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THE HIT PROBLEM OF FIVE VARIABLES IN THE DEGREE THIRTY

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Abstract

Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \dots, x_k]$ over the prime field of two elements, $\mathbb{F}_2$, with the degree of each x_i being 1. We study the *hit problem*, set up by Frank Peterson, of finding a minimal set of generators for P_k as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we explicitly determine a minimal set of $\mathcal{A}$-generators for P_k in the case $k = 5$ and the degree $2^{d+1} - 2$ with $d \leqslant 4$.

1 Introduction

Let E^k be an elementary abelian 2-group of rank k and let BE^k be the classifying space of E^k. Then,

$$P_k := H^*(BE^k) \cong \mathbb{F}_2[x_1, x_2, \dots, x_k],$$

a polynomial algebra in k generators $x_1, x_2, \dots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

Being the cohomology of a topological space, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod squares Sq^i and subject to the Cartan formula (see Steenrod and Epstein [16]).

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An element g in P_k is called *hit* if it belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ is the augmentation ideal of $\mathcal{A}$. That means g can be written as a finite sum $g = \sum_{u\geqslant 0} Sq^{2^u}(g_u)$ for suitable polynomials $g_u \in P_k$.

We study the *Peterson hit problem* of determining a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. In other words, we want to determine a basis of the $\mathbb{F}_2$-vector space $QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. This problem was first studied by Peterson [7], Wood [23], Singer [14], and Priddy [10], who showed its relation to several classical problems the homotopy theory. Then, this problem was investigated by Carlisle and Wood [1], Crabb and Hubbuck [2], Janfada and Wood [3], Kameko [4], Mothebe [5], Nam [6], Repka and Selick [11], Silverman [12], Silverman and Singer [13], Singer [15], Walker and Wood [22], Wood [24], the first named author [17, 18] and others. Recently, the hit problem and its applications to representations of general linear groups have been presented in the monographs of Walker and Wood [20, 21].

From the results of Wood [23] and Kameko [4], the hit problem is reduced to the case of degree n of the form

$$n = s(2^d - 1) + 2^d m, \tag{1.1}$$

where s, d, m are non-negative integers and $1 \leqslant s < k$ (see [18].) For $s = k - 1$ and $m > 0$, the problem was studied by Crabb and Hubbuck [2], Nam [6], Repka and Selick [11] and the first named author [17, 18].

In the present paper, we study the hit problem in degree n of the form (1.1) with $k = 5$, $s = 2$, $m = 0$ and $d \leqslant 4$.

In Section 2, we recall some needed information on the admissible monomials in P_k, Singer's criterion on the hit monomials and Kameko's homomorphism. The main results of the paper are presented in Section 3.

2 Preliminaries

In this section, we recall some needed information from Kameko [4], Singer [15] and the first named author [18] which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1, 2, \ldots, k\}$ and

$$X_{\mathbb{J}} = X_{\{j_1,j_2,\ldots,j_s\}} = \prod_{j\in\mathbb{N}_k\setminus\mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_k,$$

In particular, $X_{\mathbb{N}_k} = 1$, $X_\emptyset = x_1x_2\ldots x_k$, $X_j = x_1\ldots\hat{x}_j\ldots x_k$, $1 \leqslant j \leqslant k$, and $X := X_k \in P_{k-1}$.

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 with $i \geqslant 0$.

Let $x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \in P_k$. Denote $\nu_j(x) = a_j, 1 \leqslant j \leqslant k$ and $\nu(x) = \max\{\nu_j(x) : 1 \leqslant j \leqslant k\}$. Set

$$\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geqslant 0$. Then, we have $x = \prod_{t\geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2. A weight vector ω is a sequence of non-negative integers $(\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ such that $\omega_i = 0$ for $i \gg 0$.

For a monomial x in P_k, define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots),$$
$$\sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)},\ i \geqslant 1$. The sequences $\omega(x)$ and $\sigma(x)$ are respectively called the weight vector and the exponent vector of x.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

For any weight vector $\omega = (\omega_1, \omega_2, \ldots)$, we define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$ and the length $\ell(\omega) = \max\{i : \omega_i > 0\}$. We write $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$ if $\ell(\omega) = r$. For a weight vector $\eta = (\eta_1, \eta_2, \ldots)$, we define the concatenation of weight vectors

$$\omega|\eta = (\omega_1, \ldots, \omega_r, \eta_1, \eta_2, \ldots)$$

if $\ell(\omega) = r$ and $(a)|^b = (a)|(a)|\ldots|(a)$, ($b$ times of (a)'s), where a, b are positive integers. Denote by $P_k(\omega)$ the subspace of P_k spanned by monomials y such that $\deg y = \deg \omega$ and $\omega(y) \leqslant \omega$, and by $P_k^-(\omega)$ the subspace of $P_k(\omega)$ spanned by monomials y such that $\omega(y) < \omega$.

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k.

i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+P_k$. If $f \equiv 0$ then f is called *hit*.

ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then, we have

$$QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

For a polynomial $f \in P_k$, we denote by $[f]$ the class in QP_k represented by f. If ω is a weight vector and $f \in P_k(\omega)$, then denote by $[f]_\omega$ the class in $QP_k(\omega)$ represented by f. Denote by $|S|$ the cardinal of a set S.

It is easy to see that

$$QP_k(\omega) \cong QP_k^\omega := \langle\{[x] \in QP_k : x \text{ is admissible and } \omega(x) = \omega\}\rangle.$$

So, we get
$$(QP_k)_n = \bigoplus_{\deg\omega = n} QP_k^{\omega} \cong \bigoplus_{\deg\omega = n} QP_k(\omega). \tag{2.1}$$

Hence, we can identify the vector space $QP_k(\omega)$ with $QP_k^{\omega} \subset QP_k$.

Definition 2.4. Let x, y be monomials of the same degree in P_k. We say that $x < y$ if and only if one of the following holds:
i) $\omega(x) < \omega(y)$;
ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.5. A monomial x in P_k is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$ for $j = 1, 2, \ldots, t$ and $x - \sum_{j=1}^{t} y_j \in \mathcal{A}^+ P_k$. A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n.

Definition 2.6. A monomial x in P_k is said to be strictly inadmissible if and only if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$, for $j = 1, 2, \ldots, t$ and $x = \sum_{j=1}^{t} y_j + \sum_{u=1}^{2^s - 1} Sq^u(h_u)$ with $s = \max\{i : \omega_i(x) > 0\}$ and suitable polynomials $h_u \in P_k$.

It is easy to see that if x is strictly inadmissible, then it is inadmissible.

Theorem 2.7 (See Kameko [4], Sum [17])**.** *Let x, y, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*
i) *If w is inadmissible, then xw^{2^r} is also inadmissible.*
ii) *If w is strictly inadmissible, then wy^{2^s} is also strictly inadmissible.*

Now, we recall a result of Singer [15] on the hit monomials in P_k.

Definition 2.8. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{d_j} - 1$ for d_j a non-negative integer and $j = 1, 2, \ldots, k$. If z is a spike with $d_1 > d_2 > \ldots > d_{r-1} \geqslant d_r > 0$ and $d_j = 0$ for $j > r$, then it is called the minimal spike.

For a positive integer n, by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1 \leqslant i \leqslant r} (2^{d_i} - 1)$, where $d_i > 0$. In [15], Singer showed that if $\mu(n) \leqslant k$, then there exists uniquely a minimal spike of degree n in P_k. The following is a criterion for the hit monomials in P_k.

Theorem 2.9 (See Singer [15])**.** *Suppose $x \in P_k$ is a monomial of degree n, where $\mu(n) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$, then x is hit.*

This result implies the following, which originally is a conjecture of Peterson [7].

Theorem 2.10 (See Wood [23]). *If $\mu(n) > k$, then $(QP_k)_n = 0$.*

One of the main tools in the study of the hit problem is Kameko's homomorphism $\widetilde{Sq}^0_* : QP_k \to QP_k$. This homomorphism is induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_* : P_k \to P_k$, given by

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if } x = x_1x_2\ldots x_ky^2,\\ 0, & \text{otherwise,}\end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However, $\widetilde{Sq}^0_*Sq^{2t} = Sq^t\widetilde{Sq}^0_*$, and $\widetilde{Sq}^0_*Sq^{2t+1} = 0$ for any non-negative integer t.

Denote by $(\widetilde{Sq}^0_*)_{(k,m)} : (QP_k)_{2m+k} \longrightarrow (QP_k)_m$ Kameko's homomorphism in degree $2m + k$.

Theorem 2.11 (See Kameko [4]). *Let m be a positive integer. If $\mu(2m+k) = k$, then*

$$(\widetilde{Sq}^0_*)_{(k,m)} : (QP_k)_{2m+k} \longrightarrow (QP_k)_m$$

is an isomorphism of the $\mathbb{F}_2$-vector spaces.

From the Theorems 2.10 and 2.11, the hit problem is reduced to the case of degree n of the form (1.1). We set

$$P_k^0 = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \ : \ a_1a_2\ldots a_k = 0\}\rangle,$$
$$P_k^+ = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \ : \ a_1a_2\ldots a_k > 0\}\rangle.$$

It is easy to see that P_k^0 and P_k^+ are the $\mathcal{A}$-submodules of P_k. Furthermore, we have the following.

Proposition 2.12. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $QP_k = QP_k^0 \oplus QP_k^+$. Here $QP_k^0 = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0$ and $QP_k^+ = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+$.*

Notation 2.13. From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k,

$$B_k^0(n) = B_k(n) \cap P_k^0,\ B_k^+(n) = B_k(n) \cap P_k^+.$$

For a weight vector ω of degree n, we set

$$B_k(\omega) = B_k(n) \cap P_k(\omega),\ B_k^+(\omega) = B_k^+(n) \cap P_k(\omega).$$

For a subset $S \subset P_k$, we denote $[S] = \{[f] : f \in S\}$. If $S \subset P_k(\omega)$, then we set $[S]_\omega = \{[f]_\omega : f \in S\}$. Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are respectively the basses of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$.

For $1 \leqslant i \leqslant k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ x_{j+1}, & \text{if } i \leqslant j < k. \end{cases}$$

Proposition 2.14 (See Mothebe and Uys [5])**.** *Let i, d be positive integers such that $1 \leqslant i \leqslant k$. If x is an admissible monomial in P_{k-1} then $x_i^{2^d-1} f_i(x)$ is also an admissible monomial in P_k.*

Denote $\mathcal{N}_k = \{(i; I) : I = (i_1, i_2, \ldots, i_r), 1 \leqslant i < i_1 < \ldots < i_r \leqslant k,\ 0 \leqslant r < k\}$. For any $(i; I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i;I)} : P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ \sum_{s \in I} x_{s-1}, & \text{if } j = i, \\ x_{j-1}, & \text{if } i < j \leqslant k. \end{cases}$$

Then $p_{(i;I)}$ is a homomorphism of $\mathcal{A}$-modules.

Lemma 2.15 (See Phúc-Sum [9])**.** *If x is a monomial in P_k, then*

$$p_{(i;I)}(x) \in P_{k-1}(\omega(x)).$$

This lemma implies that if ω is a weight vector and $x \in P_k(\omega)$, then $p_{(i;I)}(x) \in P_{k-1}(\omega)$. Moreover, $p_{(i;I)}$ passes to a homomorphism from $QP_k(\omega)$ to $QP_{k-1}(\omega)$. So, these homomorphisms can be used to prove certain subset of QP_k is linearly independent.

