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SOME UNIQUENESS THEOREMS FOR HOLOMORPHIC CURVES ON ANNULUS SHARING HYPERSURFACES IN GENERAL POSITION

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Abstract

Recently H. T. Phuong and L. Vilaisavanh proved a second main theorem for algebraically non-degenerate holomorphic curves on annulus intersecting hypersurfaces. Based on this theorem, here we prove some uniqueness theorems for holomorphic curves on annulus in the case of hypersurfaces in general position in complex projective.

1 Introduction and results

The uniqueness problem for holomorphic curves on domain in complex plane $\mathbb{C}$ has been studied intensively many mathematicians. The first is the work of H. Fujimoto ([4]) in 1975, which generalized Nevanlinna's five point theorem to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Next is the works of Fujimoto

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([5]), Ru ([13]), Dethloff-Tan ([2],[3]), Yan-Chen ([1],[14]), Phuong ([8],[9],[10]), and others. For holomorphic curves on an annulus, in 2013, H. T. Phuong and T. H. Minh ([7]) proved a uniqueness theorem in the case of hyperplanes. And in 2019, H. T. Phuong and L. Vilaisavanh ([11]) introduced some uniqueness theorem in the case of hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$. Our idea here is to give some uniqueness results for algebraically non-degenerate holomorphic maps on an annulus sharing hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$ by using the second main theorem with ramification of H. T. Phuong and L. Vilaisavanh ([12]). First of all, we introduce some notations.

Let $R > 1$ be a fixed positive real number or $+\infty$, set

$$\Delta = \big\{z \in \mathbb{C} : \frac{1}{R} < |z| < R\big\},$$

be an annulus in $\mathbb{C}$.

Now let $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve on Δ, where $f_0, \ldots, f_n$ be holomorphic functions on Δ without common zeros. The mapping $(f_0, \ldots, f_n) : \Delta \to \mathbb{C}^{n+1}$ is said to be reduced representative of f. For $1 < r < R$, characteristic function $T_f(r)$ of f is defined by

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log \|f(r^{-1}e^{i\theta})\| d\theta,$$

where $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f. We set

$$O_f(r) = \begin{cases} O(\log r + \log T_f(r)) & \text{if } R = +\infty \\ O(\log \dfrac{1}{R-r} + \log T_f(r)) & \text{if } R < +\infty. \end{cases}$$

Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and Q be the homogeneous polynomial of degree d in $n+1$ variables with coefficients in $\mathbb{C}$ defining D, we define

$$\overline{E}_f(D) := \{z \in \Delta \mid Q \circ f(z) = 0 \ \text{ ignoring multiplicity }\};$$
$$E_f(D) := \{(z, m) \in \Delta \times \mathbb{N} \mid Q \circ f(z) = 0 \ \text{ and } \operatorname{ord}_{Q\circ f}(z) = m\},$$

where $\operatorname{ord}_g(z)$ is the order of zero on z of a holomorphic functions g. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of hypersurfaces, we define

$$\overline{E}_f(\mathcal{D}) := \bigcup_{D\in\mathcal{D}} \overline{E}_f(D) \quad \text{ and } \quad E_f(\mathcal{D}) := \bigcup_{D\in\mathcal{D}} E_f(D).$$

We recall that a collection of $q > n$ hypersurfaces $\mathcal{D} = \{D_1, \ldots, D_q\}$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in general position if for any distinct $i_1, \ldots, i_{n+1} \in$

$\{1, \ldots, q\}$,

$$\bigcap_{k=1}^{n+1} D_{i_k} = \emptyset.$$

In 2019, H. T. Phuong and L. Vilaisavanh ([11]) given some uniqueness theorem for holomorphic curves on annulus sharing hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ as following

Theorem A. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n_{\mathcal{D}} + 1 + 2n_{\mathcal{D}}^2/m_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.*

Theorem B. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n_{\mathcal{D}} + 1 + 2n_{\mathcal{D}}/m_{\mathcal{D}}$ hypersurfaces in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ such that*

(a) *$f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$,*

(b) *$\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ and $\overline{E}_g(D_i) \cap \overline{E}_g(D_j) = \emptyset$ for all $i \neq j \in \{1, \ldots, q\}$.*

Then $f \equiv g$.

Our contribution in this paper is to give some uniqueness results for algebraically non-degenerate holomorphic maps on an annulus sharing hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$.

