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ON THE LIE DERIVATIVE OF SYMMETRIC CONNECTIONS

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Abstract

The aim of this work is to study some properties of the normal connection and the Lie derivative of the symmetric connection on Riemannian submanifold M.

1 Introduction

The concept of Lie derivative appeared in the early 30s and was related to the works of Slebodzinski, Dantzig, Schouten and Van Campen ([17]). The Lie differentiation theory plays an important role in studying automorphisms of differential geometric structures. Moreover, the Lie derivative also is an essential tool in the Riemannian geometry. The Lie derivative of forms and its application was investigated by many authors (see [14], [15], [16], [21],[23], [24],[29] and the references given therein). In 2010, Sultanov used the Lie derivative of the linear connection to study the curvature tensor and the sorsion tensor on linear algebras (see [24], pp. 362-412). In 2012, basing on the Lie derivative of real-valued forms on the Riemannian $n-$dimensional manifold, N. H. Quang, K. P. Chi and B. C. Van constructed the Lie derivative of the currents on Riemann manifolds and given some applications on Lie groups (see [5]). In 2015, B. C. Van and T. T. K. Ha studed some properties of the Lie derivative of the linear connection ∇, the conjugate derivative d_{∇} with the linear connection and using them for searching the curvature, the torsion of a space $\mathbb{R}^n$ along the linear flat connection ∇ (see [28]). In 2007, Jeong-Sik Kim, Mohit Kumar Dwivedi and Mukut Mani Tripathi used derivatives on

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the module of normal vector fields to study the Gauss curvature, the Ricci curvature on the Riemannian $k-$dimensional manifold (see [13], pp. 395-406). The primary goal of our work is the extension of the operations of Lie derivative to objects defined on the vector-valued differential forms of a manifold. The main goal of the present work is to investigate some properties on the Lie derivative of the flat connection $\nabla^{\perp}$ and the normal curvature tensor, the normal connection on the submanifold M.

In section 3, we introduce some properties of normal connection on the submanifold M in $\widetilde{M}$ and by using the conjugate derivative with the normal connection for presenting the normal curvature of the submanifold M in $\widetilde{M}$. In section 4, we construct the Lie derivative of a linear connection on the Riemannian manifold M and given some properties of the Lie derivative of the symmetric connection on M.

2 Notation and Preliminaries

Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$ equipped with a Riemannian metric $\widetilde{g}$. We denote the vector space of all smooth vector fields on M and $\widetilde{M}$ by $\mathfrak{B}(M)$ and $\mathfrak{B}(\widetilde{M})$ respectively. We denote $\widetilde{\triangledown}$, $\triangledown$ and $\triangledown^{\perp}$ are the Levi-Civita, induced Levi-Civita induced normal connections in $\widetilde{M}$, M and the normal bundle $\mathfrak{N}(M)$ of M respectively. We use the inner product notation $\langle , \rangle$ (or $\cdot$) for both the metrics $\widetilde{g}$ of $\widetilde{M}$ and the induced metric g on the submanifold M.

At each $p \in M$, the ambient tangent space $T_p\widetilde{M}$ splits as an orthogonal direct sum $T_p\widetilde{M} = T_pM \bigoplus N_pM$, where $N_pM := (T_pM)^{\perp}$ is the normal space at p with respect to the inner product $\widetilde{g}$ on $T_p\widetilde{M}$. The set $\mathfrak{N}(M) = \bigcup\limits_{p\in M} N_pM$ is called the normal bundle of M. If X, Y are vector fields in $\mathfrak{B}(M)$, we can extend them to vector fields on $\widetilde{M}$, apply the ambient covariant derivative operator $\widetilde{\triangledown}$, and then decompose at points of M to get

$$\widetilde{\triangledown}_X Y = (\widetilde{\triangledown}_X Y)^{\top} + (\widetilde{\triangledown}_X Y)^{\perp} \tag{2.1}$$

