East-West J. of Mathematics: Vol. 23, No 1 (2021) pp. 51-56
https://doi.org/10.36853/ewjm0384

SOME RESULTS RELATED TO JACOBSON CONJECTURE

Bui Phuoc Minh, and **Nguyen van Sanh***

Department of Mathematics, Faculty of Science
Mahidol University, Bangkok 10400, Thailand
e-mail: nguyen.san@mahidol.ac.th; buiphuocminh94@gmail.com

Abstract

In this paper, we study the endomorphism rings of bounded and fully bounded modules and extended some results related to Jacobson's Conjecture.

Introduction and Preliminaries

The Jacobson radical of R, denoted by $J(R)$, is defined to be the intersection of all maximal ideals of R. The equality $\cap_{k=1}^{\infty} J^k(R) = 0$ is well-known as Jacobson conjecture, which had been appeared in ([8], pp200). This had been proved for commutative rings by W. Krull. In 1965 I. N. Herstein showed that it is false if the ring is right but not left Noetherian. The Jacobson conjecture is not true by a result of Jategaonkar in 1974, following this, there is a right Noetherian right serial ring with $\cap_{k=1}^{\infty} J^k(R) \neq 0$.

Throughout this paper, all rings are associative with identity, and all modules are unitary right R-modules. We write M_R (resp. ${}_RM$) to indicate that M is a right (resp. left) R-modules. We also write $J(R)$ (resp. $rad(M)$) for the Jacobson radical of R (resp. Jacobson radical of M_R) and $S = End(M_R)$, its endomorphism ring. A submodule X of M is called a *fully invariant* submodule of M if for any $f \in S$, we have $f(X) \subset X$. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R. Following [17], a

*Corresponding author
Key words: bounded modules, fully bounded module, Jacobson conjecture.
2010 AMS Mathematics Classification: 16D50, 16D70, 16D80.

fully invariant proper submodule X of M is called *prime submodule* of M if for any ideal I of S and any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. In particular, an ideal P of R is a prime ideal if for any ideals I, J of R, if $IJ \subset P$, then either $I \subset P$ or $J \subset P$. A non-zero submodule U of M is called *essential* in M if U has non-zero intersection with any non-zero submodule of M. A right R-module M is called a *self-generator* if it generates all its submodules. General background materials can be found in [1], [3], [4], [5], [6], [10], [11], [13], [19], [22].

Following [16], a right R-module M is called a *bounded module* if every essential submodule contains a fully invariant submodule which is essential in M as a submodule. A ring R is a right bounded if every essential right ideal of R contains an ideal which is essential as a right ideal.

Clearly, every commutative ring is right bounded. A simple Artinian ring has no proper essential right ideals and hence is right bounded.

A right R-module M_R is fully bounded if for every prime submodule X of M, the factor module M/X is a bounded module. A ring R is right fully bounded if for every prime ideal I of R, the factor ring R/I is a right bounded ring.

Theorem 1 (Cauchon [2], 1974) *Let $J(R)$ be the Jacobson radical of a left and right fully bounded ring R. Then $\cap_{k=1}^{\infty} J^k(R) = 0$.*

Theorem 2 ([20], Theorem 1) *Let R be a ring. If R is a left Noetherian, right distributive, then $\cap_{k=1}^{\infty} J^k(R) = 0$.*

Theorem 3([20]) *If R is a right Noetherian, left distributive, then $\cap_{k=1}^{\infty} J^k(R) = 0$.*

We can consider M as a left S-module. It easy to see that if M is quasi-projective and ${}_SM$ is a Noetherian module, then S is a left Noetherian ring. From this we have a following theorems.

Theorem 4 *Let M be quasi-projective, finitely generated right R-module which is a self-generator. If ${}_SM$ and M_R are Noetherian, and M_R is a fully bounded module, then $\cap_{k=1}^{\infty} J^k(S) = 0$.*

Proof. Since ${}_SM$ and M_R are Noetherian, we see that S is a Noetherian ring. By [16, Theorem 2.16], S is a right fully bounded ring. From Theorem 1, it follows that $\cap_{k=1}^{\infty} J^k(S) = 0$. □

Following [15], if M is a self-generator, then we have $I_{J(M)} = J(S)$. Then the following corollary is a direct consequence of the above theorem.

Corollary 5 *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If ${}_SM$ and M_R are Noetherian, and M_R is a fully* *bounded module, then* $\cap_{k=1}^{\infty} J^k(M) = 0$*, where* $J = J(S)$.