Now let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of q hypersurfaces in general position. We denote the degree of D_j by d_j for $j = 1, \ldots, q$ and let d be the least common multiple of the d_j. Set

$$\delta_{\mathcal{D}} := \min\{d_1, \ldots, d_q\},$$

and

$$M = (d(n+1)^2 2^{n+1} + 1)^n.$$

Out results are stated as follows:

Theorem 1. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n + 1 + 2Mn/\delta_{\mathcal{D}}$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$ such that $f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$. Then $f \equiv g$.*

Theorem 2. *Let f and g be algebraically non-degenerate holomorphic curves from Δ into $\mathbb{P}^n(\mathbb{C})$ such that $O_f(r) = o(T_f(r))$ and $O_g(r) = o(T_g(r))$. Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of $q > n + 1 + 2M/\delta_{\mathcal{D}}$ hypersurfaces in general position in $\mathbb{P}^n(\mathbb{C})$ such that*

(a) *$f(z) = g(z)$ for all $z \in \overline{E}_f(\mathcal{D}) \cup \overline{E}_g(\mathcal{D})$,*

(b) $\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$ *and* $\overline{E}_g(D_i) \cap \overline{E}_g(D_j) = \emptyset$ *for all* $i \neq j \in \{1, \dots, q\}$.
Then $f \equiv g$.

We know that, holomorphic function $h : \Delta \to \mathbb{C}$ is transcendental function if $\limsup\limits_{r\to+\infty} \dfrac{T_0(r,h)}{\log r} = \infty$ in the case $R = \infty$ and $\limsup\limits_{r\to R} \dfrac{T_0(r,h)}{-\log(R-r)} = \infty$ in the case $R < +\infty$. So holomorphic curve on an annulus $f = (f_0 : f_1 : \cdots : f_n)$ satisfy $O_f(r) = o(T_f(r))$ if one of $f_j, 0 \leqslant j \leqslant n$, is transcendental function.

Theorem 1 and Theorem 2 are uniqueness theorems for non-constant holomorphic curves on annulus Δ in the case of hypersurfaces. They give some sufficient conditions for two algebraically non-degenerate holomorphic curves on Δ are equivalent and the collections of hypersurfaces in these theorems are in general position.

2 Preliminaries and Some Lemmas

In this section, we introduce some notations and recall some results in Nevanlinna theory for meromorphic functions and holomorphic curves on annulus, which are necessary for proofs of our results. For any real number r such that $1 < r < R$, we denote

$$\Delta_{1,r} = \left\{z \in \mathbb{C} : \frac{1}{r} < |z| \leqslant 1\right\}, \quad \Delta_{2,r} = \left\{z \in \mathbb{C} : 1 < |z| < r\right\},$$

and set

$$\Delta_r = \Delta_{1,r} \cup \Delta_{2,r}.$$

Let f be a meromorphic function on Δ, we denote $n_1(r,\infty)$ the number of poles in $\Delta_{1,r}$ and $n_2(r,\infty)$ the number of poles of f in $\Delta_{2,r}$.

For $c \in \mathbb{C}$, we denote $n_1\left(r, \dfrac{1}{f-c}\right)$ the number of zeros of $f - c$ in $\Delta_{1,r}$, $n_2\left(r, \dfrac{1}{f-c}\right)$ the number of zeros of $f - c$ in $\Delta_{2,r}$. We put

$$N_1\left(r, \frac{1}{f-c}\right) = \int_{1/r}^{1} \frac{n_1(t, \frac{1}{f-c})}{t} dt,$$
$$N_2\left(r, \frac{1}{f-c}\right) = \int_{1}^{r} \frac{n_2(t, \frac{1}{f-c})}{t} dt,$$

and

$$N_1(r, f) = N_1(r, \infty) = \int_{1/r}^{1} \frac{n_1(t,\infty)}{t} dt,$$
$$N_2(r, f) = N_2(r, \infty) = \int_{1}^{r} \frac{n_2(t,\infty)}{t} dt.$$

Let

$$N_0\left(r, \frac{1}{f-c}\right) = N_1\left(r, \frac{1}{f-c}\right) + N_2\left(r, \frac{1}{f-c}\right),$$
$$N_0(r, f) = N_1(r, f) + N_2(r, f).$$

It is easy to see that the functions $N_0\left(r, \frac{1}{f-c}\right)$ and $N_0(r, f)$ are non-negative.

