur at times $d, d+1, \ldots, 2d-1$. In three-dimensional game, if all three currencies were used in the game, then there is only one currency that was used in exactly $d$ rounds and the number of rounds play on other two currencies is less than $d$. Therefore, all possible values of the duration time $T$ are $d \leq T \leq 3d-2$.

If the ruin occurs while playing dollar at time $t$, then the players must play $d$ rounds on dollar, $k$ rounds on euro, and $t-d-k$ rounds on baht where $0 \leq k \leq t-d$. Since the number of rounds played on euro and baht must be less than $d$, $0 \leq k \leq d-1$ and $0 \leq t-d-k \leq d-1$. The probability of this event is $\binom{t-1}{t-d}\binom{t-d}{k}p^d q^k r^{t-d-k}$. Similar argument can be made in the case that ruin occurs while playing euro or baht. Then, the expected duration time is

$$E(T) = \sum_{t=d}^{3d-2}\Big[\sum_{k=0}^{\min(t-d,d-1)} t\binom{t-1}{t-d}\binom{t-d}{k}(p^d q^k r^{t-d-k} + q^d p^k r^{t-d-k} + r^d p^k q^{t-d-k})\Big]. \tag{1}$$

### 2.2 The Game with Initial Fortunes of $d = n$

In this section, we consider the game when the initial fortunes are equal to $n$ dollars, $n$ euros and $n$ bahts. From Rocha and Stern [9],

$$P(T > n| \text{ play dollar for } n \text{ rounds}) = n! p_1 p_2 \cdots p_n = \alpha.$$

Since ruin cannot occur before time $n$, we have

$$P(T = n| \text{ play dollar for } n \text{ rounds}) = 1 - \alpha.$$

Following Hashemiparast and Sabzevari [6], we can generalize the two equations above to compute $P(T=n)$ as follows:

$$\begin{aligned} P(T=n) = {} & P(T=n| \text{ play dollar for } n \text{ rounds})P(\text{play dollar for } n \text{ rounds}) \\ & + P(T=n| \text{ play euro for } n \text{ rounds})P(\text{play euro for } n \text{ rounds}) \\ & + P(T=n| \text{ play baht for } n \text{ rounds})P(\text{play baht for } n \text{ rounds}) \\ = {} & p^n(1-\alpha) + q^n(1-\alpha) + r^n(1-\alpha). \end{aligned}$$

To compute the expected duration time, we first consider when the game ends at time $2n$ $(T = 2n)$. In each round, the players randomly select one currency to play. There are three possible cases: all $2n$ rounds were played on only one currency; two currencies; and all three currencies.

**case 1.** All $2n$ rounds were played on only one currency, say dollar. In this case, ruin does not occur in the first $n$ rounds (with probability $1-\alpha$) but in the next $n$ rounds (with probability $\alpha$). This event occurs with probability $p^{2n}\alpha(1-\alpha)$.

**case 2.** The $2n$ rounds were played on two currencies, say dollar and euro. Without loss of generality, assume that the ruin occurs while playing dollar. Then, among the first $2n-1$ rounds, there are $n$ rounds that played on euro but a ruin does not occur (with probability $\binom{2n-1}{n}p^{n-1}q^n\alpha$). The $2n^{th}$ round was played on the dollar and a ruin occur (with probability $p(1-\alpha)$). Then the probability of this event is $\binom{2n-1}{n}p^nq^n\alpha(1-\alpha)$.

**case 3.** The $2n$ rounds were played on all three currencies. Without loss of generality, assume that the ruin occurs while playing dollar. Then, among the first $2n-1$ rounds, there are $m$ rounds $(1 \le m \le n-1)$ that played on euro, and $n-m$ rounds that played on baht and the ruin has not occurred (with probability $\binom{2n-1}{n}\binom{n}{m}p^{n-1}q^mr^{n-m}$). The $2n^{th}$ round was played on the dollar and a ruin occur (with probability $p(1-\alpha)$). Then the probability of this event is $\binom{2n-1}{n}\binom{n}{m}(p^nq^mr^{n-m})(1-\alpha)$.