For $J = (j_1, j_2, \ldots, j_r) : 1 \leqslant j_1 < \ldots < j_r \leqslant k$, we define a monomorphism $\theta_J : P_r \to P_k$ of $\mathcal{A}$-algebras by substituting $\theta_J(x_t) = x_{j_t}$ for $1 \leqslant t \leqslant r$. It is easy to see that, for any weight vector ω of degree n,

$$Q\theta_J(P_r^+)(\omega) \cong QP_r^+(\omega) \text{ and } (Q\theta_J(P_r^+))_n \cong (QP_r^+)_n$$

for $1 \leqslant r \leqslant k$, where $Q\theta_J(P_r^+) = \theta_J(P_r^+)/\mathcal{A}^+\theta_J(P_r^+)$ and $QP_r^+ = P_r^+/\mathcal{A}^+P_r^+$. So, by a simple computation using Theorem 2.10 and (2.1), we get the following.

Proposition 2.16 (See Walker and Wood [21])**.** *For a weight vector ω of degree n, we have direct summand decompositions of the $\mathbb{F}_2$-vector spaces*

$$QP_k(\omega) = \bigoplus_{\mu(n) \leqslant r \leqslant k} \bigoplus_{\ell(J) = r} Q\theta_J(P_r^+)(\omega),$$

where $\ell(J)$ is the length of J. Consequently

$$\dim QP_k(\omega) = \sum_{\mu(n) \leqslant r \leqslant k} \binom{k}{r} \dim QP_r^+(\omega),$$

$$\dim(QP_k)_n = \sum_{\mu(n)\leqslant r\leqslant k}\binom{k}{r}\dim(QP_r^+)_n.$$

3 Main Results

First of all, we recall the results on the admissible monomials of degree $2^{d+1}-2$ in P_k with $k\leqslant 4$. From Peterson [7] and Kameko [4], we have

$B_2(2^{d+1}-2) = \{x_1^{2^d-1}x_2^{2^d-1}\}$.

For $d\geqslant 2$, $B_3(2^{d+1}-2)$ is the set of the following monomials:

$$\begin{array}{llll} x_2^{2^d-1}x_3^{2^d-1} & x_1^{2^d-1}x_3^{2^d-1} & x_1^{2^d-1}x_2^{2^d-1} & x_1x_2^{2^d-2}x_3^{2^d-1}\\ x_1x_2^{2^d-1}x_3^{2^d-2} & x_1^{2^d-1}x_2x_3^{2^d-2} & x_1^3x_2^{2^d-3}x_3^{2^d-2}. & \end{array}$$

According to a result in [18], $B_4(2^{d+1}-2) = B_4((2)|^d)\cup B_4(4,(3)|^{d-2},1)$, where $B_4((2)|^d) = B_4^0((2)|^d)\cup B_4^+((2)|^d)$ is determined as follows:

Proposition 3.1 (See [18]).

i) $B_4(2) = B_4^0(2) = \{x_ix_j;\ 1\leqslant i<j\leqslant 4\}$.

ii) *For $d\geqslant 2$, $B_4^0(2^{d+1}-2) = B_4^0((2)|^d)$ is the set of the monomials $a_t = a_{d,t}$ which are determined as follows:*

$$\begin{array}{lll} a_1 = x_3^{2^d-1}x_4^{2^d-1} & a_2 = x_2^{2^d-1}x_4^{2^d-1} & a_3 = x_2^{2^d-1}x_3^{2^d-1}\\ a_4 = x_1^{2^d-1}x_4^{2^d-1} & a_5 = x_1^{2^d-1}x_3^{2^d-1} & a_6 = x_1^{2^d-1}x_2^{2^d-1}\\ a_7 = x_2x_3^{2^d-2}x_4^{2^d-1} & a_8 = x_2x_3^{2^d-1}x_4^{2^d-2} & a_9 = x_2^{2^d-1}x_3x_4^{2^d-2}\\ a_{10} = x_1x_3^{2^d-2}x_4^{2^d-1} & a_{11} = x_1x_3^{2^d-1}x_4^{2^d-2} & a_{12} = x_1x_2^{2^d-2}x_4^{2^d-1}\\ a_{13} = x_1x_2^{2^d-2}x_3^{2^d-1} & a_{14} = x_1x_2^{2^d-1}x_4^{2^d-2} & a_{15} = x_1x_2^{2^d-1}x_3^{2^d-2}\\ a_{16} = x_1^{2^d-1}x_3x_4^{2^d-2} & a_{17} = x_1^{2^d-1}x_2x_4^{2^d-2} & a_{18} = x_1^{2^d-1}x_2x_3^{2^d-2}. \end{array}$$

For $d\geqslant 3$.

$$\begin{array}{ll} a_{19} = x_2^3x_3^{2^d-3}x_4^{2^d-2} & a_{20} = x_1^3x_3^{2^d-3}x_4^{2^d-2}\\ a_{21} = x_1^3x_2^{2^d-3}x_4^{2^d-2} & a_{22} = x_1^3x_2^{2^d-3}x_3^{2^d-2}. \end{array}$$

Proposition 3.2 (See [18]).

i) $B_4^+((2)|^2) = \{x_1x_2x_3^2x_4^2,\ x_1x_2^2x_3x_4^2\}$.

ii) *For $d\geqslant 3$, $B_4^+((2)|^d)$ is the set of the monomials $a_t = a_{d,t}$, $23\leqslant t\leqslant 35$, which are determined as follows:*

$$
\begin{array}{lll}
a_{23} = x_1x_2x_3^{2^d-2}x_4^{2^d-2} & a_{24} = x_1x_2^{2^d-2}x_3x_4^{2^d-2} & a_{25} = x_1x_2^2x_3^{2^d-4}x_4^{2^d-1} \\
a_{26} = x_1x_2^2x_3^{2^d-1}x_4^{2^d-4} & a_{27} = x_1x_2^{2^d-1}x_3^2x_4^{2^d-4} & a_{28} = x_1^{2^d-1}x_2x_3^2x_4^{2^d-4} \\
a_{29} = x_1x_2^2x_3^{2^d-3}x_4^{2^d-2} & a_{30} = x_1x_2^3x_3^{2^d-4}x_4^{2^d-2} & a_{31} = x_1x_2^3x_3^{2^d-2}x_4^{2^d-4} \\
a_{32} = x_1^3x_2x_3^{2^d-4}x_4^{2^d-2} & a_{33} = x_1^3x_2x_3^{2^d-2}x_4^{2^d-4} & a_{34} = x_1^3x_2^{2^d-3}x_3^2x_4^{2^d-4}.
\end{array}
$$

For $d = 3$, $a_{35} = x_1^3x_2^3x_3^4x_4^4$.

For $s \geqslant 4$, $a_{35} = x_1^3x_2^5x_3^{2^d-6}x_4^{2^d-4}$.

Proposition 3.3 (See [18]). *For* $d \geqslant 4$, $B_4(4,(3)|^{d-2},1) = B_4^+(4,(3)|^{d-2},1)$ *is the set of monomials* $\phi(c_t)$, $t \geqslant 1$, *with* $\phi : P_5 \to P_5$, $\phi(f) = x_1x_2x_3x_4x_5f^2$ *for all* $f \in P_5$, *and the monomial* $c_t = c_{d,t}$ *is determined as follows:*

$$
\begin{array}{ll}
c_1 = x_2^{2^{d-2}-1}x_3^{2^{d-2}-1}x_4^{2^{d-1}-1} & c_2 = x_2^{2^{d-2}-1}x_3^{2^{d-1}-1}x_4^{2^{d-2}-1} \\
c_3 = x_2^{2^{d-1}-1}x_3^{2^{d-2}-1}x_4^{2^{d-2}-1} & c_4 = x_1^{2^{d-2}-1}x_3^{2^{d-2}-1}x_4^{2^{d-1}-1} \\
c_5 = x_1^{2^{d-2}-1}x_3^{2^{d-1}-1}x_4^{2^{d-2}-1} & c_6 = x_1^{2^{d-2}-1}x_2^{2^{d-2}-1}x_4^{2^{d-1}-1} \\
c_7 = x_1^{2^{d-2}-1}x_2^{2^{d-2}-1}x_3^{2^{d-1}-1} & c_8 = x_1^{2^{d-2}-1}x_2^{2^{d-1}-1}x_4^{2^{d-2}-1} \\
c_9 = x_1^{2^{d-2}-1}x_2^{2^{d-1}-1}x_3^{2^{d-2}-1} & c_{10} = x_1^{2^{d-1}-1}x_3^{2^{d-2}-1}x_4^{2^{d-2}-1} \\
c_{11} = x_1^{2^{d-1}-1}x_2^{2^{d-2}-1}x_4^{2^{d-2}-1} & c_{12} = x_1^{2^{d-1}-1}x_2^{2^{d-2}-1}x_3^{2^{d-2}-1} \\
c_{13} = x_1x_2^{2^{d-2}-2}x_3^{2^{d-2}-1}x_4^{2^{d-1}-1} & c_{14} = x_1x_2^{2^{d-2}-2}x_3^{2^{d-1}-1}x_4^{2^{d-2}-1} \\
c_{15} = x_1x_2^{2^{d-2}-1}x_3^{2^{d-2}-2}x_4^{2^{d-1}-1} & c_{16} = x_1x_2^{2^{d-2}-1}x_3^{2^{d-1}-1}x_4^{2^{d-2}-2} \\
c_{17} = x_1x_2^{2^{d-1}-1}x_3^{2^{d-2}-2}x_4^{2^{d-2}-1} & c_{18} = x_1x_2^{2^{d-1}-1}x_3^{2^{d-2}-1}x_4^{2^{d-2}-2} \\
c_{19} = x_1^{2^{d-2}-1}x_2x_3^{2^{d-2}-2}x_4^{2^{d-1}-1} & c_{20} = x_1^{2^{d-2}-1}x_2x_3^{2^{d-1}-1}x_4^{2^{d-2}-2} \\
c_{21} = x_1^{2^{d-2}-1}x_2^{2^{d-1}-1}x_3x_4^{2^{d-2}-2} & c_{22} = x_1^{2^{d-1}-1}x_2x_3^{2^{d-2}-2}x_4^{2^{d-2}-1} \\
c_{23} = x_1^{2^{d-1}-1}x_2x_3^{2^{d-2}-1}x_4^{2^{d-2}-2} & c_{24} = x_1^{2^{d-1}-1}x_2^{2^{d-2}-1}x_3x_4^{2^{d-2}-2} \\
c_{25} = x_1x_2^{2^{d-2}-1}x_3^{2^{d-2}-1}x_4^{2^{d-1}-2} & c_{26} = x_1x_2^{2^{d-2}-1}x_3^{2^{d-1}-2}x_4^{2^{d-2}-1} \\
c_{27} = x_1x_2^{2^{d-1}-2}x_3^{2^{d-2}-1}x_4^{2^{d-2}-1} & c_{28} = x_1^{2^{d-2}-1}x_2x_3^{2^{d-2}-1}x_4^{2^{d-1}-2} \\
c_{29} = x_1^{2^{d-2}-1}x_2x_3^{2^{d-1}-2}x_4^{2^{d-2}-1} & c_{30} = x_1^{2^{d-2}-1}x_2^{2^{d-2}-1}x_3x_4^{2^{d-1}-2}.
\end{array}
$$

For $d = 4$,

$$
\begin{array}{l}
c_{31} = x_1^3x_2^3x_3^5x_4^2,\ c_{32} = x_1^3x_2^5x_3^2x_4^3,\ c_{33} = x_1^3x_2^5x_3^3x_4^2, \\
c_{34} = x_1^3x_2^3x_3^3x_4^4,\ c_{35} = x_1^3x_2^3x_3^4x_4^3.
\end{array}
$$