Lemma 2.1 ([6]) *Let f be a non-constant meromorphic function on Δ. Then for any $r \in (1, R)$, we have*

$$N_0\left(r, \frac{1}{f}\right) - N_0(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|f(r^{-1}e^{i\theta})|d\theta - \frac{1}{\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta.$$

Let $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve on Δ, where $f_0, \ldots, f_n$ be holomorphic functions on Δ without common zeros. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and let Q is the homogeneous polynomial of degree d defining D. Let $n_{1,f}(r, D)$ be the number of zeros of $Q \circ f$ in $\Delta_{1,r}$, $n_{2,f}(r, D)$ be the number of zeros of $Q \circ f$ in $\Delta_{2,r}$, counting multiplicity. The integrated counting and truncated functions are defined by

$$N_{1,f}(r, D) = \int_{r^{-1}}^{1} \frac{n_{1,f}(t, D)}{t}dt, \quad N_{2,f}(r, D) = \int_{1}^{r} \frac{n_{2,f}(t, D)}{t}dt,$$

and we set

$$N_f(r, D) = N_{1,f}(r, D) + N_{2,f}(r, D).$$

Let α be a positive integer, we denote $n^{\alpha}_{1,f}(r, D)$ the number of zeros with multiplicity truncated by M of $Q \circ f$ in $\Delta_{1,r}$, $n^{\alpha}_{2,f}(r, D)$ be the number of zeros of $Q \circ f$ in $\Delta_{2,r}$ where any zero is counted with multiplicity if its multiplicity is less than or equal to α, and α times otherwise, namely

$$n^{\alpha}_{1,f}(r, D) = \sum_{z \in \Delta_{1,r}, Q\circ f(z)=0} \min\{\mathrm{ord}_{Q\circ f}(z), \alpha\},$$

$$n^{\alpha}_{2,f}(r, D) = \sum_{z \in \Delta_{2,r}, Q\circ f(z)=0} \min\{\mathrm{ord}_{Q\circ f}(z), \alpha\}.$$

The integrated counting and truncated functions are defined by

$$N^{\alpha}_{1,f}(r, D) = \int_{r^{-1}}^{1} \frac{n^{\alpha}_{1,f}(t, D)}{t}dt, \quad N^{\alpha}_{2,f}(r, D) = \int_{1}^{r} \frac{n^{\alpha}_{2,f}(t, D)}{t}dt,$$

and we set

$$N_f^\alpha(r, D) = N_{1,f}^\alpha(r, D) + N_{2,f}^\alpha(r, D).$$

For a positive integer k, we denote

$$n_{1,f}^\alpha(r, D, > k) = \sum_{z \in \Delta_{1,r}, \mathrm{ord}_{Q\circ f}(z) > k} \min\{\mathrm{ord}_{Q\circ f}(z), \alpha\},$$

$$n_{2,f}^\alpha(r, D > k) = \sum_{z \in \Delta_{2,r}, \mathrm{ord}_{Q\circ f}(z) > k} \min\{\mathrm{ord}_{Q\circ f}(z), \alpha\}.$$

The integrated counting and truncated functions are defined by

$$N_{1,f}^\alpha(r, D, > k) = \int_{r^{-1}}^{1} \frac{n_{1,f}^\alpha(t, D, > k)}{t} dt,$$
$$N_{2,f}^\alpha(r, D, > k) = \int_{1}^{r} \frac{n_{2,f}^\alpha(t, D, > k)}{t} dt.$$

Put

$$N_f^\alpha(r, D, > k) = N_{1,f}^\alpha(r, D) + N_{2,f}^\alpha(r, D, > k).$$

Similarly, we denote

$$n_{1,f}^\alpha(r, D, \leqslant k) = \sum_{z \in \Delta_{1,r}, 0 < \mathrm{ord}_{Q\circ f}(z) \leqslant k} \min\{\mathrm{ord}_{Q\circ f}(z), \alpha\},$$
$$n_{2,f}^\alpha(r, D \leqslant k) = \sum_{z \in \Delta_{2,r}, 0 < \mathrm{ord}_{Q\circ f}(z) \leqslant k} \min\{\mathrm{ord}_{Q\circ f}(z), \alpha\},$$
$$N_{1,f}^\alpha(r, D, \leqslant k) = \int_{r^{-1}}^{1} \frac{n_{1,f}^\alpha(t, D, \leqslant k)}{t} dt,$$
$$N_{2,f}^\alpha(r, D, \leqslant k) = \int_{1}^{r} \frac{n_{2,f}^\alpha(t, D, \leqslant k)}{t} dt,$$

and we put

$$N_f^\alpha(r, D, \leqslant k) = N_{1,f}^\alpha(r, D, \leqslant k) + N_{2,f}^\alpha(r, D, \leqslant k).$$