Therefore,

$$\begin{aligned}
P(T=2n) &= (p^{2n}+q^{2n}+r^{2n})\alpha(1-\alpha) \\
&\quad + 2\binom{2n-1}{n}(p^nq^n+p^nr^n+r^nq^n)\alpha(1-\alpha) \\
&\quad + \sum_{m=1}^{n-1}\binom{2n-1}{n}\binom{n}{m}(p^nq^mr^{n-m}+q^np^mr^{n-m}+r^np^mq^{n-m})(1-\alpha).
\end{aligned}$$

Following the same method, we can derive the probability, $P(T=kn)$, $k = 2,3,\ldots$, given as follows:

$$\begin{aligned}
P(T=kn) &= (p^{kn}+q^{kn}+r^{kn})\alpha^{k-1}(1-\alpha) \\
&\quad + \sum_{i=1}^{k-1}\sum_{j=0}^{i}\binom{kn-1}{in}\binom{in}{jn}\Phi_{k,i,j,0}\alpha^{k-1}(1-\alpha) \\
&\quad + \sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\sum_{m=1}^{n-1}\binom{kn-1}{in}\binom{in}{jn+m}\Phi_{k,i,j,m}\alpha^{k-2}(1-\alpha),
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{k,i,j,m} &= p^{(k-i)n}q^{jn+m}r^{(i-j)n-m}+q^{(k-i)n}p^{jn+m}r^{(i-j)n-m} \\
&\quad + r^{(k-i)n}p^{jn+m}q^{(i-j)n-m}.
\end{aligned}$$

We now consider the general case when the game ends at time $T = kn + l$ where $k = 2, 3, \ldots$ and $0 \leq l \leq n - 1$. First, we assume that the ruin occurs while playing dollar game, we then can find the probability $P(T = kn + l,$ ruin occured on dollar$)$ as follows:

**case 1.** Play $kn - 1$ rounds on dollar, play $l - m$ rounds on euro, and play $m$ rounds on baht where $m = 0, 1, \ldots, l$ and play on dollar in the $(kn + l)^{th}$ round. Then the probability of this event is

$$\sum_{m=0}^{l} \binom{kn+l-1}{l} \binom{l}{m} p^{kn} q^{l-m} r^m \alpha^{k-1}(1-\alpha).$$

**case 2.** Play $(k - i)n - 1$ rounds on dollar, play $jn + m$ rounds on euro, and play $(i - j)n + l - m$ rounds on baht where $i = 1, \ldots, k - 1$, $j = 0, \ldots, i$ and $m = 0, 1, \ldots, l$ and play on dollar in the $(kn + l)^{th}$ round. Then the probability of this event is

$$\sum_{i=1}^{k-1} \sum_{j=0}^{i} \sum_{m=0}^{l} \binom{kn+l-1}{in+l} \binom{in+l}{jn+m} p^{(k-i)n} q^{jn+m} r^{(i-j)n+l-m} \alpha^{k-1}(1-\alpha).$$

**case 3.** Play $(k - i)n - 1$ rounds on dollar, play $jn + m$ rounds on euro, and play $(i - j)n + l - m$ rounds on baht where $i = 1, \ldots, k - 1$, $j = 0, \ldots, i - 1$ and $m = l + 1, l + 2, \ldots, n - 1$ and play on dollar in the $(kn + l)^{th}$ round. Then the probability of this event is

$$\sum_{i=1}^{k-1} \sum_{j=0}^{i-1} \sum_{m=l+1}^{n-1} \binom{kn+l-1}{in+l} \binom{in+l}{jn+m} p^{(k-i)n} q^{jn+m} r^{(i-j)n+l-m} \alpha^{k-2}(1-\alpha).$$

