

A NOTE ON GENERALIZED EGOROV'S THEOREM

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Abstract

We prove that the following generalized version of Egorov's theorem is independent from the ZFC axioms of the set theory.

Let $\{f_n\}_{n \in \omega}$, $f_n : \langle 0, 1 \rangle \rightarrow R$, be a sequence of functions (not necessarily measurable) converging pointwise to zero for every $x \in \langle 0, 1 \rangle$. Then for every $\varepsilon > 0$, there are a set $A \subset \langle 0, 1 \rangle$ of Lebesgue outer measure $m^* > 1 - \varepsilon$ and a sequence of integers $\{n_k\}_{k \in \omega}$ with $\{f_{n_k}\}_{k \in \omega}$ converging uniformly on A .

The following question was asked by F. Di Biase in connection with some problem related to the behaviour of bounded harmonic functions on the open unit disc in R^2 (see [3]):

Suppose that $\{f_n\}_{n \in \omega}$, $f_n : \langle 0, 1 \rangle \rightarrow R$, is a sequence of functions converging pointwise to zero for each $x \in \langle 0, 1 \rangle$. Is it true that for every $\varepsilon > 0$, there are a set $A \subset \langle 0, 1 \rangle$ of outer measure $m^ > 1 - \varepsilon$ and a sequence $\{n_k\}_{k \in \omega}$ such that $\{f_{n_k}\}_{k \in \omega}$ converges uniformly on A ?*

Notice that, by the well-known Egorov's theorem, the answer is positive, if we assume that $\{f_n\}_{n \in \omega}$ is a sequence of measurable functions.

This paper is a slightly renewed version of the older article by the author (see [7]) which was accepted for the publication in the East-West Journal of Mathematics in 2004, but it has not appeared yet. Since [7] receives a lot of citations (see [2]–[6]), the author has decided to publish its original 2003 version and to enlarge it by a list of some later articles related to Egorov's theorem in which the subject is studied more thoroughly.

Key words: Egorov's Theorem, generalized Egorov's statement, infinite combinatorics on ω

2000 AMS Mathematics classification: Primary: 26A03, 28A20, 40A35, 03E35, 03E20; Secondary: 03E17, 40A30.

In this paper, we show that Di Base's conjecture is independent from the ZFC axioms. Throughout the proof we assume that the reader is familiar with basic facts from set theory and forcing.

Theorem 1. *In the Laver real model M of ZFC, Di Base's conjecture holds.*

Proof. Let M be a model obtained by an $\aleph_2$ -iteration with countable supports of Laver forcing over a countable standard model M_0 of ZFC and the continuum hypothesis (CH) (see [1] for details). Suppose that $\{f_n\}_{n \in \omega}$, with $f_n : 2^\omega \rightarrow R$, is a sequence of functions converging pointwise to zero. For $x \in 2^\omega$, define an increasing functions $g_x \in \mathcal{N}$ (Baire space) as follows.

$$g_x(n) = \text{the last } m \text{ such that } \forall k \geq m |f_k(x)| < \frac{1}{n}.$$

Let $G = \{g_x : x \in 2^\omega \cap M_0\}$. Clearly, $|G| \leq \aleph_1$. Thus there is an intermediate model M_α , where $\alpha < \aleph_2$, such that $G \subseteq M_\alpha$. Let $g \in \mathcal{N} \cap M_{\alpha+1}$ be a dominating function added by Laver forcing, that is, $\forall h \in \mathcal{N} \cap M_\alpha \ h \leq_* g$, where $h \leq_* g$ iff $\exists m \forall n \geq m \ h(n) \leq g(n)$.

For $n \in \omega$, let $D_n = \{x \in 2^\omega \cap M_0 : \forall m \geq n \ g_x(m) \leq g(m)\}$. Clearly, $\bigcup_{n \in \omega} D_n = 2^\omega \cap M_0$. Put $D'_n = D_0 \cup \dots \cup D_n$.

By the factor lemma (see [1]), we may assume without loss of generality that each D'_n belongs to M_0 . Since the outer measure m^* of any set $A \in M_0$, calculated in M_0 , is the same as the outer measure m^* of A calculated in M (see [1]), we have that for some $n_0 \in \omega$, $m^*(D'_{n_0}) > 1 - \varepsilon$.