So it is to see that

$$N_f^\alpha(r, D) = N_f^\alpha(r, D, \leqslant k) + N_f^\alpha(r, D, > k)$$

holds for any integer α and k.

In 2021, H.T. Phuong and L. Vilaisavanh ([12]) proved the following

Lemma 2.2. *Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d and $f = (f_0 : \cdots : f_n) : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained D. Then we have for any $1 < r < R$,*

$$m_f(r, D) + N_f(r, D) = dT_f(r) + O(1).$$

Lemma 2.3. *Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve, and let $D_j, 1 \leqslant j \leqslant q$, be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Let d be the least common multiple of the $d_1, d_2, \ldots, d_q$. Let $0 < \varepsilon < 1$ and*

$$\alpha \geqslant \left(d[(n+1)^2 2^n)\varepsilon^{-1}] + 1\right)^n.$$

Then for any $1 < r < R$, we have

$$\| \quad (q - (n+1) - \varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} d_j^{-1} N_f^{\alpha}(r, D_j) + O_f(r).$$

3 Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. Assume that $f \not\equiv g$, then there are two numbers $l, t \in \{0, \ldots, n\},\ l \neq t$ such that $f_l g_t \not\equiv f_t g_l$. Let d_j be the degree of $D_j,\ j = 1, \ldots, q$, and let d be the least common multiple of the d_j. Let k be a sufficiently large positive integer, which will be chosen later. With the hypothesis in Theorem 1, we have

$$\begin{aligned}
N_f^M(r, D_j) &= N_f^M(r, D_j, \leqslant k) + N_f^M(r, D_j, > k) \\
&= \frac{k}{k+1} N_f^M(r, D_j, \leqslant k) + \frac{1}{k+1} N_f^M(r, D_j, \leqslant k) + N_f^M(r, D_j, > k) \\
&\leqslant \frac{k}{k+1} N_f^M(r, D_j, \leqslant k) + \frac{M}{k+1} N_f^1(r, D_j, \leqslant k) + M N_f^1(r, D_j, > k) \\
&\leqslant \frac{k}{k+1} N_f^M(r, D_j, \leqslant k) + \frac{M}{k+1} N_f^1(r, D_j, \leqslant k) + \frac{M}{k+1} N_f(r, D_j, > k) \\
&\leqslant \frac{k}{k+1} N_f^M(r, D_j, \leqslant k) + \frac{M}{k+1} N_f(r, D_j, \leqslant k) + \frac{M}{k+1} N_f(r, D_j, > k) \\
&= \frac{k}{k+1} N_f^M(r, D_j, \leqslant k) + \frac{M}{k+1} N_f(r, D_j) \\
&\leqslant \frac{k}{k+1} N_f^M(r, D_j, \leqslant k) + \frac{M d_j}{k+1} T_f(r) + O(1),
\end{aligned}$$

so

$$\frac{1}{d_j} N_f^M(r, D_j) \leqslant \frac{k}{d_j(k+1)} N_f^M(r, D_j, \leqslant k) + \frac{M}{k+1} T_f(r) + O(1).$$

This implies that

$$\sum_{j=1}^{q}\frac{1}{d_j}N_f^M(r,D_j)\leqslant\frac{k}{k+1}\sum_{j=1}^{q}\frac{1}{d_j}N_f^M(r,D_j,\leqslant k)+\frac{qM}{k+1}T_f(r)+O(1). \tag{3.1}$$

On the other hand, by Lemma 2.3 with $\varepsilon = 1/2$, we have

$$(q-n-\frac{3}{2})T_f(r)\leqslant\sum_{j=1}^{q}\frac{1}{d_j}N_f^M(r,D_j)+O_f(r). \tag{3.2}$$