By using similar techniques, we can calculate $P(T = kn+l,$ ruin occured on euro$)$ and $P(T = kn + l,$ ruin occured on baht$)$. Then,

$$\begin{aligned} P(T = kn + l) &= \sum_{m=0}^{l} \binom{kn+l-1}{l} \binom{l}{m} \Phi_{k,l,0,0,m} \alpha^{k-1}(1-\alpha) \\ &+ \sum_{i=1}^{k-1} \sum_{j=0}^{i} \sum_{m=0}^{l} \binom{kn+l-1}{in+l} \binom{in+l}{jn+m} \Phi_{k,l,i,j,m} \alpha^{k-1}(1-\alpha) \\ &+ \sum_{i=1}^{k-1} \sum_{j=0}^{i-1} \sum_{m=l+1}^{n-1} \binom{kn+l-1}{in+l} \binom{in+l}{jn+m} \Phi_{k,l,i,j,m} \alpha^{k-2}(1-\alpha), \end{aligned} \quad (2)$$

where

$$\begin{aligned} \Phi_{k,l,i,j,m} &= p^{(k-i)n} q^{jn+m} r^{(i-j)n+l-m} + q^{(k-i)n} p^{jn+m} r^{(i-j)n+l-m} \\ &+ r^{(k-i)n} p^{jn+m} q^{(i-j)n+l-m}. \end{aligned}$$

If $k=1$, there is only one possible case: play $n-1$ rounds on dollar, play $l-m$ rounds on euro, and play $m$ rounds on baht where $m=0,1,\ldots,l$ and in the round $n+l$, dollar have come up. This event occurs with probability

$$P(T=n+l)=\sum_{m=0}^{l}\binom{n+l-1}{l}\binom{l}{m}\Phi_{1,l,0,0,m}(1-\alpha). \tag{3}$$

By (2) and (3), we have

$$\begin{aligned}
E(T) = &\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m=0}^{l}(kn+l)\binom{kn+l-1}{l}\binom{l}{m}\Phi_{k,l,0,0,m}\alpha^{k-1}(1-\alpha)\\
&+\sum_{k=2}^{\infty}\sum_{l=0}^{n-1}\sum_{i=1}^{k-1}\sum_{j=0}^{i}\sum_{m=0}^{l}(kn+l)\binom{kn+l-1}{in+l}\binom{in+l}{jn+m}\Phi_{k,l,i,j,m}\alpha^{k-1}(1-\alpha)\\
&+\sum_{k=2}^{\infty}\sum_{l=0}^{n-1}\sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\sum_{m=l+1}^{n-1}(kn+l)\binom{kn+l-1}{in+l}\binom{in+l}{jn+m}\Phi_{k,l,i,j,m}\alpha^{k-2}(1-\alpha).
\end{aligned} \tag{4}$$

## 2.3 The Game with Initial Fortunes of $d=n+1$

In this section, we consider the game when the initial fortunes are equal to $n+1$ dollars, $n+1$ euros and $n+1$ bahts.

**Proposition 2.1.** *(Rocha and Stern, [9]) In one-dimensional game with equal initial fortunes of $n+1$ dollars, for $\beta=\frac{(n+1)!}{2}(p_1p_2\cdots p_n)$ and all $k=1,2,\ldots,$*
$P(T>kn+1)=\beta^k$,
$P(T=kn+1)=\beta^{k-1}(1-\beta)$.

Following Hashemiparast and Sabzevari [6], we can generalize the two equations above to compute $P(T=n+1)$ as follows:

$$P(T=n+1)=\ p^{n+1}(1-\beta)+q^{n+1}(1-\beta)+r^{n+1}(1-\beta).$$

Next, we consider the case $T=2n+1$. Assuming that the ruin occurs while playing dollar games, we then can calculate the probability
$P(T=2n+1,\text{ ruin occured on dollar})$ as follows:

**case 1.** All $2n+1$ rounds were played only on dollar. Then there is no ruin occur in the first $n+1$ rounds but a ruin occurs in the next $n$ rounds. Then, the probability of this event is $p^{2n+1}\beta(1-\beta)$.

**case 2.** There are $n+1$ rounds were played on dollar; $m$ $(0\le m\le n)$ rounds were played on euro and $n-m$ rounds were played on baht. Then, the probability of this event is $\binom{2n}{n}\binom{n}{m}(p^{n+1}q^mr^{n-m})(1-\beta)$.