Let M be a right R-module. The module M is called a *uniserial module* if the lattice of its submodules is linearly ordered by inclusion. A ring R is called a right *uniserial ring* if R_R is a uniserial module. A module M is called a *serial module* if it is a direct sum of uniserial modules. In particular, A ring R is called a right serial ring if R_R is a serial module. For the following two propositions, we refer to [22].

Proposition 6 ([22, 55.1]) *Let* M *be a quasi-projective* R*-module. If* M_R *is a uniserial module then* $S = End(M)$ *is a right serial ring.*

Proposition 7 ([22], 55.2) *Let* M *be a quasi-projective, finitely generated* R*-module. If* M_R *is a serial module, then* S *is a right serial ring.*

In ([3], Theorem 6.7) we have a result related to Jacobson conjecture.

Proposition 8 ([3], Theorem 6.7) *Let* R *be a left and right Noetherian right serial ring with Jacobson radical* $J(R)$*. Then* $\bigcap_{k=1}^{\infty} J^k(R) = 0$

Motivated this result we can prove the following theorem.

Theorem 9 *Let* M_R *be a quasi-projective, finitely generated* R*-module. If* ${}_SM$ *and* M_R *are Noetherian module and* M_R *is a serial module, then* $\bigcap_{k=1}^{\infty} J^k(S) = 0$*, where* $J(S)$ *is the Jacobson radical of* S*.*

Proof. Since ${}_SM$ and M_R are Noetherian, we see that S is a Noetherian ring. If M_R is a serial module, then S is a right serial ring, by Proposition 7. From Proposition 8, it follows that $\bigcap_{k=1}^{\infty} J^k(S) = 0$. □

Corollary 10 *Let* M_R *be a quasi-projective, finitely generated* R*-module which is a self-generator. If* ${}_SM$ *and* M_R *are Noetherian module and* M_R *is a serial module, then* $\bigcap_{k=1}^{\infty} J^k(M) = 0$*.*

A right R-module M is said to be *distributive* if the lattice of its submodules is *distributive*, i.e., $F \cap (G + H) = F \cap G + F \cap H$ for all submodules F, G and H of the module M. A ring R is *a right distributive ring* if R_R is a distributive module. Next we study some properties of distributive modules.

Proposition 11 *Let* M *be a quasi-projective, finitely generated right* R*-module which is a self-generator. If* M_R *is a distributive module, then* S *is a right distributive ring.*

Proof. Let I be a right ideal of S. By [22,18.4], $I = Hom(M, I(M))$. Put $X = I(M)$, then $I = I_X = Hom(M, I(M)) = \{f \in S | f(M) \subset X\}$. Let I_1 and I_2 be right ideal of S. It follows that $I_1 = I_X$ and $I_2 = I_Y$ for $X = I_1(M)$, $Y = I_2(M)$. Hence $I_{X \cap Y} = I_X \cap I_Y$. Since M is a self-generator, $I_X(M) = X$

for any $X \subset_{>} M$ and from [22, 18.4], it follows that, $I_{X+Y}(M) = (I_X + I_Y)(M)$ and $I_X \cap (I_Y + I_Z) = I_X \cap I_{Y+Z} = I_{X\cap(Y+Z)} = I_{(X\cap Y)+(X\cap Z)} = I_{(X\cap Y)} + I_{(X\cap Z)} = (I_X \cap I_Y) + (I_X \cap I_Z)$. Hence S is a right distributive ring as required. □

Any direct sum of distributive modules is called a *semidistributive* module. Next we will show that if M is a quasi-projective, finitely generated R-module which is a semidistributive, then S is a right semidistributive ring by following theorem.

Theorem 11 *Let M_R be a quasi-projective, finitely generated R-module. If M_R is a semidistributive module, then S is a right semidistributive ring.*

Proof. Suppose that M_1, M_2 are distributive submodules of M. We may without loss of generality assume that $M = M_1 \oplus M_2$. Let p_1 be the natural homomorphism from M to M_1. We will prove that Sp_1 is a distributive module of S_S. Suppose that Ip_1, Jp_1, Kp_1 are ideals of Sp_1. Then $Ip_1 \cap (Jp_1 + Kp_1) = Hom(M; (Ip_1 \cap Ip_2 + Ip_1 \cap Kp_1)(M)) = Ip_1 \cap Jp_1 + Ip_1 \cap Kp_1$. Hence Sp_1 is a distributive module of S_S. Since $S = Sp_1 + Sp_2$, we have S is a right semidistributive ring, proving the theorem. □