The Gauss and Weingarten formulas are given respectively by ([18], pp. 135)

$$\widetilde{\triangledown}_X Y = \triangledown_X Y + \sigma(X, Y) \text{ and } \widetilde{\triangledown}_X N = -A_N X + \triangledown^{\perp}_X N, \tag{2.2}$$

for all X, Y $\in \mathfrak{B}(M)$ and N $\in \mathfrak{N}(M)$, where σ is the second fundamental form of M from $\mathfrak{B}(M) \times \mathfrak{B}(M)$ to $\mathfrak{N}(M)$ given by

$$\sigma(X, Y) := (\widetilde{\triangledown}_X Y)^{\perp}, \tag{2.3}$$

where X and Y are extended arbitrarily to $\widetilde{M}$ and the shape operator A_N : $X \mapsto A_N X$, for all $X \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$.

The Weingarten Equation is given by (see [18], pp. 136)

$$\left\langle \widetilde{\nabla}_X N, Y \right\rangle = -\left\langle N, \sigma(X,Y) \right\rangle . \tag{2.4}$$

Thus, σ is the second fundamental form related to the shape operator A by

$$\langle \sigma(X,Y), N \rangle = \langle A_N X, Y \rangle . \tag{2.5}$$

The equation of Gauss is given by (see [18], pp. 136)

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - \langle \sigma(X,W), \sigma(Y,Z) \rangle + \langle \sigma(X,Z), \sigma(Y,W) \rangle , \tag{2.6}$$

for all X, Y, Z, W $\in \mathfrak{B}(M)$, where $\widetilde{R}$ and R are the Riemann curvature tensors of $\widetilde{M}$ and M respectively. The curvature tensor $R^{\perp}$ of the normal bundle of M is defied by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N, \tag{2.7}$$

for any $X, Y \in \mathfrak{B}(M)$ and $N \in \mathfrak{N}(M)$. If $R^{\perp} = 0$, then the normal connection $\nabla^{\perp}$ of M is said to be *flat*.

The mean curvature vector H is given by $H = \frac{1}{n} trace(\sigma)$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if H = 0.

Let $\{E_1, ..., E_n\}$ and $\{N_{n+1}, ..., N_m\}$ be an orthonormal basis of $\mathfrak{B}(M)$ and $\mathfrak{N}(M)$. The map

$$\begin{aligned} h_j : \mathfrak{B}(M) &\to \mathfrak{B}(M) \\ X &\mapsto h_j(X) = -\left(\widetilde{\nabla}_X N_j\right)^{\top} \end{aligned} \tag{2.8}$$

for any $j = n+1, ..., m$ is called the Weingarten mapping partially .

By using Weingarten Equation (2.4) and for any $j = n+1, ..., m$, we have

$$\langle \sigma(E_i, E_i), N_j \rangle = -\left\langle \left(\widetilde{\nabla}_{E_i} N_j\right)^{\top}, E_i \right\rangle . \tag{2.9}$$

Hence

$$\begin{aligned} H_j &= \left\langle \left(\sum_{i=1}^{n} \sigma(E_i, E_j) \right), N_j \right\rangle = \sum_{i=1}^{n} \langle \sigma(E_i, E_j), N_j \rangle \\ &= \sum_{i=1}^{n} \left\langle -\left(\widetilde{\nabla}_{E_i} N_j)\right)^{\top}, E_i \right\rangle = \sum_{i=1}^{n} \langle h_j(E_i), E_i \rangle = Trace(h_j). \end{aligned} \tag{2.10}$$

So that

$$\begin{aligned} H &= \frac{1}{n}Trace(\sigma) = \frac{1}{n}\sum_{i=1}^{n}\sigma(E_i,E_i) \\ &= \frac{1}{n}\sum_{j=n+1}^{m}\langle H_j, N_j\rangle = \frac{1}{n}\sum_{j=n+1}^{m}Trace(h_j).N_j, \end{aligned} \tag{2.11}$$

where $H=(H_1,...,H_n)$ is the mean curvature vector of M at p.