Combining the formulas (3.1) and (3.2) together, we have

$$\Big(q-\frac{qM}{k+1}-n-\frac{3}{2}\Big)T_f(r)\leqslant\frac{k}{k+1}\sum_{j=1}^{q}\frac{1}{d_j}N_f^M(r,D_j,\leqslant k)+O_f(r).$$

This implies that

$$\begin{aligned}\big(q(k+1-M)-&(n+\frac{3}{2})(k+1)\big)T_f(r) \qquad (3.3)\\ &\leqslant k\sum_{j=1}^{q}\frac{1}{d_j}N_f^M(r,D_j,\leqslant k)+O_f(r)\\ &\leqslant Mk\sum_{j=1}^{q}\frac{1}{d_j}N_f^1(r,D_j,\leqslant k)+O_f(r)\\ &\leqslant\frac{Mk}{\delta}\sum_{j=1}^{q}N_f^1(r,D_j,\leqslant k)+O_f(r).\end{aligned}$$

Assume that $z_0\in\Delta$ is a zero of $D_j\circ f$ with multiplicity not greater than k, then $z_0\in\overline{E}_f(\mathcal{D})\cup\overline{E}_g(\mathcal{D})$. This implies that $g(z_0)=f(z_0)$, so

$$f_l(z_0)g_t(z_0)=f_t(z_0)g_l(z_0),$$

namely z_0 is the zero of the holomorphic function $h=f_lg_t-f_tg_l$. Note that by the hypothesis that the hypersurfaces in $\mathcal{D}$ are in general position with Δ, then there exist at most n hypersurfaces D_j in $\mathcal{D}$ such that $D_j\circ f(z_0)=0$. This implies that

$$\sum_{j=1}^{q}N_f^1(r,D_j,\leqslant k)\leqslant nN_0\Big(r,\frac{1}{h}\Big).$$

Since h is holomorphic function, from Lemma 2.1 we have

$$\begin{aligned}
N_0\Big(r, \frac{1}{h}\Big) &= \frac{1}{2\pi}\int_0^{2\pi} \log|(f_l g_t - f_t g_l)(r^{-1}e^{i\theta})|d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \log|(f_l g_t - f_t g_l)(re^{i\theta})|d\theta + O(1) \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log\big(2.\max_{j=0,\dots,n}|f_j(r^{-1}e^{i\theta})|\max_{j=0,\dots,n}|g_j(r^{-1}e^{i\theta})|\big)d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \log\big(2.\max_{j=0,\dots,n}|f_j(re^{i\theta})|\max_{j=0,\dots,n}|g_j(re^{i\theta})|\big)d\theta + O(1) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \big(\log\max_{j=0,\dots,n}|f_j(r^{-1}e^{i\theta})|d\theta + \log\max_{j=0,\dots,n}|g_j(r^{-1}e^{i\theta})|\big)d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \big(\log\max_{j=0,\dots,n}|f_j(re^{i\theta})|d\theta + \log\max_{j=0,\dots,n}|g_j(re^{i\theta})|\big)d\theta + O(1) \\
&= T_f(r) + T_g(r) + O(1).
\end{aligned}$$

Therefore, (3.3) becomes

$$\begin{aligned}
\big(q(k+1-M) - (n+\frac{3}{2})(k+1)\big)T_f(r)& \\
\leqslant \frac{nMk}{\delta}(T_f(r)+T_g(r)) + O_f(r).&
\end{aligned} \tag{3.4}$$

Similarly for the holomorphic map g we have

$$\begin{aligned}
\big(q(k+1-M) - (n+\frac{3}{2})(k+1)\big)T_g(r)& \\
\leqslant \frac{nMk}{\delta}(T_f(r)+T_g(r)) + O_f(r).&
\end{aligned} \tag{3.5}$$

Adding the inequalities (3.4) and (3.5) together, we have

$$\begin{aligned}
\big(q(k+1-M) - (n+\frac{3}{2})(k+1)\big)(T_f(r)+T_g(r))& \\
\leqslant \frac{2nMk}{\delta}(T_f(r)+T_g(r)) + O_f(r) + O_g(r).&
\end{aligned}$$