Then,

$$
\begin{aligned}
&P(T = 2n+1,\ \text{ruin occured on dollar}) \\
&\quad = p^{2n+1}\beta(1-\beta) + \sum_{m=0}^{n} \binom{2n}{n}\binom{n}{m} p^{n+1} q^m r^{n-m}(1-\beta).
\end{aligned}
$$

By using similar techniques, we can calculate $P(T = 2n+1,\ \text{ruin occured on euro})$ and $P(T = 2n+1,\ \text{ruin occured on baht})$. Then,

$$
\begin{aligned}
P(T = 2n+1) &= (p^{2n+1} + q^{2n+1} + r^{2n+1})\beta(1-\beta) \\
&+ \sum_{m=0}^{n} \binom{2n}{n}\binom{n}{m}(p^{n+1}q^m r^{n-m} + q^{n+1}p^m r^{n-m} + r^{n+1}p^m q^{n-m})(1-\beta).
\end{aligned}
$$

Using the same method, we can derive the probability, $P(T = kn+1)$, $k = 3, 4, \dots$, given as follows:

$$
\begin{aligned}
P(T = kn+1) = {} & (p^{kn+1} + q^{kn+1} + r^{kn+1})\beta^{k-1}(1-\beta) \\
& + \sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\sum_{m=1}^{n-1} \binom{kn}{in}\binom{in}{jn+m}\Psi_{k,0,i,j,m}\beta^{k-2}(1-\beta) \\
& + \sum_{i=1}^{k-1} \binom{kn}{in}(\Psi_{k,0,i,i,0} + \Psi_{k,0,i,0,0})\beta^{k-2}(1-\beta) \\
& + \sum_{i=1}^{k-1}\sum_{j=1}^{i-1} \binom{kn}{in}\binom{in}{jn}\Psi_{k,0,i,j,0}\beta^{k-3}(1-\beta), \qquad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_{k,0,i,j,m} = {} & p^{(k-i)n+1}q^{jn+m}r^{(i-j)n-m} + q^{(k-i)n+1}p^{jn+m}r^{(i-j)n-m} \\
& + r^{(k-i)n+1}p^{jn+m}q^{(i-j)n-m}.
\end{aligned}
$$

Next, we consider the case $T = kn + l + 1$ ($k \geq 2$ and $1 \leq l \leq n-1$). First, we assume that the ruin occurs while playing dollar game, then we can find the probability $P(T = kn+l+1,\ \text{ruin occured on dollar})$ as follows:

**case 1.** Play $kn$ rounds on dollar, play $m$ rounds on euro, and play $l-m$ rounds on baht where $m = 1, \dots, l$ and play on dollar in the $(kn+l+1)^{th}$ round. Then the probability of this event is

$$
\sum_{m=0}^{l} \binom{kn+l}{l}\binom{l}{m} p^{kn+1} q^m r^{l-m}\beta^{k-1}(1-\beta).
$$

**case 2.** Play $(k-i)n$ rounds on dollar, play $jn+m$ rounds on euro, and play $(i-j)n+l-m$ rounds on baht where $i=1,\ldots,k-1$, $j=1,\ldots,i-1$ and $m=1,\ldots,l-1$ and play on dollar in the $(kn+l+1)^{th}$ round. Then the probability of this event is

$$\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}\sum_{m=1}^{l-1}\binom{kn+l}{in+l}\binom{in+l}{jn+m}p^{(k-i)n+1}q^{jn+m}r^{(i-j)n+l-m}\beta^{k-1}(1-\beta).$$

**case 3.** Play $(k-i)n$ rounds on dollar, play $in+m$ rounds on euro, and play $l-m$ rounds on baht where $i=1,\ldots,k-1$ and $m=1,\ldots,l$ and play on dollar in the $(kn+l+1)^{th}$ round. Then the probability of this event is

$$\sum_{i=1}^{k-1}\sum_{m=1}^{l}\binom{kn+l}{in+l}\binom{in+l}{in+m}p^{(k-i)n+1}q^{in+m}r^{l-m}\beta^{k-1}(1-\beta).$$