Let $N \in \mathfrak{N}(M)$, the Weingarten map $h_N : \mathfrak{B}(M) \to \mathfrak{B}(M)$ give by

$$h_N(X) = -\left(\widetilde{\nabla}_X N\right)^{\top}, \tag{2.12}$$

for all $X \in \mathfrak{B}(M)$. We easily get the following properties of the Weingarten mapping h_N

$$h_N\left(X+Y\right) = h_N\left(X\right) + h_N\left(Y\right), h_N\left(\varphi X\right) = \varphi h_N\left(X\right), \tag{2.13}$$

and

$$h_N\left(\varphi X\right).Y = \varphi h_N\left(Y\right).X, \tag{2.14}$$

for all $X, Y \in \mathfrak{B}(M), \varphi \in \mathfrak{F}(M)$. Suppose that $N=\sum_{j=n+1}^{m}\varphi_j N_j$ be a normal vector field. Then the mean curvature of M give by the following formula

$$\begin{aligned} \frac{1}{n}Trace(h_N) &= \frac{1}{n}\sum_{i=1}^{n}\langle h_N(E_i), E_i\rangle = \frac{1}{n}\sum_{i=1}^{n}\left\langle -(\widetilde{\nabla}_{E_i}N)^{\top}, E_i\right\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\left\langle -\left(\widetilde{\nabla}_{E_i}(\sum_{j=n+1}^{m}\varphi_j N_j)\right)^{\top}, E_i\right\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=n+1}^{m}\left\langle -\left(\widetilde{\nabla}_{E_i}(\varphi_j N_j)\right)^{\top}, E_i\right\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=n+1}^{m}\langle\sigma(E_i,E_i), \varphi_j N_j\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\left\langle \sigma(E_i,E_i), \sum_{j=n+1}^{m}\varphi_j N_j\right\rangle \\ &= \frac{1}{n}\sum_{i=1}^{n}\langle\sigma(E_i,E_i), N\rangle = \langle H, N\rangle = H\cdot N. \end{aligned} \tag{2.15}$$

Now, we define the Weingarten normal mapping and the derivative of the map

$h_N^\perp$ along a vector field X.

$$\begin{aligned} h_N^\perp : \mathfrak{B}(M) &\to \mathfrak{N}(M) \\ X &\mapsto h_N^\perp(X) = \triangledown_X^\perp N \end{aligned} \tag{2.16}$$

is called the Weingarten normal mapping. We get the following properties (2.17), (2.18) of Weingarten normal mapping $h_N^\perp$

$$h_N^\perp(X+Y) = h_N^\perp(X) + h_N^\perp(Y), \ \forall X, Y \in \mathfrak{B}(M). \tag{2.17}$$

$$h_N^\perp(\varphi X) = \varphi h_N^\perp(X), \ \forall X \in \mathfrak{B}(M), \ \forall \varphi \in \mathfrak{F}(M). \tag{2.18}$$

Let $N, K \in \mathfrak{N}(M)$ and $\varphi \in \mathfrak{F}(M)$, we obtain

$$h_{N+K}^\perp = h_N^\perp + h_K^\perp, \text{ and } h_{\varphi N}^\perp = \varphi h_N^\perp. \tag{2.19}$$

Next, the derivative of the mapping $h_N^\perp$ along a vector field X is the mapping

$$\begin{aligned} \triangledown_X h_N^\perp : \mathfrak{B}(M) &\to \mathfrak{N}(M) \\ Y &\mapsto (\triangledown_X h_N^\perp)(Y) = \triangledown_X^\perp(h_N^\perp(Y)) - h_N^\perp(\triangledown_X Y) \end{aligned} \tag{2.20}$$

We easily get the mapping $h_N^\perp$ and $\triangledown_X h_N^\perp$ are modular homomorphics. Indeed, for all X, Y, Z $\in \mathfrak{B}(M), \varphi \in \mathfrak{F}(M)$, we have

$$(\triangledown_X h_N^\perp)(Y+Z) = (\triangledown_X h_N^\perp)(Y) + (\triangledown_X h_N^\perp)(Z), \tag{2.21}$$

and

$$(\triangledown_X h_N^\perp)(\varphi \cdot Y) = \varphi \cdot (\triangledown_X h_N^\perp)(Y). \tag{2.22}$$