Clearly,

$$\forall n \geq n_0 \forall m \geq g(n) \forall x \in D'_{n_0} \ |f_m(x)| < \frac{1}{n}.$$

Thus $\{f_n\}_{n \in \omega}$ converges uniformly to zero on D'_{n_0} .

To see that this finishes the proof of Theorem 1, notice that the standard surjective function $f : 2^\omega \rightarrow \langle 0, 1 \rangle$, where $f(x) = \sum_{i \in \omega} \frac{f(i)}{2^{i+1}}$, preserves measure. $\square$

Theorem 2. *Assume that the continuum hypothesis (CH) holds. Then there is a sequence $\{f_n\}_{n \in \omega}$, $f_n : \langle 0, 1 \rangle \rightarrow R$, converging pointwise to zero, such that there are no set $A \subset \langle 0, 1 \rangle$ of positive outer measure and no sequence $\{n_k\}_{k \in \omega}$, so that $\{f_{n_k}\}_{k \in \omega}$ converges uniformly on A*

Proof. Let $\{x_\alpha\}_{\alpha < \mathfrak{c}}$ be a bijective enumeration of $\langle 0, 1 \rangle$. We define a sequence $\{f_n\}_{n \in \omega}$, $f_n : \langle 0, 1 \rangle \rightarrow R$, converging to zero, by constructing values $f_n(x_\alpha)$, $n \in \omega$, for each real number x_α . To do this, we apply the following easy lemma.

Lemma 3. *Suppose that $F \subseteq \mathcal{N}$, $G \subseteq [\omega]^\omega$ (the set of all infinite subsets of ω) are such that $|F| \leq \aleph_0$, $|G| \leq \aleph_0$. Then there is an increasing function h with the property*

- a) $\forall f \in F \ f \leq_* h$,
- b) $\forall g \in G \ g \cap \text{range}(h)$ is finite.

Proof of Lemma 3. Let $\{a_n\}_{n \in \omega}$ be a partition of ω into infinite disjoint subsets. Suppose that $F = \{f_0, f_1, \dots\}$ and $G = \{g_0, g_1, \dots\}$. If $n \in \omega$ is such that $n \in a_k$, define $h(n)$ to be equal to the least m , so that

- a) $m \geq \max\{f_0(n), \dots, f_n(n)\}$,
- b) $m \in g_k$ and $m > h(n-1)$.

□

By Lemma 3 and by CH, we may assume that there exists a sequence $\{h_\alpha\}_{\alpha < \mathfrak{c}} \subseteq \mathcal{N}$ of increasing functions satisfying the following conditions.

- a) $\forall \alpha < \beta < \mathfrak{c} \ h_\alpha \leq_* h_\beta$,
- b) $\forall g \in [\omega]^\omega \forall f \in \mathcal{N} \exists \alpha \forall \beta > \alpha \ f \leq_* h_\beta$ and $g \cap \text{range}(h_\beta)$ is infinite.

Let x_{α_0} be a fixed real number. We define $f_n(x_{\alpha_0}) = \frac{1}{n-1}$ iff $h_{\alpha_0}(n) = m$, $n \geq 2$, and we put zero otherwise. Clearly, $\{f_n(x_{\alpha_0})\}_{n \in \omega}$ converges to zero. We apply the same procedure to define $\{f_n(x)\}_{n \in \omega}$, for every $x \in \langle 0, 1 \rangle$.

Now assume that there exist $A \subseteq \langle 0, 1 \rangle$, $m^*(A) > 0$, $\{n_k\}_{k \in \omega}$, so that $\{f_{n_k}\}_{k \in \omega}$ converges uniformly on A .

Then

$$\forall n \in \omega \exists n_{k(n)} \forall n_k \geq n_{k(n)} \forall x \in A \ f_{n_k}(x) < \frac{1}{n}. \quad (*)$$

Suppose that h enumerates bijectively the sequence $\{n_{k(n)}\}_{n \in \omega}$. Let α be such that $x_\alpha \in A$, $h \leq_* h_\alpha$ and $\{n_k\}_{k \in \omega} \cap \text{range}(h_\alpha)$ is infinite. Then clearly $\{f_{n_k}(x_\alpha)\}_{k \in \omega}$ does not satisfy $(*)$, which is a contradiction. □

To finish the paper, we show that in Theorem 2, the continuum hypothesis can be replaced by some lighter axioms.