For $d \geqslant 5$,

$$\begin{array}{ll}
c_{31} = x_1^3x_2^{2^{d-2}-3}x_3^{2^{d-2}-2}x_4^{2^{d-1}-1} & c_{32} = x_1^3x_2^{2^{d-2}-3}x_3^{2^{d-1}-1}x_4^{2^{d-2}-2}\\
c_{33} = x_1^3x_2^{2^{d-1}-1}x_3^{2^{d-2}-3}x_4^{2^{d-2}-2} & c_{34} = x_1^{2^{d-1}-1}x_2^3x_3^{2^{d-2}-3}x_4^{2^{d-2}-2}\\
c_{35} = x_1^3x_2^{2^{d-2}-3}x_3^{2^{d-2}-1}x_4^{2^{d-1}-2} & c_{36} = x_1^3x_2^{2^{d-2}-3}x_3^{2^{d-1}-2}x_4^{2^{d-2}-1}\\
c_{37} = x_1^3x_2^{2^{d-2}-1}x_3^{2^{d-2}-3}x_4^{2^{d-1}-2} & c_{38} = x_1^{2^{d-2}-1}x_2^3x_3^{2^{d-2}-3}x_4^{2^{d-1}-2}\\
c_{39} = x_1^3x_2^{2^{d-2}-1}x_3^{2^{d-1}-3}x_4^{2^{d-2}-2} & c_{40} = x_1^3x_2^{2^{d-1}-3}x_3^{2^{d-2}-2}x_4^{2^{d-2}-1}\\
c_{41} = x_1^3x_2^{2^{d-1}-3}x_3^{2^{d-2}-1}x_4^{2^{d-2}-2} & c_{42} = x_1^{2^{d-2}-1}x_2^3x_3^{2^{d-1}-3}x_4^{2^{d-2}-2}\\
c_{43} = x_1^7x_2^{2^{d-1}-5}x_3^{2^{d-2}-3}x_4^{2^{d-2}-2}. &
\end{array}$$

For $d = 5$,

$$c_{44} = x_1^7x_2^7x_3^9x_4^6,\ c_{45} = x_1^7x_2^7x_3^7x_4^8.$$

For $d \geqslant 6$,

$$c_{44} = x_1^7x_2^{2^{d-2}-5}x_3^{2^{d-1}-3}x_4^{2^{d-2}-2},\ c_{45} = x_1^7x_2^{2^{d-2}-5}x_3^{2^{d-2}-3}x_4^{2^{d-1}-2}.$$

We now determine $B_5(2^{d+1}-2)$. It is easy to see that for $d = 1$, we have $\dim(QP_5)_2 = 10$ and $B_5(2) = \{x_ix_j; 1 \leqslant i < j \leqslant 5\}$.

3.1 The case $d = 2$

Lemma 3.1.1. *If x is an admissible monomial of degree 6 in P_5, then either $\omega(x) = (2,2)$ or $\omega(x) = (4,1)$.*

Lemma 3.1.2. *Let (i,j,t,u,v) be an arbitrary permutation of $(1,2,3,4,5)$. The following monomials are strictly inadmissible:*

$$x_i^2x_jx_t^3,\ i<j;\ x_i^2x_jx_t^2x_u,\ i<\min\{j,t,u\};$$
$$x_ix_j^2x_t^2x_u,\ i<j<t<u;\ x_1^2x_2x_3x_4x_5.$$

From the result in [18], we have $\dim(QP_5^0)_6 = 70$. By a direct computation using Lemmas 3.1.1, 3.1.2 and Theorem 2.7 we obtain

Proposition 3.1.3. $B_5^+(6) = B_5^+(4,1)$ *is the set of the following monomials:*

$$x_1x_2x_3x_4x_5^2,\ x_1x_2x_3x_4^2x_5, x_1x_2x_3^2x_4x_5, x_1x_2^2x_3x_4x_5.$$

Consequently $\dim(QP_5)_6 = 74$.

3.2 The case $d = 3$

For $d = 3$, the space $(QP_5)_{14}$ has been determined by Ly-Tin [19]. From the results of Peterson [7], Kameko [4] and the first named author [18] we have $\dim(QP_2^+)_{14} = 1$, $\dim(QP_3^+)_{14} = 4$, $\dim(QP_4^+)_{14} = 28$. Hence, we obtain

$$\dim(QP_5^0)_{14} = \binom{5}{2} + 4\binom{5}{3}4 + 28\binom{5}{4} = 190.$$

Lemma 3.2.1. *If x is an admissible monomial of degree 14 in P_5, then either $\omega(x) = ((2)|^3)$ or $\omega(x) = (2,4,1)$ or $\omega(x) = (4,3,1)$.*

Lemma 3.2.2. *Let (i,j,t,u,v) be an arbitrary permutation of $(1,2,3,4,5)$. The following monomials are strictly inadmissible:*

$$x_i^2x_j^2x_t^3x_u^3;\ x_i^2x_jx_t^2x_u^3x_v^3,\ i<j;\ x_i^2x_jx_tx_u^3x_v^3,\ i<j<t,$$
$$x_i^3x_j^4x_t^7;\ x_ix_j^6x_t^3x_u^4,\ x_i^3x_j^4x_tx_u^6,\ x_i^3x_j^4x_t^3x_u^4,\ i<j<t<u,$$
$$x_ix_j^2x_t^2x_u^6x_v^3, u<v.$$

Based on Lemmas 3.1.1, 3.1.2, 3.2.2 and Theorem 2.7 we obtain

Theorem 3.2.3 (Ly-Tin [19]). *We have*

i) $B_5^+((2)|^3)$ *is the set of the following monomials:*

$$x_1x_2^2x_3^5x_4^4x_5^4,\ x_ix_j^2x_t^4x_ux_v^6,\ x_ix_j^2x_t^4x_u^3x_v^4,\ i<j<t, u<v$$

ii) $B_5^+((2,4,1))$ *is the set of the following monomials:*

$$x_1^3x_2^5x_3^2x_4^2x_5^2;\ x_ix_j^2x_t^2x_u^2x_v^7,\ i<j<t<u;\ x_ix_j^2x_t^2x_u^3x_v^6,\ i<j<t,\ u<v,\ i<v.$$

iii) $B_5^+((4,3,1))$ *is the set of the following monomials:*

$$x_ix_j^2x_tx_u^3x_v^7,\ i<j;\ x_ix_jx_t^3x_u^3x_v^6,\ i<v;\ x_ix_j^2x_t^5x_u^3x_v^3,\ i<j<t;$$
$$x_i^3x_j^5x_t^2x_ux_v^3,\ i<j<t;\ x_ix_j^3x_t^4x_u^3x_v^3,\ x_i^3x_j^4x_tx_u^3x_v^3,\ i<j<t.$$

Here (i,j,t,u,v) is an arbitrary permutation of $(1,2,3,4,5)$. Consequently

$$\dim QP_5^+((2)|^3) = 15,\ \dim QP_5^+(2,4,1) = 15,\ \dim QP_5^+(4,3,1) = 100,$$
$$\dim(QP_5^+)_{14} = 130,\ \dim(QP_5)_{14} = 320.$$

3.3 The case $d = 4$

First we determine the weight vectors of the admissible monomials of degree 30.

Lemma 3.3.1. *If x be an admissible monomial of degree 30 in P_5, then either $\omega(x) = ((2)|^4)$ or $\omega(x) = (2,4,3,1)$ or $\omega(x) = (4,3,3,1)$.*

We need the following lemma.

Lemma 3.3.2. *Let (i,j,t,u,v) be an arbitrary permutation of $(1,2,3,4,5)$. The following monomials are strictly inadmissible:*

$$x_i^3x_j^4x_t^4x_u^4x_v^7,\ x_i^3x_j^4x_t^4x_u^5x_v^6.$$

Proof. By a direct computation we have

$$\begin{aligned}x_i^3x_j^4x_t^4x_u^4x_v^7 &= Sq^1\big(x_i^3x_jx_t^2x_u^8x_v^7 + x_i^3x_jx_t^2x_u^4x_v^{11}\big) + Sq^2\big(x_i^5x_j^2x_t^2x_u^4x_v^7\big)\\ &\quad + Sq^4\big(x_i^3x_j^2x_t^2x_u^4x_v^7\big) \bmod(P_5^-(2,2,4)).\\ x_i^3x_j^5x_t^4x_u^4x_v^6 &= Sq^1\big(x_i^3x_j^3x_tx_u^8x_v^6 + x_i^3x_j^3x_tx_u^4x_v^{10}\big) + Sq^2\big(x_i^5x_j^3x_t^2x_u^4x_v^6\big)\\ &\quad + sq^4\big(x_i^3x_j^3x_t^2x_u^4x_v^6\big) \bmod(P_5^-(2,2,4)).\end{aligned}$$

The above equalities show that $x_i^3x_j^4x_t^4x_u^4x_v^7$ and $x_i^3x_j^5x_t^4x_u^4x_v^6$ are strictly inadmissible. □

Proof of Lemma 3.3.1. Observe that $z = x_1^{15}x_2^{15}$ is the minimal spike of degree 30 in P_5 and $\omega(z) = ((2)|^4)$. Since 30 is even, using Theorem 2.9, we obtain $\omega_1(x) = 2$ or $\omega_1(x) = 4$. Suppose $\omega_1(x) = 2$. Then, $x = x_ix_jy^2$ with $1 \leqslant i < j \leqslant 5$ and y an admissible monomial of degree 14. By Lemma 3.2.1, either $\omega(y) = ((2)|^3)$ or $\omega(y) = (2,4,1)$ or $\omega(y) = (4,3,1)$. By a direct computation we see that if $\omega(y) = (2,4,1)$ then there is a monomial w which is given in one of Lemmas 3.2.1, 3.3.2 such that $x = x_ix_jy^2 = wh^{2^r}$ with $r = 2,3$ and h suitable monomial. By Theorem 2.7, x is inadmissible. So, we get either $\omega(x) = ((2)|^4)$ or $\omega(x) = (2,4,3,1)$.