This concludes that

$$q(k+1-M) - (n+\frac{3}{2})(k+1) - \frac{2Mnk}{\delta} \leqslant \frac{O_f(r)+O_g(r)}{T_f(r)+T_g(r)}$$

holds for a sufficiently large positive real number r. Let $r \to \infty$ we have

$$q(k+1-M) - (n+\frac{3}{2})(k+1) - \frac{2Mnk}{\delta} \leqslant 0.$$

This is equivalent to

$$k(q\delta - (n+\frac{3}{2})\delta - 2Mn) + (q - qM - (n+\frac{3}{2}))\delta \leqslant 0. \tag{3.6}$$

If we take

$$k > \frac{(qM - q + n + \frac{3}{2})\delta}{q\delta - (n+\frac{3}{2})\delta - 2nM},$$

then since the hypothesis that $q \geqslant n+2+\dfrac{2nM}{\delta}$ we have a contradiction. Hence $f_i g_j \equiv f_j g_i$ for any $i \neq j \in \{0, \dots, n\}$, namely $f \equiv g$. This is the conclusion of the proof of Theorem 1. □

Proof of Theorem 2. We assume that $f \not\equiv g$ too. Then there are two numbers $l, t \in \{0, \dots, n\},\ l \neq t$ such that $f_l g_t - f_t g_l \not\equiv 0$. Let k be a sufficiently large positive integer, which will be chosen later. With the hypothesis in Theorem 2 and the proof of Theorem 1, we have

$$\begin{aligned}\big(q(k+1-M) - &(n+\frac{3}{2})(k+1)\big)T_f(r) \\ &\leqslant \frac{Mk}{\delta}\sum_{j=1}^{q} N_f^1(r, D_j, \leqslant k) + O_f(r).\end{aligned} \tag{3.7}$$

We know that, if $z_0 \in \Delta$ is a zero of $D_j \circ f$ with multiplicity less than or equal to k, then z_0 will be a zero of the function $f_l g_t - f_t g_l$. By the hypothesis we have

$$\overline{E}_f(D_i) \cap \overline{E}_f(D_j) = \emptyset$$

for any pair $i \neq j \in \{1, \dots, q\}$. So if z_0 is a zero of $D_j \circ f$ then z_0 will not be a zero of $D_i \circ f$ for all $i \neq j \in \{1, \dots, q\}$. Hence

$$\sum_{j=1}^{q} N_f^1(r, D_j, \leqslant k) \leqslant N_0\Big(r, \frac{1}{f_l g_t - f_t g_l}\Big) \leqslant T_f(r) + T_g(r) + O_f(r).$$

Therefore, (3.7) becomes

$$\begin{aligned}\big(q(k+1-M) - &(n+\frac{3}{2})(k+1)\big)T_f(r) \\ &\leqslant \frac{Mk}{\delta}(T_f(r) + T_g(r)) + O_f(r).\end{aligned} \tag{3.8}$$

Similarly for the holomorphic map g we have

$$\begin{aligned}\big(q(k+1-M) - &(n+\frac{3}{2})(k+1)\big)T_g(r) \\ &\leqslant \frac{Mk}{\delta}(T_f(r) + T_g(r)) + O_f(r).\end{aligned} \tag{3.9}$$

Adding the inequalities (3.8) and (3.9) together, we have

$$\big(q(k+1-M)-(n+\frac{3}{2})(k+1)\big)(T_f(r)+T_g(r)) \leqslant \frac{2Mk}{\delta}(T_f(r)+T_g(r))+O_f(r)+O_g(r).$$

This concludes that

$$q(k+1-M)-(n+\frac{3}{2})(k+1)-\frac{2Mk}{\delta} \leqslant \frac{O_f(r)+O_g(r)}{T_f(r)+T_g(r)}$$

holds for a sufficiently large positive real number r. Let $r \to \infty$ we have

$$q(k+1-M)-(n+\frac{3}{2})(k+1)-\frac{2Mk}{\delta} \leqslant 0.$$

This is equivalent to

$$k(q\delta-(n+\frac{3}{2})\delta-2M)+(q-qM-(n+\frac{3}{2}))\delta \leqslant 0. \tag{3.10}$$

If we take

$$k > \frac{(qM-q+n+\frac{3}{2})\delta}{q\delta-(n+\frac{3}{2})\delta-2M},$$

then since the hypothesis that $q \geqslant n+2+\dfrac{2M}{\delta}$ we have a contradiction. Hence $f_ig_j \equiv f_jg_i$ for any $i \neq j \in \{0,\ldots,n\}$, namely $f \equiv g$. This is the conclusion of the proof of Theorem 2. □

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