**case 4.** Play $(k-i)n$ rounds on dollar, play $jn+m$ rounds on euro, and play $(i-j)n+l-m$ rounds on baht where $i=1,\ldots,k-1$, $j=0,\ldots,i-1$ and $m=l,\ldots,n-1$ and play on dollar in the $(kn+l+1)^{th}$ round. Then the probability of this event is

$$\sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\sum_{m=l}^{n-1}\binom{kn+l}{in+l}\binom{in+l}{jn+m}p^{(k-i)n+1}q^{jn+m}r^{(i-j)n+l-m}\beta^{k-2}(1-\beta).$$

**case 5.** Play $(k-i)n$ rounds on dollar, play $jn$ rounds on euro, and play $(i-j)n+l$ rounds on baht where $i=1,\ldots,k-1$, $j=1,\ldots,i$ and play on dollar in the $(kn+l+1)^{th}$ round. Then the probability of this event is

$$\sum_{i=1}^{k-1}\sum_{j=1}^{i}\binom{kn+l}{in+l}\binom{in+l}{jn}p^{(k-i)n+1}q^{jn}r^{(i-j)n+l}\beta^{k-2}(1-\beta).$$

**case 6.** Play $(k-i)n$ rounds on dollar, play $m$ rounds on euro, and play $in+l-m$ rounds on baht where $i=1,\ldots,k-1$ and $m=0,\ldots,l-1$ and play on dollar in the $(kn+l+1)^{th}$ round. Then the probability of this event is

$$\sum_{i=1}^{k-1}\sum_{m=1}^{l-1}\binom{kn+l}{in+l}\binom{in+l}{m}p^{(k-i)n+1}q^{m}r^{in+l-m}\beta^{k-1}(1-\beta).$$

Then,

$$
\begin{aligned}
P(T=kn+l+1) = &\sum_{m=0}^{l} \binom{kn+l}{l}\binom{l}{m}\Psi_{k,l,0,0,m}\beta^{k-1}(1-\beta) \\
&+ \sum_{i=1}^{k-1}\sum_{j=1}^{i-1}\sum_{m=1}^{l-1} \binom{kn+l}{in+l}\binom{in+l}{jn+m}\Psi_{k,l,i,j,m}\beta^{k-1}(1-\beta) \\
&+ \sum_{i=1}^{k-1}\sum_{m=1}^{l} \binom{kn+l}{in+l}\binom{in+l}{in+m}\Psi_{k,l,i,i,m}\beta^{k-1}(1-\beta) \\
&+ \sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\sum_{m=l}^{n-1} \binom{kn+l}{in+l}\binom{in+l}{jn+m}\Psi_{k,l,i,j,m}\beta^{k-2}(1-\beta) \\
&+ \sum_{i=1}^{k-1}\sum_{j=1}^{i} \binom{kn+l}{in+l}\binom{in+l}{jn}\Psi_{k,l,i,j,0}\beta^{k-2}(1-\beta) \\
&+ \sum_{i=1}^{k-1}\sum_{m=1}^{l-1} \binom{kn+l}{in+l}\binom{in+l}{m}\Psi_{k,l,i,0,m}\beta^{k-1}(1-\beta), \quad (6)
\end{aligned}
$$

where $k = 2, 3, \ldots,\ l = 1, \ldots, n-1$ and

$$
\begin{aligned}
\Psi_{k,l,i,j,m} = &\, p^{(k-i)n+1}q^{jn+m}r^{(i-j)n+l-m} + q^{(k-i)n+1}p^{jn+m}r^{(i-j)n+l-m} \\
&+ r^{(k-i)n+1}p^{jn+m}q^{(i-j)n+l-m}.
\end{aligned}
$$

In the special case $k = 1$, there is only one possible case: play $n$ rounds on dollar, play $m$ rounds on euro, and play $l-m$ rounds on baht where $m = 0, 1, \ldots, l$ and play on dollar in the $(n+l+1)^{th}$ round. This event occurs with probability

$$
P(T=n+l+1) = \sum_{m=0}^{l} \binom{n+l}{l}\binom{l}{m}\Psi_{1,l,0,0,m}(1-\beta). \quad (7)
$$