Since $\triangledown, \triangledown^\perp, h_N^\perp$ are modular homomorphics, thus for all $X, Y, Z \in \mathfrak{B}(M)$, we have

$$\begin{aligned} (\triangledown_{X+Y} h_N^\perp)(Z) &= \triangledown_{X+Y}^\perp(h_N^\perp(Z)) - h_N^\perp(\triangledown_{X+Y}(Z)) \\ &= \triangledown_X^\perp(h_N^\perp(Z)) + \triangledown_Y^\perp(h_N^\perp(Z)) - h_N^\perp(\triangledown_X(Z) + \triangledown_Y(Z)) \\ &= \triangledown_X^\perp(h_N^\perp(Z)) + \triangledown_Y^\perp(h_N^\perp(Z)) - h_N^\perp(\triangledown_X(Z)) - h_N^\perp(\triangledown_Y(Z)) \\ &= \left[\triangledown_X^\perp(h_N^\perp(Z)) - h_N^\perp(\triangledown_X(Z))\right] + \left[\triangledown_Y^\perp(h_N^\perp(Z)) - h_N^\perp(\triangledown_Y(Z))\right] \\ &= (\triangledown_X h_N^\perp)(Z) + (\triangledown_Y h_N^\perp)(Z) \\ &= (\triangledown_X h_N^\perp + \triangledown_Y h_N^\perp)(Z), \ \forall Z \in \mathfrak{B}(M). \end{aligned} \tag{2.23}$$

Thus,

$$\triangledown_{X+Y} h_N^\perp = \triangledown_X h_N^\perp + \triangledown_Y h_N^\perp. \tag{2.24}$$

Let $X, Y \in \mathfrak{B}(M)$ and $\varphi \in \mathfrak{F}(M)$, we have

$$\begin{aligned}(\nabla_{\varphi X} h_N^{\perp})(Y) &= \nabla_{\varphi X}^{\perp}(h_N^{\perp}(Y)) - h_N^{\perp}(\nabla_{\varphi X}(Y)) \\ &= \varphi \nabla_X^{\perp} (h_N^{\perp}(Y)) - h_N^{\perp}(\varphi \nabla_X (Y)) \\ &= \varphi \nabla_X^{\perp} (h_N^{\perp}(Y)) - \varphi h_N^{\perp}(\nabla_X(Y)) \\ &= \varphi \left[\nabla_X^{\perp}(h_N^{\perp}(Y)) - h_N^{\perp}(\nabla_X(Y))\right] \\ &= (\varphi \nabla_X h_N^{\perp})(Y)\ \forall Y \in \mathfrak{B}(M).\end{aligned} \tag{2.25}$$

Hence,

$$\nabla_{\varphi X} h_N^{\perp} = \varphi \nabla_X h_N^{\perp}. \tag{2.26}$$

Let $N, K \in \mathfrak{N}(M)$, for any $X, Y \in \mathfrak{B}(M)$, and using Equation (2.19) we have

$$\begin{aligned}(\nabla_X h_{N+K}^{\perp})(Y) &= \nabla_X^{\perp}(h_{N+K}^{\perp}(Y)) - h_{N+K}^{\perp}(\nabla_X(Y)) \\ &= \nabla_X^{\perp}(h_N^{\perp}(Y) + h_K^{\perp}(Y)) - h_N^{\perp}(\nabla_X(Y)) - h_K^{\perp}(\nabla_X(Y)) \\ &= \nabla_X^{\perp}(h_N^{\perp}(Y)) + \nabla_X^{\perp}(h_K^{\perp}(Y)) - h_N^{\perp}(\nabla_X(Y)) - h_K^{\perp}(\nabla_X(Y)) \\ &= \left[\nabla_X^{\perp}(h_N^{\perp}(Y)) - h_N^{\perp}(\nabla_X(Y))\right] + \left[\nabla_X^{\perp}(h_K^{\perp}(Y)) - h_K^{\perp}(\nabla_X(Y))\right] \\ &= (\nabla_X h_N^{\perp})(Y) + (\nabla_X h_K^{\perp})(Y) \\ &= (\nabla_X h_N^{\perp} + \nabla_X h_K^{\perp})(Y),\ \forall Y \in \mathfrak{B}(M).\end{aligned} \tag{2.27}$$