Let $\mathfrak{b}$ denote the cardinality of the smallest unbounded family in the sense of $\leq_*$ relation, that is,

$$\mathfrak{b} = \min\{|F| : \forall g \in \omega^\omega \exists f \in F \exists n^\infty \ f(n) \geq g(n)\}.$$

Assume that $\mathcal{M}$ is the ideal of meager subsets of R . We define

$$\begin{aligned} \text{add}(\mathcal{M}) &= \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M} \text{ and } \bigcup \mathcal{A} \notin \mathcal{M}\right\} \text{ and} \\ \text{cov}(\mathcal{M}) &= \min\left\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{M} \text{ and } \bigcup \mathcal{B} \neq R\right\}. \end{aligned}$$

Fact 4 (Miller, Truss). $\text{add}(\mathcal{M}) = \min\{\text{cov}(\mathcal{M}), \mathfrak{b}\}$.

Proof. See [1]. □

Fact 5 (Bartoszyński, Miller). *For any cardinal κ the following are equivalent.*

1. $\text{cov}(\mathcal{M}) \geq \kappa$,
2. $\forall F \in [\omega^\omega]^{<\kappa} \exists g \in \omega \forall f \in F \exists n^\infty f(n) = g(n)$.
3. $\forall F \in [\omega^\omega]^{<\kappa} \forall G \in ([\omega]^\omega)^{<\kappa} \exists g \in \omega^\omega \forall f \in F \forall X \in G \exists n^\infty \in X \ f(n) = g(n)$.

Proof. See [1]. □

Lemma 6. *Assume that $\text{add}(\mathcal{M}) = \mathfrak{c}$ holds. Then for each $F \subseteq \mathcal{N}$ and $G \subseteq [\omega]^\omega$, with $|F| < \mathfrak{c}$, $|G| < \mathfrak{c}$, there is $h \in \mathcal{N}$ such that*

- a) $\forall f \in F \ f \leq_* h$,
- b) $\forall g \in G \ g \cap \text{range}(h)$ *is infinite.*

Proof. Define a sequence $\{I_n\}_{n \in \omega}$ of finite disjoint intervals in ω as follows.

$I_n = [\bar{f}(n), \bar{f}(n+1))$, for $n \in \omega$, where $\bar{f}$ is any increasing function with $\forall f \in F \ f \leq_* \bar{f}$.

For $g \in G$, let $h_g \in \mathcal{N}$ be such that $h_g(n) = \min\{m : m \in I_n \text{ and } m \in g\}$, or $h_g(n) = 0$ if $g \cap I_n = \emptyset$, $n \in \omega$.

Notice that we may assume without loss of generality that for every $n \in \omega$ and every h_g , $g \in G$, $h_g(n) \in \{s \in 2^{I_n} : |\{n : s(n) = 1\}| \leq 1\}$. Let X_g be equal to $\{n : h_g(n) \neq 0\}$, for $g \in G$, and $G' = \{X_g : g \in G\}$.

By Fact 5, there is $\bar{h} \in \mathcal{N}$ such that

$$\forall g \in G \forall X \in G' \exists n^\infty n \in X \text{ and } h_g(n) = \bar{h}(n).$$

Put $h(n) = m$, if $\bar{h}(n)(m) = 1$, and define $h(n) = k$, where k is any element of I_n , if $\bar{h}(n) = 0$. □

Theorem 7. *Assume that $\text{add}(\mathcal{M}) = \mathfrak{c}$ holds. Then there is a sequence $\{f_n\}_{n \in \omega}$, $f_n : \langle 0, 1 \rangle \rightarrow \mathbb{R}$, converging pointwise to zero such that for any set $A \subseteq \langle 0, 1 \rangle$ of positive outer measure, and every sequence $\{n_k\}_{k \in \omega}$, $\{f_{n_k}\}_{k \in \omega}$ does not converge uniformly on A .*

Proof. Use Lemma 6 and the same construction as in the proof of Theorem 2. Notice that $\text{add}(\mathcal{M}) = \mathfrak{c}$ implies that every subset of $\langle 0, 1 \rangle$ of cardinality smaller than $\mathfrak{c}$ has measure zero. □

Remark 8. It is easy to see that to prove Theorem 1 we can apply the following property which holds in the iterated Laver real model (see [1]): the cardinality of the smallest subset of 2^ω of full outer measure is smaller than $\mathfrak{b}$. This fact was known also to I. Reclaw.

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