Suppose that $\omega_1(x) = 4$. Then $x = X_iy^2$ with y an admissible monomial of degree 13 in P_5. By Phuc [8], $\omega(y) = (3,3,1)$, so $\omega(x) = (4,3,3,1)$. The lemma is proved. □

By Lemma 3.3.1, we have

$$(QP_5)_{30} = (QP_5^0)_{30} \oplus QP_5^+((2)|^4) \oplus QP_5^+(2,4,3,1) \oplus QP_5^+(4,3,3,1).$$

We have $\dim(QP_2^+)_{30} = \dim QP_2^+((2)|^4) = 1$, $\dim(QP_3^+)_{30} = \dim QP_3^+((2)|^4) = 4$, $QP_4^+(2,4,3,1) = 0$, $\dim QP_4^+((2)|^4) = 13$, $\dim QP_4^+(4,3,3,1) = 35$. So, we get $QP_5^0(2,4,3,1) = 0$ and

$$\begin{aligned}\dim QP_5^0((2)|^4) &= \binom{5}{2} + 4\binom{5}{3} + 13\binom{5}{4} = 115,\\ \dim QP_5^0(4,3,3,1) &= 35\binom{5}{4} = 175.\end{aligned}$$

Theorem 3.3.3. i) $B_5^+((2)|^4)$ *is the set of the monomials* $b_t = b_{4,t}$, $1 \leqslant t \leqslant 39$, *which are determined as follows:*

$b_1 = x_1x_2x_3^2x_4^{12}x_5^{14}$	$b_2 = x_1x_2x_3^2x_4^{14}x_5^{12}$	$b_3 = x_1x_2x_3^{14}x_4^2x_5^{12}$
$b_4 = x_1x_2^2x_3x_4^{12}x_5^{14}$	$b_5 = x_1x_2^2x_3x_4^{14}x_5^{12}$	$b_6 = x_1x_2^2x_3^{12}x_4x_5^{14}$
$b_7 = x_1x_2^{14}x_3x_4^2x_5^{12}$	$b_8 = x_1x_2x_3^6x_4^{10}x_5^{12}$	$b_9 = x_1x_2^2x_3^{13}x_4^2x_5^{12}$
$b_{10} = x_1x_2^2x_3^3x_4^{12}x_5^{12}$	$b_{11} = x_1x_2^3x_3^2x_4^{12}x_5^{12}$	$b_{12} = x_1x_2^3x_3^{12}x_4^2x_5^{12}$
$b_{13} = x_1^3x_2x_3^2x_4^{12}x_5^{12}$	$b_{14} = x_1^3x_2x_3^{12}x_4^2x_5^{12}$	$b_{15} = x_1x_2^2x_3^4x_4^8x_5^{15}$
$b_{16} = x_1x_2^2x_3^4x_4^{15}x_5^8$	$b_{17} = x_1x_2^2x_3^{15}x_4^4x_5^8$	$b_{18} = x_1x_2^{15}x_3^2x_4^4x_5^8$
$b_{19} = x_1^{15}x_2x_3^2x_4^4x_5^8$	$b_{20} = x_1x_2^2x_3^4x_4^9x_5^{14}$	$b_{21} = x_1x_2^2x_3^5x_4^8x_5^{14}$
$b_{22} = x_1x_2^2x_3^5x_4^{14}x_5^8$	$b_{23} = x_1x_2^2x_3^5x_4^{10}x_5^{12}$	$b_{24} = x_1x_2^3x_3^4x_4^8x_5^{14}$
$b_{25} = x_1x_2^3x_3^4x_4^{14}x_5^8$	$b_{26} = x_1x_2^3x_3^{14}x_4^4x_5^8$	$b_{27} = x_1^3x_2x_3^4x_4^8x_5^{14}$
$b_{28} = x_1^3x_2x_3^4x_4^{14}x_5^8$	$b_{29} = x_1^3x_2x_3^{14}x_4^4x_5^8$	$b_{30} = x_1x_2^3x_3^4x_4^{10}x_5^{12}$
$b_{31} = x_1^3x_2x_3^4x_4^{10}x_5^{12}$	$b_{32} = x_1x_2^3x_3^6x_4^8x_5^{12}$	$b_{33} = x_1x_2^3x_3^6x_4^{12}x_5^8$
$b_{34} = x_1^3x_2x_3^6x_4^8x_5^{12}$	$b_{35} = x_1^3x_2x_3^6x_4^{12}x_5^8$	$b_{36} = x_1^3x_2^{13}x_3^2x_4^4x_5^8$
$b_{37} = x_1^3x_2^5x_3^2x_4^8x_5^{12}$	$b_{38} = x_1^3x_2^5x_3^2x_4^{12}x_5^8$	$b_{39} = x_1^3x_2^5x_3^{10}x_4^4x_5^8.$

ii) $B_5^+(2,4,3,1) = \{b_{40} = x_1^3x_2^5x_3^6x_4^6x_5^{10}\}$.

Consequently $\dim QP_5((2)|^4) = 154$ *and* $\dim QP_5(2,4,3,1) = 1$.

We need some lemmas for the proof of Theorem 3.3.3.

Lemma 3.3.4. *The following monomials are strictly inadmissible:*

$$x_ix_j^2x_t^6x_u^6x_v^7,\ \ x_ix_j^6x_t^3x_u^6x_v^6,\ j < t.$$

Here (i,j,t,u,v) *is an arbitrary permutation of* $(1,2,3,4,5)$.

Proof. By a direct computation we have

$$\begin{aligned}
x_ix_j^2x_t^6x_u^6x_v^7 &= Sq^1\big(x_ix_j^2x_t^5x_u^6x_v^7 + x_ix_j^4x_t^3x_u^6x_v^7 + x_i^4x_jx_t^3x_u^6x_v^7\big)\\
&\quad + Sq^2\big(x_i^2x_j^2x_t^3x_u^6x_v^7\big) \ \mathrm{mod}(P_5^-(2,4,3)).\\
x_ix_j^6x_t^3x_u^6x_v^6 &= x_ix_j^3x_t^6x_u^6x_v^6 + Sq^1\big(x_ix_j^3x_t^5x_u^6x_v^6 + x_ix_j^5x_t^3x_u^6x_v^6 + x_i^4x_j^3x_t^3x_u^5x_v^6\big)\\
&\quad + Sq^2\big(x_i^2x_j^3x_t^3x_u^6x_v^6\big) \ \mathrm{mod}(P_5^-(2,4,3)).
\end{aligned}$$

The above equalities show that the monomials $x_ix_j^2x_t^6x_u^6x_v^7$ and $x_ix_j^6x_t^3x_u^6x_v^6$ are strictly inadmissible. □

Lemma 3.3.5. *Let* (i,j,t,u,v) *be an arbitrary permutation of* $(1,2,3,4,5)$. *The following monomials are strictly inadmissible:*

i) $x_ix_j^7x_t^{10}x_u^{12}$, $i<j<t<u$; $x_i^7x_jx_t^{10}x_u^{12}$, $i<j<t<u$; $x_i^3x_j^3x_t^{12}x_u^{12}$, $x_i^3x_j^5x_t^8x_u^{14}$, $x_i^3x_j^5x_t^{14}x_u^8$, $x_i^7x_j^7x_t^8x_u^8$, $i<j<t<u$.

ii)

$x_1x_2^6x_3x_4^{10}x_5^{12}$	$x_1x_2^2x_3^{12}x_4^3x_5^{12}$	$x_1^3x_2^{12}x_3x_4^2x_5^{12}$	$x_1x_2^2x_3^4x_4^{11}x_5^{12}$
$x_1x_2^2x_3^7x_4^8x_5^{12}$	$x_1x_2^2x_3^7x_4^{12}x_5^8$	$x_1x_2^7x_3^2x_4^8x_5^{12}$	$x_1x_2^7x_3^2x_4^{12}x_5^8$
$x_1^7x_2x_3^2x_4^8x_5^{12}$	$x_1^7x_2x_3^2x_4^{12}x_5^8$	$x_1^3x_2^4x_3x_4^{10}x_5^{12}$	$x_1x_2^7x_3^{10}x_4^4x_5^8$
$x_1^7x_2x_3^{10}x_4^4x_5^8$	$x_1x_2^6x_3^7x_4^8x_5^8$	$x_1x_2^7x_3^6x_4^8x_5^8$	$x_1^7x_2x_3^6x_4^8x_5^8$
$x_1^3x_2^4x_3^9x_4^2x_5^{12}$	$x_1^3x_2^5x_3^8x_4^2x_5^{12}$	$x_1^7x_2^9x_3^2x_4^4x_5^8$	$x_1^3x_2^3x_3^4x_4^8x_5^{12}$
$x_1^3x_2^3x_3^4x_4^{12}x_5^8$	$x_1^3x_2^3x_3^{12}x_4^4x_5^8$	$x_1^3x_2^5x_3^6x_4^8x_5^8$	$x_1^3x_2^5x_3^8x_4^6x_5^8.$

Proof. Part i) follows from the results in [18]. We prove Part ii) for some monomials, the other cases can be proved by a similar computation. By a direct computation using the Cartan formula and Theorem 2.9 we obtain

$$\begin{aligned}x_1x_2^6x_3x_4^{10}x_5^{12} &= x_1x_2^2x_3^4x_4^9x_5^{14} + x_1x_2^3x_3^4x_4^8x_5^{14} + x_1x_2^3x_3^4x_4^{10}x_5^{12} + x_1x_2^4x_3^2x_4^{11}x_5^{12}\\ &+ x_1x_2^4x_3^4x_4^{10}x_5^{11} + x_1x_2^5x_3^2x_4^8x_5^{14} + x_1x_2^5x_3^2x_4^{10}x_5^{12} + x_1x_2^6x_3x_4^8x_5^{14}\\ &+ Sq^1\big(x_1^2x_2^3x_3^4x_4^7x_5^{13} + x_1^2x_2^3x_3^4x_4^9x_5^{11} + x_1^2x_2^5x_3x_4^8x_5^{13} + x_1^2x_2^5x_3x_4^9x_5^{12}\\ &+ x_1^2x_2^5x_3^4x_4^7x_5^{11}\big) + Sq^2\big(x_1x_2^2x_3^4x_4^7x_5^{14} + x_1x_2^3x_3^4x_4^7x_5^{13} + x_1x_2^3x_3^4x_4^9x_5^{11}\\ &+ x_1x_2^5x_3x_4^8x_5^{13} + x_1x_2^5x_3x_4^9x_5^{12} + x_1x_2^5x_3^4x_4^7x_5^{11} + x_1x_2^6x_3^2x_4^7x_5^{12}\\ &+ x_1x_2^6x_3^2x_4^8x_5^{11}\big) + Sq^4\big(x_1x_2^4x_3^2x_4^7x_5^{12}\big) \ \mathrm{mod}(P_5^-((2|^4))).\end{aligned}$$

This implies $x_1x_2^6x_3x_4^{10}x_5^{12}$ is strictly inadmissible.

$$\begin{aligned}x_1^3x_2^4x_3^9x_4^2x_5^{12} &= x_1x_2^2x_3^9x_4^4x_5^{14} + x_1^2x_2x_3^{11}x_4^4x_5^{12} + x_1^2x_2x_3^{13}x_4^2x_5^{12} + x_1^2x_2^4x_3^9x_4^4x_5^{11}\\ &+ x_1^3x_2x_3^{12}x_4^2x_5^{12} + x_1^3x_2^2x_3^9x_4^4x_5^{12} + x_1^3x_2^4x_3^8x_4^4x_5^{11} + Sq^1\big(x_1x_2^4x_3^7x_4^4x_5^{13}\\ &+ x_1^3x_2x_3^7x_4^2x_5^{16} + x_1^3x_2x_3^{11}x_4^2x_5^{12} + x_1^3x_2^4x_3^7x_4^4x_5^{11}\big) + Sq^2\big(x_1x_2^2x_3^7x_4^4x_5^{14}\\ &+ x_1^2x_2x_3^{11}x_4^2x_5^{12} + x_1^2x_2^4x_3^7x_4^4x_5^{11} + x_1^5x_2^2x_3^7x_4^2x_5^{12}\big)\\ &+ Sq^4\big(x_1^3x_2^2x_3^7x_4^2x_5^{12}\big) \ \mathrm{mod}(P_5^-((2|^4))).\end{aligned}$$

Hence, $x_1^3x_2^4x_3^9x_4^2x_5^{12}$ is strictly inadmissible.