Theorem 12 ([21, 3.107]) *Let R be a Noetherian right semidistributive ring. Then $\bigcap_{k=1}^{\infty} J^k(R) = 0$.*

Theorem 13 *Let M be a quasi-projective, finitely generated module. If ${}_SM$ and M_R are Noetherian and M_R is a semidistributive module. Then $\bigcap_{k=1}^{\infty} J^k(S) = 0$.*

Proof. Since ${}_SM$ and M_R are Noetherian, we have S is a Noetherian ring. Since M_R is a semidistributive module, we can see that S is a right semidistributive ring, by Theorem 12. It follows from Theorem 3.13, $\bigcap_{k=1}^{\infty} J^k(S) = 0$. □

As a consequence, we immediately get the following result for the semidistributive modules.

Corollary 14 *Let M be a quasi-projective, finitely generated right R-module which is self-generator. If ${}_SM$ and M_R are Noetherian, M_R is a semidistributive module. Then $\bigcap_{k=1}^{\infty} J^k(M) = 0$.*

References

[1] Anderson F.W., Fuller K.R., *Rings and Categories of Modules.* Graduate Texts in Mathematics. No.13, Springer-Verlag, New York - Heidelberg - Berlin, 1974.

[2] Cauchon, *Sur l'intersection des puissance du radical d'un T-anneau Noetherian.* C. R. Acad. Sci. Paris,t.279(1974), Series A, 91-93.

[3] Chatters A.W. and Hajanavis C.R., *Rings with chain conditions.* Pitman advanced Publishing Program, 1980.

[4] Faith C., *Algebra I : Rings, Modules and Categories.* Springer-Verlag, 1980.

[5] Faith C., *Algebra II. Ring Theory,* Springer-Verlag, 1976.

[6] Goodearl K.R., Warfield R.B Jr. *An Introduction to Noncommutative Noetherian Rings,* (London Mathematical Society Student Texts No. 16), Cambridge University Press, 1989.

[7] Herstein I.N., *A Counter-example in Noetherian rings* Proc. N. A. S., 54(1965), 1036-1037.

[8] Jacobson N., *The Structure of Rings.* American Mathematics Society, Providence, 1956.

[9] Jategaonkar A.V., *Jacobson's conjecture and modules over fully bounded Noetherian rings,* J. of Algebra, 30(1974), 103-121.

[10] Kasch F. *Modules and Rings.* Academic Press, London, New York, 1982.

[11] Lam T.Y., *Lectures on Modules and Rings.* Graduate Texts in Mathematics, No.189, Springer-Verlag, 1998.

[12] Lenagan T.H., *Noetherian Rings with Krull Dimension One.* J. London Math. Soc., **2**(15)(1977), 41-47.

[13] McConnell J.C. and Robson J.C., *Noncommutative Noetherian Rings,* Graduate Studies in Mathematics 30, Amer. Math. Soc., Providence, Rhode Island, 2001.

[14] Sanh N.V., Asawasamrit S., Ahmed K.F.U. and Thao L.P., *On prime and semiprime Goldie modules.*, Asian-European Journal of Mathematics, 3(2)(2011), 321-334.

[15] Sanh N.V. and Thao L.P., *A Generalization of Hopkins-Levitzki Theorem,* Southeast Asian Bull. Math. 37 (2013), no. 4, 591-600.

[16] N.V. Sanh, O. Arunphalungsanti, N. T. Bac and S. Chairat, *On fully bounded Noetherian modules and their endomorphism rings,* East-West J. Math, **22**(1)(2020), 95-101.

[17] Sanh N.V., Vu N.A., Ahmed K.F.U., Asawasamrit S. and Thao L.P., *Primeness in module category,* Asian-European Journal of Mathematics, 3(1)(2010), 145-154.

[18] N. V. Sanh and S. Sanpinij, *On Serial Artinian Modules and Their Endomorphism Rings,* Southeast Asian Bull. Math. 37 (2013), no. 3, 401-404.

[19] Stenstrom Bo., *Rings of Quotients.* Springer-Verlag, Berlin Heidelberg-New York, 1975.

[20] Tuganbaev A.A., *A remark on the intersection of powers of the Jacobson radical,* Journal of Mathematical Sciences, 15(2)(2009), 207-209.

[21] Tuganbaev A.A., *Semidistributive modules,* Journal of Mathematical Sciences, 94(6), 1999.

[22] Wisbauer R., *Foundations of Module and Ring Theory,* Gordon and Breach, Tokyo, e.a. 1991.