From the equations $(5)-(7)$, we have

$$
\begin{aligned}
E(T) &= \sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\sum_{m=0}^{l}\binom{kn+l}{l}\binom{l}{m}(kn+l+1)\Psi_{k,l,0,0,m}\beta^{k-1}(1-\beta)\\
&+\sum_{k=2}^{\infty}\sum_{l=1}^{n-1}\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}\sum_{m=1}^{l-1}\binom{kn+l}{in+l}\binom{in+l}{jn+m}(kn+l+1)\Psi_{k,l,i,j,m}\beta^{k-1}(1-\beta)\\
&+\sum_{k=2}^{\infty}\sum_{l=1}^{n-1}\sum_{i=1}^{k-1}\sum_{m=1}^{l}\binom{kn+l}{in+l}\binom{in+l}{in+m}(kn+l+1)\Psi_{k,l,i,i,m}\beta^{k-1}(1-\beta)\\
&+\sum_{k=2}^{\infty}\sum_{l=1}^{n-1}\sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\sum_{m=l}^{n-1}\binom{kn+l}{in+l}\binom{in+l}{jn+m}(kn+l+1)\Psi_{k,l,i,j,m}\beta^{k-2}(1-\beta)\\
&+\sum_{k=2}^{\infty}\sum_{l=1}^{n-1}\sum_{i=1}^{k-1}\sum_{j=1}^{i}\binom{kn+l}{in+l}\binom{in+l}{jn}(kn+l+1)\Psi_{k,l,i,j,0}\beta^{k-2}(1-\beta)\\
&+\sum_{k=2}^{\infty}\sum_{l=1}^{n-1}\sum_{i=1}^{k-1}\sum_{m=1}^{l-1}\binom{kn+l}{in+l}\binom{in+l}{m}(kn+l+1)\Psi_{k,l,i,0,m}\beta^{k-1}(1-\beta)\\
&+\sum_{k=2}^{\infty}\sum_{i=1}^{k-1}\sum_{j=0}^{i-1}\sum_{m=1}^{n-1}\binom{kn}{in}\binom{in}{jn+m}(kn+1)\Psi_{k,0,i,j,m}\beta^{k-2}(1-\beta)\\
&+\sum_{k=2}^{\infty}\sum_{i=1}^{k-1}\binom{kn}{in}(kn+1)(\Psi_{k,0,i,i,0}+\Psi_{k,0,i,0,0})\beta^{k-2}(1-\beta)\\
&+\sum_{k=2}^{\infty}\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}\binom{kn}{in}\binom{in}{jn}(kn+1)\Psi_{k,0,i,j,0}\beta^{k-3}(1-\beta).
\end{aligned}
\tag{8}
$$

## 3 Discussion and Conclusion

In this section, we first show some numerical results of the expected duration time in the three-dimensional game with initial fortunes of $d$, $1 \le d \le n-1$. We obtain the expected duration time for the following games.

$(i)$ The three-dimensional $n$-player game with $p = q = r$.

$(ii)$ The three-dimensional $n$-player game with $p = q = \frac{4}{9}, r = \frac{1}{9}$.

$(iii)$ The three-dimensional $n$-player game with $p = q = \frac{5000}{10001}, r = \frac{1}{10001}$.

$(iv)$ The two-dimensional $n$-player game with $p = q = \frac{1}{2}$.

TABLE I
Expected duration time in the three-dimensional 11-player game when the players use equal initial fortunes of $d$ in each currency where $1 \leq d \leq 10$.

| $d$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| *(i)* | 3.333 | 6.016 | 8.876 | 11.846 | 14.894 | 18.001 | 21.513 | 24.342 | 27.563 |
| *(ii)* | 2.812 | 4.686 | 6.591 | 8.525 | 10.488 | 12.474 | 14.481 | 16.505 | 18.543 |
| *(iii)* | 2.500 | 4.125 | 5.813 | 7.540 | 9.294 | 11.069 | 12.859 | 14.663 | 16.478 |
| *(iv)* | 2.500 | 4.125 | 5.813 | 7.539 | 9.293 | 11.067 | 12.858 | 14.662 | 16.476 |

From Table I, we see that the expected duration time of the three-dimensional game is closer to the expected duration time of the two-dimensional game when the probability in choosing baht closer to zero. For example, the expected duration time of the three-dimensional 11-player game with $r = \frac{1}{10001}$ when $d = 7$ is 11.069. It is closer to 11.067 which is the expected duration time of the two-dimensional 11-player game when $d = 7$. The proof of the general case will be shown in the following corollary.