$$\begin{aligned}x_1^3x_2^5x_3^8x_4^2x_5^{12} &= x_1^2x_2^3x_3^5x_4^8x_5^{12} + x_1^2x_2^5x_3^5x_4^8x_5^{10} + x_1^2x_2^5x_3^9x_4^8x_5^6 + x_1^3x_2^3x_3^8x_4^4x_5^{12}\\ &+ x_1^3x_2^4x_3^5x_4^8x_5^{10} + x_1^3x_2^4x_3^9x_4^8x_5^6 + x_1^3x_2^5x_3^4x_4^8x_5^{10} + x_1^3x_2^5x_3^6x_4^8x_5^8\\ &+ Sq^1\big(x_1^3x_2^3x_3^5x_4^8x_5^{10} + x_1^3x_2^3x_3^9x_4^8x_5^6\big) + Sq^2\big(x_1^2x_2^3x_3^5x_4^8x_5^{10} + x_1^2x_2^3x_3^9x_4^8x_5^6\\ &+ x_1^5x_2^3x_3^6x_4^8x_5^6 + x_1^5x_2^3x_3^8x_4^2x_5^{10}\big) + Sq^4\big(x_1^3x_2^3x_3^6x_4^8x_5^6 + x_1^3x_2^3x_3^8x_4^2x_5^{10}\\ &+ x_1^3x_2^9x_3^4x_4^4x_5^6\big) + Sq^8\big(x_1^3x_2^5x_3^4x_4^4x_5^6\big) \ \mathrm{mod}(P_5^-((2|^4))).\end{aligned}$$

$$\begin{aligned}x_1^3x_2^3x_3^{12}x_4^4x_5^8 &= x_1^2x_2^3x_3^{13}x_4^4x_5^8 + x_1^2x_2^5x_3^{11}x_4^4x_5^8 + x_1^2x_2^5x_3^7x_4^8x_5^8 + x_1^2x_2^8x_3^7x_4^5x_5^8\\ &+ Sq^1\big(x_1^3x_2^3x_3^7x_4^8x_5^8 + x_1^3x_2^3x_3^{11}x_4^4x_5^8 + x_1^3x_2^8x_3^7x_4^3x_5^8\big) + Sq^2\big(x_1^2x_2^3x_3^7x_4^8x_5^8\\ &+ x_1^2x_2^3x_3^{11}x_4^4x_5^8 + x_1^2x_2^8x_3^7x_4^3x_5^8\big) + Sq^4\big(x_1^3x_2^4x_3^7x_4^4x_5^8\big) \ \mathrm{mod}(P_5^-((2|^4))).\end{aligned}$$

$$\begin{aligned}x_1x_2^2x_3^4x_4^{11}x_5^{12} &= x_1x_2x_3^6x_4^8x_5^{14} + x_1x_2x_3^6x_4^{10}x_5^{12} + x_1x_2^2x_3^2x_4^{12}x_5^{13} + x_1x_2^2x_3^3x_4^{12}x_5^{12}\\ &+ x_1x_2^2x_3^4x_4^9x_5^{14} + Sq^1\big(x_1^2x_2x_3^3x_4^7x_5^{16} + x_1^2x_2x_3^3x_4^9x_5^{14} + x_1^2x_2x_3^3x_4^{10}x_5^{13}\\ &+ x_1^2x_2x_3^3x_4^{12}x_5^{11} + x_1^2x_2x_3^5x_4^7x_5^{14} + x_1^2x_2x_3^5x_4^8x_5^{13} + x_1^2x_2x_3^5x_4^9x_5^{12}\\ &+ x_1^2x_2x_3^5x_4^{10}x_5^{11} + x_1^2x_2x_3^6x_4^7x_5^{13} + x_1^2x_2x_3^6x_4^9x_5^{11} + x_1^2x_2x_3^8x_4^7x_5^{11}\big)\\ &+ Sq^2\big(x_1x_2x_3^3x_4^7x_5^{16} + x_1x_2x_3^3x_4^9x_5^{14} + x_1x_2x_3^3x_4^{10}x_5^{13} + x_1x_2x_3^3x_4^{12}x_5^{11}\\ &+ x_1x_2x_3^5x_4^7x_5^{14} + x_1x_2x_3^5x_4^8x_5^{13} + x_1x_2x_3^5x_4^9x_5^{12} + x_1x_2x_3^5x_4^{10}x_5^{11}\\ &+ x_1x_2x_3^6x_4^7x_5^{13} + x_1x_2x_3^6x_4^9x_5^{11} + x_1x_2x_3^8x_4^7x_5^{11} + x_1x_2^4x_3^2x_4^{10}x_5^{11}\big)\end{aligned}$$

$$+ Sq^4\big(x_1x_2^2x_3^2x_4^{10}x_5^{11} + x_1x_2^2x_3^4x_4^7x_5^{12}\big) \bmod(P_5^-((2|^4))).$$

So, $x_1^3x_2^5x_3^8x_4^2x_5^{12}$, $x_1^3x_2^3x_3^{12}x_4^4x_5^8$, $x_1x_2^2x_3^4x_4^{11}x_5^{12}$ are strictly inadmissible.

$$\begin{aligned}
x_1x_2^2x_3^7x_4^8x_5^{12} &= x_1x_2x_3^4x_4^{10}x_5^{14} + x_1x_2x_3^6x_4^{10}x_5^{12} + x_1x_2x_3^{10}x_4^4x_5^{14} + x_1x_2x_3^{10}x_4^6x_5^{12}\\
&+ x_1x_2^2x_3^3x_4^{12}x_5^{12} + x_1x_2^2x_3^4x_4^{10}x_5^{13} + x_1x_2^2x_3^4x_4^{12}x_5^{11} + x_1x_2^2x_3^6x_4^8x_5^{13}\\
&+ Sq^1\big(x_1^2x_2x_3^3x_4^3x_5^{20} + x_1^2x_2x_3^3x_4^5x_5^{18} + x_1^2x_2x_3^3x_4^{10}x_5^{13} + x_1^2x_2x_3^3x_4^{12}x_5^{11}\\
&+ x_1^2x_2x_3^5x_4^3x_5^{18} + x_1^2x_2x_3^5x_4^{10}x_5^{11} + x_1^2x_2x_3^7x_4^3x_5^{16} + x_1^2x_2x_3^7x_4^5x_5^{14}\\
&+ x_1^2x_2x_3^7x_4^6x_5^{13} + x_1^2x_2x_3^7x_4^8x_5^{11} + x_1^2x_2x_3^9x_4^3x_5^{14} + x_1^2x_2x_3^9x_4^6x_5^{11}\big)\\
&+ Sq^2\big(x_1x_2x_3^3x_4^3x_5^{20} + x_1x_2x_3^3x_4^5x_5^{18} + x_1x_2x_3^3x_4^{10}x_5^{13} + x_1x_2x_3^3x_4^{12}x_5^{11}\\
&+ x_1x_2x_3^5x_4^3x_5^{18} + x_1x_2x_3^5x_4^{10}x_5^{11} + x_1x_2x_3^7x_4^3x_5^{16} + x_1x_2x_3^7x_4^5x_5^{14}\\
&+ x_1x_2x_3^7x_4^6x_5^{13} + x_1x_2x_3^7x_4^8x_5^{11} + x_1x_2x_3^9x_4^3x_5^{14} + x_1x_2x_3^9x_4^6x_5^{11}\\
&+ x_1x_2^4x_3^3x_4^6x_5^{14} + x_1x_2^4x_3^6x_4^3x_5^{14} + x_1x_2^4x_3^6x_4^6x_5^{11}\big)\\
&+ Sq^4\big(x_1x_2^2x_3^3x_4^6x_5^{14} + x_1x_2^2x_3^5x_4^4x_5^{14} + x_1x_2^2x_3^5x_4^6x_5^{12}\\
&+ x_1x_2^2x_3^6x_4^3x_5^{14} + x_1x_2^2x_3^6x_4^6x_5^{11}\big) \bmod(P_5^-((2|^4))).\\
x_1^3x_2^4x_3x_4^{10}x_5^{12} &= x_1x_2^6x_3^2x_4^9x_5^{12} + x_1x_2^8x_3^2x_4^7x_5^{12} + x_1^2x_2x_3^2x_4^{13}x_5^{12} + x_1^2x_2x_3^4x_4^{11}x_5^{12}\\
&+ x_1^2x_2^3x_3^4x_4^9x_5^{12} + x_1^3x_2x_3^2x_4^{12}x_5^{12} + x_1^3x_2^2x_3^4x_4^9x_5^{12} + x_1^3x_2^3x_3^4x_4^8x_5^{12}\\
&+ Sq^1\big(x_1x_2^5x_3^4x_4^7x_5^{12} + x_1^3x_2x_3^2x_4^7x_5^{16} + x_1^3x_2x_3^2x_4^{11}x_5^{12} + x_1^3x_2^3x_3^4x_4^7x_5^{12}\\
&+ x_1^3x_2^4x_3x_4^9x_5^{12}\big) + Sq^2\big(x_1x_2^6x_3^2x_4^7x_5^{12} + x_1^2x_2x_3^2x_4^{11}x_5^{12} + x_1^2x_2^3x_3^4x_4^7x_5^{12}\\
&+ x_1^5x_2^2x_3^2x_4^7x_5^{12}\big) + Sq^4\big(x_1^3x_2^2x_3^2x_4^7x_5^{12}\big) \bmod(P_5^-((2|^4))).\\
x_1^7x_2x_3^{10}x_4^4x_5^8 &= x_1^4x_2^4x_3^{11}x_4^3x_5^8 + x_1^4x_2^8x_3^7x_4^3x_5^8 + x_1^5x_2^2x_3^7x_4^8x_5^8 + x_1^5x_2^2x_3^{11}x_4^4x_5^8\\
&+ x_1^7x_2x_3^8x_4^6x_5^8 + Sq^1\big(x_1^7x_2x_3^8x_4^5x_5^8 + x_1^7x_2x_3^9x_4^4x_5^8 + x_1^7x_2^4x_3^7x_4^3x_5^8\big)\\
&+ Sq^2\big(x_1^7x_2^2x_3^7x_4^4x_5^8 + x_1^7x_2^4x_3^8x_4^3x_5^8\big)\\
&+ Sq^4\big(x_1^4x_2^4x_3^7x_4^3x_5^8 + x_1^5x_2^2x_3^7x_4^4x_5^8\big) \bmod(P_5^-((2|^4))).\\
x_1x_2^7x_3^6x_4^8x_5^8 &= x_1x_2^4x_3^6x_4^8x_5^{11} + x_1x_2^4x_3^{10}x_4^8x_5^7 + x_1x_2^6x_3^4x_4^8x_5^{11} + x_1x_2^6x_3^8x_4^8x_5^7\\
&+ x_1x_2^7x_3^4x_4^8x_5^{10} + Sq^1\big(x_1^2x_2^7x_3^3x_4^8x_5^9 + x_1^2x_2^7x_3^5x_4^8x_5^7 + x_1^2x_2^9x_3^3x_4^8x_5^7\big)\\
&+ Sq^2\big(x_1x_2^7x_3^3x_4^8x_5^9 + x_1x_2^7x_3^5x_4^8x_5^7 + x_1x_2^9x_3^3x_4^8x_5^7\big)\\
&+ Sq^4\big(x_1x_2^4x_3^6x_4^8x_5^7 + x_1x_2^6x_3^4x_4^8x_5^7\big) \bmod(P_5^-((2|^4))).
\end{aligned}$$

The above equalities show that the monomials

$$x_1x_2^2x_3^7x_4^8x_5^{12},\ x_1^3x_2^4x_3x_4^{10}x_5^{12},\ x_1^7x_2x_3^{10}x_4^4x_5^8,\ x_1x_2^7x_3^6x_4^8x_5^8$$

are strictly inadmissible. □

Lemma 3.3.6. *The following monomials are strictly inadmissible:*

$$x_1^3x_2^5x_3^6x_4^{10}x_5^6,\ x_1^3x_2^5x_3^{10}x_4^6x_5^6,\ x_i^3x_j^5x_t^2x_u^6x_v^{14},\ x_i^3x_j^{13}x_t^2x_u^6x_v^6.$$

Here (i,j,t,u,v) *is an arbitrary permutation of* $(1,2,3,4,5)$.

Proof. By a direct computation we have

$$\begin{aligned}
x_1^3x_2^5x_3^{10}x_4^6x_5^6 &= x_1^3x_2^5x_3^6x_4^6x_5^{10} + Sq^1\big(x_1^3x_2^3x_3^5x_4^9x_5^9 + x_1^3x_2^3x_3^9x_4^9x_5^5 + x_1^3x_2^6x_3^5x_4^6x_5^9\\
&+ x_1^3x_2^6x_3^5x_4^9x_5^6 + x_1^3x_2^6x_3^6x_4^9x_5^5 + x_1^3x_2^6x_3^9x_4^6x_5^5\big) + Sq^2\big(x_1^2x_2^5x_3^3x_4^9x_5^9\\
&+ x_1^2x_2^5x_3^9x_4^9x_5^3 + x_1^5x_2^2x_3^3x_4^9x_5^9 + x_1^5x_2^2x_3^9x_4^9x_5^3 + x_1^5x_2^3x_3^3x_4^8x_5^9\\
&+ x_1^5x_2^3x_3^3x_4^9x_5^8 + x_1^5x_2^3x_3^5x_4^6x_5^9 + x_1^5x_2^3x_3^5x_4^9x_5^6 + x_1^5x_2^3x_3^6x_4^9x_5^5\\
&+ x_1^5x_2^3x_3^8x_4^9x_5^3 + x_1^5x_2^3x_3^9x_4^6x_5^5 + x_1^5x_2^3x_3^9x_4^8x_5^3\big) + Sq^4\big(x_1^2x_2^3x_3^3x_4^9x_5^9\\
&+ x_1^2x_2^3x_3^9x_4^9x_5^3 + x_1^3x_2^2x_3^3x_4^9x_5^9 + x_1^3x_2^2x_3^9x_4^9x_5^3 + x_1^3x_2^3x_3^3x_4^8x_5^9\\
&+ x_1^3x_2^3x_3^3x_4^9x_5^8 + x_1^3x_2^3x_3^5x_4^6x_5^9 + x_1^3x_2^3x_3^5x_4^9x_5^6 + x_1^3x_2^3x_3^6x_4^9x_5^5\\
&+ x_1^3x_2^3x_3^8x_4^9x_5^3 + x_1^3x_2^3x_3^9x_4^6x_5^5 + x_1^3x_2^3x_3^9x_4^8x_5^3\big) \ \mathrm{mod}(P_5^-(2,4,3,1)).
\end{aligned}$$

This equality shows that the monomial $x_1^3x_2^5x_3^{10}x_4^6x_5^6$ is strictly inadmissible. By a similar computation we also prove that $x_1^3x_2^5x_3^6x_4^{10}x_5^6$ is strictly inadmissible. We have

$$\begin{aligned}
x_i^3x_j^5x_t^2x_u^6x_v^{14} &= Sq^1\big(x_i^3x_j^5x_t^2x_u^6x_v^{13} + x_i^3x_j^5x_t^2x_u^{12}x_v^7 + x_i^3x_j^5x_t^4x_u^6x_v^{11} + x_i^3x_j^5x_t^4x_u^{10}x_v^7\\
&+ x_i^3x_j2^{10}x_t^4x_u^5x_v^7 + x_i^3x_j^{12}x_t^2x_u^5x_v^7 + x_i^9x_j^5x_t^2x_u^6x_v^7 + x_i^{10}x_j^3x_t^4x_u^5x_v^7\\
&+ x_i^{12}x_j^3x_t^2x_u^5x_v^7\big) + Sq^2\big(x_i^3x_j^6x_t^2x_u^6x_v^{11} + x_i^3x_j^6x_t^2x_u^{10}x_v^7 + x_i^3x_j^{10}x_t^2x_u^6x_v^7\\
&+ x_i^{10}x_j^3x_t^2x_u^6x_v^7\big) + Sq^4\big(x_i^5x_j^6x_t^2x_u^6x_v^7\big) \ \mathrm{mod}(P_5^-(2,4,3,1)).\\
x_i^3x_j^{13}x_t^2x_u^6x_v^6 &= Sq^1\big(x_i^3x_j^7x_t^2x_u^5x_v^{12} + x_i^3x_j^7x_t^2x_u^{12}x_v^5 + x_i^3x_j^7x_t^4x_u^5x_v^{10} + x_i^3x_j^7x_t^4x_u^{10}x_v^5\\
&+ x_i^3x_j^{11}x_t^4x_u^5x_v^6 + x_i^9x_j^7x_t^2x_u^5x_v^6 + x_i^9x_j^7x_t^4x_u^3x_v^6 + x_i^{12}x_j^7x_tx_u^3x_v^6\big)\\
&+ Sq^2\big(x_i^3x_j^7x_t^2x_u^6x_v^{10} + x_i^3x_j^7x_t^2x_u^{10}x_v^6 + x_i^3x_j^{11}x_t^2x_u^6x_v^6 + x_i^{10}x_j^7x_t^2x_u^3x_v^6\big)\\
&+ Sq^4\big(x_i^5x_j^7x_t^2x_u^6x_v^6\big) \ \mathrm{mod}(P_5^-(2,4,3,1)).
\end{aligned}$$

Hence, the monomials $x_i^3x_j^5x_t^2x_u^6x_v^{14}$ and $x_i^3x_j^{13}x_t^2x_u^6x_v^6$ are strictly inadmissible. $\square$

Proof of Theorem 3.3.3. First we prove that $QP_5((2)|^4)$ is generated by the set $\{[b_t]; 1 \leqslant t \leqslant 39\}$. Let x be an admissible monomial in P_5^+ such that $\omega(x) = ((2)|^4)$. Then $x = x_jx_\ell y^2$ with $1 \leqslant j < \ell \leqslant 5$ and y an monomial in P_5 and $\omega(y) = ((2)|^3)$. Since x is admissible, by Theorem 2.7, $y \in B_5((2)|^3)$.

Let $z \in B_5((2)|^3)$ such that $x_jx_\ell z^2 \in P_5^+$. By a direct computation, we see that if $x_jx_\ell z^2 \neq b_t$, for all $t,\ 1 \leqslant t \leqslant 39$, then there is a strictly inadmissible monomial w which is given in one of Lemmas 3.1.2, 3.2.2, 3.3.5 such that $x_jx_\ell z^2 = wz_1^{2^u}$ with suitable monomial $z_1 \in P_5$, and $u = \max\{j \in \mathbb{Z} : \omega_j(w) > 0\}$. By Theorem 2.7, $x_jx_\ell z^2$ is inadmissible. Since $x = x_jx_\ell y^2$ with $y \in B_5((2)|^3)$ and x admissible, one obtain $x = b_t$ for some $t,\ 1 \leqslant t \leqslant 39$. Hence, $QP_5^+((2)|^4)$ is spanned by the set $\{[b_t] : 1 \leqslant t \leqslant 39\}$.

Let x be a monomial in P_5^+ such that $\omega(x) = ((2,4,3,1)$. By a direct computation, we see that if $x \neq b_{40}$, then there is a strictly inadmissible monomial w which is given in one of Lemmas 3.2.2, 3.3.4, 3.3.6 such that $x = wy^{2^r}$ with suitable monomial $y \in P_5$, and $r = \max\{j \in \mathbb{Z} : \omega_j(y) > 0\}$. By Theorem 2.7, x is inadmissible. Hence, if x is admissible, then $x = b_{40}$.

We now prove the set $\{[b_t] : 1 \leqslant t \leqslant 40\}$ is linearly independent in $(QP_5)_{30}$. Suppose there is a linear relation $\mathcal{S} = \sum_{t=1}^{40} \gamma_t a_t \equiv 0$, where $\gamma_t \in \mathbb{F}_2$. Consider the homomorphism $p_{(i;j)} : (QP_5^+)_{30} \to (QP_4^+)_{30}$ with $1 \leqslant i < j \leqslant 5$. From Proposition 3.2, we see that the following monomials are admissible in P_4:

$w_1 = x_1x_2x_3^{14}x_4^{14}$ $\quad w_2 = x_1x_2^{14}x_3x_4^{14}$ $\quad w_3 = x_1x_2^{2}x_3^{12}x_4^{15}$ $\quad w_4 = x_1x_2^{2}x_3^{15}x_4^{12}$
$w_5 = x_1x_2^{15}x_3^{2}x_4^{12}$ $\quad w_6 = x_1^{15}x_2x_3^{2}x_4^{12}$ $\quad w_7 = x_1x_2^{2}x_3^{13}x_4^{14}$ $\quad w_8 = x_1x_2^{3}x_3^{12}x_4^{14}$
$w_9 = x_1x_2^{3}x_3^{14}x_4^{12}$ $\quad w_{10} = x_1^{3}x_2x_3^{12}x_4^{14}$ $\quad w_{11} = x_1^{3}x_2x_3^{14}x_4^{12}$ $\quad w_{12} = x_1^{3}x_2^{13}x_3^{2}x_4^{12}$
$w_{13} = x_1^{3}x_2^{5}x_3^{10}x_4^{12}$.

For simplicity, we denote $\gamma_J = \sum_{j\in J}\gamma_j$ with $J \subset \{1, 2, \dots, 40\}$. By a direct computation, we obtain

$$\begin{aligned} p_{(1;2)}(\mathcal{S}) \equiv\ &\gamma_{\{6,10,20,21,22\}}w_1 + \gamma_6w_2 + \gamma_{15}w_3 + \gamma_{16}w_4 + \gamma_{17}w_5\\ &+ \gamma_7w_6 + \gamma_{20}w_7 + \gamma_{21}w_8 + \gamma_{22}w_9 + \gamma_{\{4,6,20,21\}}w_{10}\\ &+ \gamma_{\{5,22\}}w_{11} + \gamma_9w_{12} + \gamma_{23}w_{13} \equiv 0. \end{aligned} \tag{3.1}$$

From (3.1), we get

$$\gamma_j = 0 \text{ for } j \in \{4,\ 5,\ 6,\ 7,\ 9,\ 10,\ 15,\ 16,\ 17,\ 20,\ 21,\ 22,\ 23\}. \tag{3.2}$$

By using (3.2) we get

$$\begin{aligned} p_{(1;3)}(\mathcal{S}) \equiv\ &\gamma_{\{8,11,24,25,31\}}w_1 + \gamma_{\{3,14,26\}}w_5 + \gamma_{18}w_6\\ &+ \gamma_{\{1,8,24,27,31,32\}}w_8 + \gamma_{\{2,8,25,28,31,33\}}w_9 + \gamma_{24}w_{10}\\ &+ \gamma_{25}w_{11} + \gamma_{12}w_{12} + \gamma_{\{30,40\}}w_{13} \equiv 0. \end{aligned} \tag{3.3}$$

Combining (3.1) and (3.3) we have

$$\gamma_j = 0 \text{ for } j \in \{12,\ 18,\ 24,\ 25\}. \tag{3.4}$$

By a direct computation using (3.2) and (3.4) we obtain

$$\begin{aligned} p_{(4;5)}(\mathcal{S}) \equiv\ &\gamma_3w_1 + \gamma_{19}w_6 + \gamma_{26}w_9 + \gamma_{14}w_{10}\\ &+ \gamma_{29}w_{11} + \gamma_{36}w_{12} + \gamma_{39}w_{13} + \equiv 0. \end{aligned} \tag{3.5}$$