**Corollary 3.1 (Limiting Results).** *If $r \to 0$, we have*

$$E(T) = \begin{cases} \displaystyle\sum_{t=d}^{2d-1} t\binom{t-1}{d-1}(p^d q^{t-d} + p^{t-d} q^d) & \text{if} \quad 1 \leq d \leq n-1, \\ \displaystyle\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\sum_{i=0}^{k-1}(kn+l)\binom{kn+l-1}{in+l}\Lambda_{k,l,i}\alpha^{k-1}(1-\alpha) & \text{if} \quad d = n, \\ \displaystyle\sum_{k=1}^{\infty}(kn+1)\Gamma_{k,0,0}\beta^{k-1}(1-\beta) & \text{if} \quad d = n+1, \\ \displaystyle +\sum_{k=1}^{\infty}\sum_{i=1}^{k-1}(kn+1)\binom{kn}{in}\Gamma_{k,0,i}\beta^{k-2}(1-\beta) & \\ \displaystyle +\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\sum_{i=0}^{k-1}(kn+l+1)\binom{kn+l}{in+l}\Gamma_{k,l,i}\beta^{k-1}(1-\beta) & \end{cases}$$

*where $\Lambda_{k,l,i} = p^{(k-i)n}q^{in+l} + p^{in+l}q^{(k-i)n}$ and $\Gamma_{k,l,i} = p^{(k-i)n+1}q^{in+l}$ $+ p^{in+l}q^{(k-i)n+1}$, which is the expected duration time of the two-dimensional game in Hashemiparast and Sabzevari [6].*

*Proof.* We note that if $r \to 0$, then $p + q \to 1$ and $r^x \to 0$ for all $x \neq 0$. Then all terms of the equation (1) are zeroes except the terms that $k = t - d$. Since $k \neq t - d$ for all $t = 2d, 2d+1, \ldots, 3d-2$, the equation (1) approaches to

$$E(T) = \sum_{t=d}^{2d-1} t\binom{t-1}{d-1}(p^d q^{t-d} + p^{t-d} q^d),$$

when $r$ approaches zero.
In the cases of $d = n$ and $d = n+1$, we notice that all terms are zeroes except

the terms in which $i = j$ and $l = m$. Then, the equations (4) and (8) approach to

$$\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\sum_{i=1}^{k-1}(kn+l)\binom{kn+l-1}{in+l}(p^{(k-i)n}q^{in+l}+p^{in+l}q^{(k-i)n})\alpha^{k-1}(1-\alpha),$$

and

$$\begin{aligned}&\sum_{k=1}^{\infty}(kn+1)(p^{(k-i)n}q^{in+l}+p^{in+l}q^{(k-i)n})\beta^{k-1}(1-\beta)\\ &+\sum_{k=1}^{\infty}\sum_{i=1}^{k-1}(kn+1)\binom{kn}{in}(p^{(k-i)n}q^{in+l}+p^{in+l}q^{(k-i)n})\beta^{k-2}(1-\beta)\\ &+\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\sum_{i=0}^{k-1}(kn+l+1)\binom{kn+l}{in+l}(p^{(k-i)n}q^{in+l}+p^{in+l}q^{(k-i)n})\beta^{k-1}(1-\beta),\end{aligned}$$

respectively, when $r$ approaches zero.

□

In this paper, the expected duration time of the three-dimensional non-uniform and asymmetric $n$-player games when the players use equal initial fortunes of $d$ dollars, $d$ euros and $d$ baht, $1 \le d \le n+1$, have been computed. Some further extensions of our results could be considered. For example, the multi-dimensional game with $n$ players and the case where ties may occur.

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