The equality (3.5) implies

$$\gamma_j = 0 \text{ for } j \in \{3,\ 19,\ 26,\ 14,\ 29,\ 36,\ 39\}. \tag{3.6}$$

Based on (3.2), (3.4) and (3.6), we get

$$\begin{aligned} p_{(3;5)}(\mathcal{S}) \equiv \gamma_{\{2,37,38\}}w_1 + \gamma_{\{11,33,37\}}w_8 \\ + \gamma_{\{13,35,37\}}w_{10} + \gamma_{28}w_{11} + \gamma_{38}w_{13} \equiv 0. \end{aligned} \tag{3.7}$$

From (3.5) we have

$$\gamma_j = 0 \text{ for } j = 28,\ 38. \tag{3.8}$$

By a similar computation using (3.2), (3.4), (3.6), (3.8), we obtain

$$\begin{aligned} p_{(1;5)}(\mathcal{S}) &\equiv \gamma_{\{2,30,32,33\}}w_1 + \gamma_{\{1,13,31,34,37\}}w_3 + \gamma_{\{2,8,30,35,40\}}w_7 \\ &\quad + \gamma_{\{8,32,35,40\}}w_8 + \gamma_{\{11,30,32\}}w_{10} + \gamma_{33}w_{13} \equiv 0. \quad (3.9) \\ p_{(2;5)}(\mathcal{S}) &\equiv \gamma_{\{2,31,34,35\}}w_1 + \gamma_{\{1,11,27,30,32,40\}}w_3 + \gamma_{\{2,8,31,33\}}w_7 \\ &\quad + \gamma_{\{8,33,34\}}w_8 + \gamma_{\{13,31,34\}}w_{10} + \gamma_{35}w_{13} \equiv 0. \quad (3.10) \\ p_{(3;4)}(\mathcal{S}) &\equiv \gamma_{\{1,40\}}w_1 + \gamma_{\{11,30,32\}}w_9 + \gamma_{27}w_{10} \\ &\quad + \gamma_{\{13,31,34\}}w_{11} + \gamma_{\{37,40\}}w_{13} \equiv 0. \quad (3.11) \end{aligned}$$

From the above equalities we get

$$\gamma_j = 0,\ j = 27,\ 31,\ 32,\ 33,\ 35; \gamma_j = \gamma_1,\ j = 2,\ 8,\ 11,\ 13,\ 30,\ 34,\ 37,\ 40. \tag{3.12}$$

By a direct computation using (3.1), (3.12), we have

$$p_{(1;4)}(\mathcal{S}) \equiv \gamma_1 w_7 + \gamma_1 w_{13} \equiv 0.$$

So, we get $\gamma_1 = 0$ hence $\gamma_t = 0$ for all $t,\ 1 \leqslant t \leqslant 40$. The proof is completed. □

Now, we determine the space $QP_5^+(4,3,3,1)$. We denote $\mathcal{C} = \{x_i^{15}f_i(u_j) : 1 \leqslant i \leqslant 5,\ 1 \leqslant j \leqslant 36\}$, where u_j, $1 \leqslant j \leqslant 36$, are the admissible monomials of degree 15 in P_4 which are determined as follows:

1. $x_1x_2x_3^6x_4^7$	2. $x_1x_2x_3^7x_4^6$	3. $x_1x_2^6x_3x_4^7$	4. $x_1x_2^6x_3^7x_4$
5. $x_1x_2^7x_3x_4^6$	6. $x_1x_2^7x_3^6x_4$	7. $x_1^7x_2x_3x_4^6$	8. $x_1^7x_2x_3^6x_4$
9. $x_1x_2^2x_3^5x_4^7$	10. $x_1x_2^2x_3^7x_4^5$	11. $x_1x_2^7x_3^2x_4^5$	12. $x_1^7x_2x_3^2x_4^5$
13. $x_1x_2^3x_3^4x_4^7$	14. $x_1x_2^3x_3^7x_4^4$	15. $x_1x_2^7x_3^3x_4^4$	16. $x_1^3x_2x_3^4x_4^7$
17. $x_1^3x_2x_3^7x_4^4$	18. $x_1^3x_2^7x_3x_4^4$	19. $x_1^7x_2x_3^3x_4^4$	20. $x_1^7x_2^3x_3x_4^4$
21. $x_1x_2^3x_3^5x_4^6$	22. $x_1x_2^3x_3^6x_4^5$	23. $x_1x_2^6x_3^3x_4^5$	24. $x_1^3x_2x_3^5x_4^6$
25. $x_1^3x_2x_3^6x_4^5$	26. $x_1^3x_2^5x_3x_4^6$	27. $x_1^3x_2^5x_3^6x_4$	28. $x_1^3x_2^5x_3^2x_4^5$
29. $x_1^3x_2^3x_3^4x_4^5$	30. $x_1^3x_2^3x_3^5x_4^4$	31. $x_1^3x_2^5x_3^3x_4^4$	32. $x_1^3x_2^4x_3x_4^7$
33. $x_1^3x_2^4x_3^7x_4$	34. $x_1^3x_2^7x_3^4x_4$	35. $x_1^7x_2^3x_3^4x_4$	36. $x_1^3x_2^4x_3^3x_4^5$.

Denote $\mathcal{D} = \{x_i^7f_i(v_j) : 1 \leqslant i \leqslant 5,\ 1 \leqslant j \leqslant 75\}$, where v_j are the admissible monomials of degree 23 in P_4 which are determined as follows:

1. $x_1x_2x_3^7x_4^{14}$	2. $x_1x_2x_3^{14}x_4^7$	3. $x_1x_2^7x_3x_4^{14}$	4. $x_1x_2^7x_3^{14}x_4$	5. $x_1x_2^{14}x_3x_4^7$
6. $x_1x_2^{14}x_3^7x_4$	7. $x_1^7x_2x_3x_4^{14}$	8. $x_1^7x_2x_3^{14}x_4$	9. $x_1x_2^2x_3^7x_4^{13}$	10. $x_1x_2^2x_3^{13}x_4^7$
11. $x_1x_2^7x_3^2x_4^{13}$	12. $x_1^7x_2x_3^2x_4^{13}$	13. $x_1x_2^3x_3^5x_4^{14}$	14. $x_1x_2^3x_3^{14}x_4^5$	15. $x_1x_2^{14}x_3^3x_4^5$
16. $x_1^3x_2x_3^5x_4^{14}$	17. $x_1^3x_2x_3^{14}x_4^5$	18. $x_1^3x_2^5x_3x_4^{14}$	19. $x_1^3x_2^5x_3^{14}x_4$	20. $x_1x_2^3x_3^6x_4^{13}$
21. $x_1x_2^3x_3^{13}x_4^6$	22. $x_1x_2^6x_3^3x_4^{13}$	23. $x_1^3x_2x_3^6x_4^{13}$	24. $x_1^3x_2x_3^{13}x_4^6$	25. $x_1^3x_2^{13}x_3x_4^6$
26. $x_1^3x_2^{13}x_3^6x_4$	27. $x_1x_2^3x_3^7x_4^{12}$	28. $x_1x_2^3x_3^{12}x_4^7$	29. $x_1x_2^7x_3^3x_4^{12}$	30. $x_1^3x_2x_3^7x_4^{12}$
31. $x_1^3x_2x_3^{12}x_4^7$	32. $x_1^3x_2^7x_3x_4^{12}$	33. $x_1^3x_2^7x_3^{12}x_4$	34. $x_1^7x_2x_3^3x_4^{12}$	35. $x_1^7x_2^3x_3x_4^{12}$
36. $x_1^7x_2^3x_3^{12}x_4$	37. $x_1x_2^7x_3^{11}x_4^4$	38. $x_1^7x_2x_3^{11}x_4^4$	39. $x_1^7x_2^{11}x_3x_4^4$	40. $x_1^7x_2^{11}x_3^4x_4$
41. $x_1x_2^6x_3^{11}x_4^5$	42. $x_1x_2^7x_3^{10}x_4^5$	43. $x_1^7x_2x_3^{10}x_4^5$	44. $x_1x_2^6x_3^7x_4^9$	45. $x_1x_2^7x_3^6x_4^9$
46. $x_1^7x_2x_3^6x_4^9$	47. $x_1x_2^7x_3^7x_4^8$	48. $x_1^7x_2x_3^7x_4^8$	49. $x_1^7x_2^7x_3x_4^8$	50. $x_1^7x_2^7x_3^8x_4$
51. $x_1^3x_2^5x_3^2x_4^{13}$	52. $x_1^3x_2^{13}x_3^2x_4^5$	53. $x_1^7x_2^9x_3^2x_4^5$	54. $x_1^3x_2^3x_3^4x_4^{13}$	55. $x_1^3x_2^3x_3^{13}x_4^4$
56. $x_1^3x_2^4x_3^3x_4^{13}$	57. $x_1^3x_2^{13}x_3^3x_4^4$	58. $x_1^3x_2^3x_3^5x_4^{12}$	59. $x_1^3x_2^3x_3^{12}x_4^5$	60. $x_1^3x_2^5x_3^3x_4^{12}$
61. $x_1^3x_2^4x_3^{11}x_4^5$	62. $x_1^3x_2^5x_3^{11}x_4^4$	63. $x_1^3x_2^4x_3^7x_4^9$	64. $x_1^3x_2^7x_3^4x_4^9$	65. $x_1^3x_2^7x_3^9x_4^4$
66. $x_1^7x_2^3x_3^4x_4^9$	67. $x_1^7x_2^3x_3^9x_4^4$	68. $x_1^7x_2^9x_3^3x_4^4$	69. $x_1^3x_2^5x_3^{10}x_4^5$	70. $x_1^3x_2^5x_3^6x_4^9$
71. $x_1^3x_2^5x_3^7x_4^8$	72. $x_1^3x_2^7x_3^5x_4^8$	73. $x_1^3x_2^7x_3^8x_4^5$	74. $x_1^7x_2^3x_3^5x_4^8$	75. $x_1^7x_2^3x_3^8x_4^5$.

By Proposition 2.14, $\mathcal{C} \subset B_5^+(4,3,3,1)$ and $\mathcal{D} \subset B_5^+(4,3,3,1)$. By a simple computation we have $|\mathcal{C}| = 180$ and $|\mathcal{D}| = 278$.

Theorem 3.3.7. $B_5^+(4,3,3,1) = \mathcal{C} \cup \mathcal{D} \cup \{s_t : 1 \leqslant t \leqslant 52\}$, *where* s_t *is determined as follows:*

1. $x_1x_2^7x_3^{11}x_4^5x_5^6$	2. $x_1x_2^7x_3^{11}x_4^6x_5^5$	3. $x_1^7x_2x_3^{11}x_4^5x_5^6$	4. $x_1^7x_2x_3^{11}x_4^6x_5^5$
5. $x_1^7x_2^{11}x_3x_4^5x_5^6$	6. $x_1^7x_2^{11}x_3x_4^6x_5^5$	7. $x_1^7x_2^{11}x_3^5x_4x_5^6$	8. $x_1^7x_2^{11}x_3^5x_4^6x_5$
9. $x_1x_2^7x_3^7x_4^9x_5^6$	10. $x_1^7x_2x_3^7x_4^9x_5^6$	11. $x_1^7x_2^7x_3x_4^9x_5^6$	12. $x_1^7x_2^7x_3^9x_4x_5^6$
13. $x_1^7x_2^7x_3^9x_4^6x_5$	14. $x_1^7x_2^{11}x_3^5x_4^2x_5^5$	15. $x_1^3x_2^3x_3^5x_4^5x_5^{14}$	16. $x_1^3x_2^3x_3^5x_4^{14}x_5^5$
17. $x_1^3x_2^5x_3^3x_4^5x_5^{14}$	18. $x_1^3x_2^5x_3^3x_4^{14}x_5^5$	19. $x_1^3x_2^5x_3^{14}x_4^3x_5^5$	20. $x_1^3x_2^3x_3^5x_4^6x_5^{13}$
21. $x_1^3x_2^3x_3^5x_4^{13}x_5^6$	22. $x_1^3x_2^3x_3^{13}x_4^5x_5^6$	23. $x_1^3x_2^3x_3^{13}x_4^6x_5^5$	24. $x_1^3x_2^5x_3^3x_4^6x_5^{13}$
25. $x_1^3x_2^5x_3^3x_4^{13}x_5^6$	26. $x_1^3x_2^5x_3^6x_4^3x_5^{13}$	27. $x_1^3x_2^{13}x_3^3x_4^5x_5^6$	28. $x_1^3x_2^{13}x_3^3x_4^6x_5^5$
29. $x_1^3x_2^{13}x_3^6x_4^3x_5^5$	30. $x_1^3x_2^7x_3^{12}x_4^3x_5^5$	31. $x_1^7x_2^3x_3^{12}x_4^3x_5^5$	32. $x_1^3x_2^7x_3^{11}x_4^4x_5^5$
33. $x_1^3x_2^7x_3^{11}x_4^5x_5^4$	34. $x_1^7x_2^3x_3^{11}x_4^4x_5^5$	35. $x_1^7x_2^3x_3^{11}x_4^5x_5^4$	36. $x_1^7x_2^{11}x_3^3x_4^4x_5^5$
37. $x_1^7x_2^{11}x_3^3x_4^5x_5^4$	38. $x_1^7x_2^{11}x_3^4x_4^3x_5^5$	39. $x_1^7x_2^{11}x_3^5x_4^3x_5^4$	40. $x_1^3x_2^5x_3^6x_4^{11}x_5^5$
41. $x_1^3x_2^5x_3^{11}x_4^5x_5^6$	42. $x_1^3x_2^5x_3^{11}x_4^6x_5^5$	43. $x_1^3x_2^5x_3^7x_4^9x_5^6$	44. $x_1^3x_2^7x_3^5x_4^9x_5^6$
45. $x_1^3x_2^7x_3^9x_4^5x_5^6$	46. $x_1^3x_2^7x_3^9x_4^6x_5^5$	47. $x_1^7x_2^3x_3^5x_4^9x_5^6$	48. $x_1^7x_2^3x_3^9x_4^5x_5^6$
49. $x_1^7x_2^3x_3^9x_4^6x_5^5$	50. $x_1^7x_2^9x_3^3x_4^5x_5^6$	51. $x_1^7x_2^9x_3^3x_4^6x_5^5$	52. $x_1^7x_2^7x_3^8x_4^3x_5^5$.

Consequently,

$$\dim QP_5^+(4,3,3,1) = 510,\ \dim QP_5(4,3,3,1) = 685,\ \dim(QP_5)_{30} = 840.$$

We need the following lemma for the proof of Theorem 3.3.7.

Lemma 3.3.8. *The following monomials are strictly inadmissible:*

$x_1^3x_2^{12}x_3x_4^7x_5^7$	$x_1^3x_2^{12}x_3^7x_4x_5^7$	$x_1^3x_2^{12}x_3^7x_4^7x_5$	$x_1^3x_2^{12}x_3^3x_4^5x_5^7$	$x_1^3x_2^{12}x_3^3x_4^7x_5^5$
$x_1^3x_2^{12}x_3^7x_4^3x_5^5$	$x_1^3x_2^4x_3^9x_4^7x_5^7$	$x_1^3x_2^5x_3^9x_4^6x_5^7$	$x_1^3x_2^5x_3^9x_4^7x_5^6$	$x_1^7x_2^9x_3^6x_4^3x_5^5$
$x_1^3x_2^5x_3^8x_4^7x_5^7$	$x_1^7x_2^8x_3^3x_4^5x_5^7$	$x_1^7x_2^8x_3^3x_4^7x_5^5$	$x_1^7x_2^8x_3^7x_4^3x_5^5$.	

Proof. By a direct computation we have

$$\begin{aligned}x_1^3x_2^{12}x_3x_4^7x_5^7 &= x_1^2x_2^7x_3x_4^7x_5^{13} + x_1^2x_2^7x_3x_4^{13}x_5^7 + x_1^2x_2^{13}x_3x_4^7x_5^7 + x_1^3x_2^7x_3x_4^7x_5^{12}\\ &+ x_1^3x_2^7x_3x_4^{12}x_5^7 + x_1^3x_2^7x_3^4x_4^7x_5^9 + x_1^3x_2^7x_3^4x_4^9x_5^7 + x_1^3x_2^9x_3^4x_4^7x_5^7\\ &+ Sq^2\big(x_1^3x_2^{11}x_3x_4^7x_5^7 + x_1^3x_2^7x_3x_4^{11}x_5^7 + x_1^3x_2^7x_3x_4^7x_5^{11}\big) + Sq^2\big(x_1^3x_2^{11}x_3x_4^7x_5^7\\ &+ x_1^3x_2^7x_3x_4^{11}x_5^7 + x_1^3x_2^7x_3x_4^7x_5^{11}\big) + Sq^4\big(x_1^3x_2^7x_3^2x_4^7x_5^7\big) \bmod(P_5^-(4,3,3)).\end{aligned}$$

This equality shows that the monomial $x_1^3x_2^{12}x_3x_4^7x_5^7$ is strictly inadmissible. We have

$$\begin{aligned}x_1^3x_2^{12}x_3^3x_4^5x_5^7 &= x_1^2x_2^7x_3^3x_4^5x_5^{13} + x_1^2x_2^7x_3^5x_4^5x_5^{11} + x_1^2x_2^7x_3^5x_4^9x_5^7 + x_1^2x_2^{11}x_3^5x_4^5x_5^7\\ &+ x_1^2x_2^{13}x_3^3x_4^5x_5^7 + x_1^3x_2^7x_3^3x_4^5x_5^{12} + x_1^3x_2^7x_3^4x_4^5x_5^{11} + x_1^3x_2^7x_3^4x_4^9x_5^7\\ &+ x_1^3x_2^7x_3^5x_4^6x_5^9 + x_1^3x_2^7x_3^5x_4^8x_5^7 + x_1^3x_2^9x_3^5x_4^6x_5^7 + x_1^3x_2^{11}x_3^4x_4^5x_5^7\\ &+ Sq^1\big(x_1^3x_2^{11}x_3^3x_4^5x_5^7 + x_1^3x_2^7x_3^3x_4^9x_5^7 + x_1^3x_2^7x_3^3x_4^5x_5^{11}\big)\\ &+ Sq^2\big(x_1^2x_2^{11}x_3^3x_4^5x_5^7 + x_1^5x_2^7x_3^3x_4^6x_5^7 + x_1^2x_2^7x_3^3x_4^5x_5^{11} + x_1^2x_2^7x_3^3x_4^9x_5^7\big)\\ &+ Sq^4\big(x_1^3x_2^7x_3^3x_4^6x_5^7\big) \bmod(P_5^-(4,3,3)).\\ x_1^7x_2^9x_3^6x_4^3x_5^5 &= x_1^5x_2^7x_3^3x_4^5x_5^{10} + x_1^5x_2^7x_3^3x_4^6x_5^9 + x_1^5x_2^7x_3^3x_4^9x_5^6 + x_1^5x_2^7x_3^3x_4^{10}x_5^5\\ &+ x_1^5x_2^7x_3^6x_4^3x_5^9 + x_1^5x_2^7x_3^{10}x_4^3x_5^5 + x_1^5x_2^{11}x_3^3x_4^5x_5^6 + x_1^5x_2^{11}x_3^3x_4^6x_5^5\\ &+ x_1^5x_2^{11}x_3^6x_4^3x_5^5 + x_1^7x_2^7x_3^3x_4^5x_5^8 + x_1^7x_2^7x_3^3x_4^8x_5^5 + x_1^7x_2^7x_3^8x_4^3x_5^5\\ &+ x_1^7x_2^9x_3^3x_4^5x_5^6 + x_1^7x_2^9x_3^3x_4^6x_5^5 + Sq^1\big(x_1^7x_2^7x_3^5x_4^5x_5^5\big) + Sq^2\big(x_1^7x_2^7x_3^6x_4^3x_5^5\\ &+ x_1^7x_2^7x_3^3x_4^5x_5^6 + x_1^7x_2^7x_3^3x_4^6x_5^5\big) + Sq^4\big(x_1^5x_2^7x_3^6x_4^3x_5^5 + x_1^5x_2^7x_3^3x_4^5x_5^6\\ &+ x_1^5x_2^7x_3^3x_4^6x_5^5\big) \bmod(P_5^-(4,3,3)).\end{aligned}$$

Hence, $x_1^3x_2^{12}x_3^3x_4^5x_5^7$ and $x_1^7x_2^9x_3^6x_4^3x_5^5$ are strictly inadmissible.

The other cases can be proved by a similar computation. □

Proof of Theorem 3.3.7. First we prove that $QP_5^+(4,3,3,1)$ is generated by the set $[\mathcal{C}\cup\mathcal{D}\cup\{s_t : 1 \leqslant t \leqslant 52\}]$. Let x be an admissible monomial in P_5^+ such that $\omega(x) = (4,3,3,1)$. Then $x = X_iy^2$ with $1 \leqslant i \leqslant 5$ and y a monomial in P_5 and $\omega(y) = (3,3,1)$. Since x is admissible, by Theorem 2.7, $y \in B_5(3,3,1)$.

Let $z \in B_5(3,3,1)$ such that $X_iz^2 \in P_5^+$. By a direct computation, we see that if $X_iz^2 \notin \mathcal{E} := \mathcal{C}\cup\mathcal{D}\cup\{s_t : 1 \leqslant t \leqslant 52\}$, then there is a strictly inadmissible monomial w which is given in one of Lemmas 3.2.2, 3.3.8 such that $X_iz^2 = wz_1^{2^u}$ with suitable monomial $z_1 \in P_5$, and $u = \max\{j \in \mathbb{Z} : \omega_j(w) > 0\}$. By Theorem 2.7, X_iz^2 is inadmissible. Since $x = X_iy^2$ with $y \in B_5(3,3,1)$ and x admissible, $x \in \mathcal{E}$. Hence, $QP_5^+(4,3,3,1)$ is spanned by the set $[\mathcal{E}]$.

Set $\mathcal{U} = \langle[\mathcal{C}]\rangle$, $\mathcal{V} = \langle[\mathcal{D}\cup\{s_t : 1 \leqslant t \leqslant 52\}]$. Since $\nu(z) = 15$ for all $z \in \mathcal{C}$ and $\nu(z) < 15$ for all $z \in \mathcal{D}$, we obtain $\mathcal{U}\cap\mathcal{V} = \{0\}$, hence $QP_5^+(4,3,3,1) = \mathcal{U}\oplus\mathcal{V}$. From Proposition 2.14 we have $\dim\mathcal{U} = 180$. By a direct computation analogous to the one in the proof of Theorem 3.3.3 we can prove that the set $[\mathcal{D}\cup\{s_t : 1 \leqslant t \leqslant 52\}]$ is linearly independent in $QP_5(4,3,3,1)$. So, the theorem is proved. □

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