Hence,

$$\nabla_X h_{N+K}^{\perp} = \nabla_X h_N^{\perp} + \nabla_X h_K^{\perp}. \tag{2.28}$$

Suppose that $X, Y \in \mathfrak{B}(M), \varphi \in \mathfrak{F}(M)$, and using Equation (2.19), we obtain

$$\begin{aligned}(\nabla_X h_{\varphi N}^{\perp})(Y) &= \nabla_X^{\perp}(h_{\varphi N}^{\perp}(Y)) - h_{\varphi N}^{\perp}(\nabla_X(Y)) \\ &= \nabla_X^{\perp}(\varphi h_N^{\perp}(Y)) - \varphi h_N^{\perp}(\nabla_X(Y)) \\ &= X[\varphi].h_N^{\perp}(Y) + \varphi \nabla_X^{\perp} (h_N^{\perp}(Y)) - \varphi h_N^{\perp}(\nabla_X(Y)) \\ &= X[\varphi].h_N^{\perp}(Y) + \varphi \left[\nabla_X^{\perp}(h_N^{\perp}(Y)) - h_N^{\perp}(\nabla_X(Y))\right] \\ &= h_{X[\varphi].N}^{\perp}(Y) + \left(\varphi(\nabla_X h_N^{\perp})\right)(Y) \\ &= \left(h_{X[\varphi].N}^{\perp} + \left(\varphi(\nabla_X h_N^{\perp})\right)\right)(Y),\ \forall Y \in \mathfrak{B}(M).\end{aligned} \tag{2.29}$$

So that

$$\nabla_X h_{\varphi N}^{\perp} = h_{X[\varphi].N}^{\perp} + \varphi \nabla_X h_N^{\perp}. \tag{2.30}$$

3 The normal connection of submanifold

In this section, we introduce some properties of the normal connection on an n-dimensional submanifold M of an m-dimensional Riemannian manifold $\widetilde{M}$ and by using the conjugate derivative with the normal connection for presenting the normal curvature of the submanifold M in $\widetilde{M}$.

Let $X \in \mathfrak{B}(M)$ be a vector field on the submanifold M in $\widetilde{M}$. The normal connection $\nabla^{\perp}_X$ along a vector field X, is defined by the mapping

$$\begin{aligned} \nabla^{\perp}_X : \mathfrak{N}(M) &\to \mathfrak{N}(M) \\ N &\mapsto \nabla^{\perp}_X N \end{aligned}$$

We denote by $K = \left\{\nabla^{\perp}_X \mid X \in \mathfrak{B}(M)\right\}$ the space of the normal connection along a vector field. The operators on K is defined by:

i) $(\nabla^{\perp}_X + \nabla^{\perp}_Y)(N) = \nabla^{\perp}_X(N) + \nabla^{\perp}_Y(N)$, for all $X, Y \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$;

ii) $(\varphi.\nabla^{\perp}_X)(N) = \varphi.(\nabla^{\perp}_X N)$, for all $X \in \mathfrak{B}(M), \varphi \in \mathfrak{F}(M)$;

iii) $\left[\nabla^{\perp}_X, \nabla^{\perp}_Y\right](N) = \nabla^{\perp}_X(\nabla^{\perp}_Y(N)) - \nabla^{\perp}_Y(\nabla^{\perp}_X(N))$, for all $X, Y \in \mathfrak{B}(M)$, for all $N \in \mathfrak{N}(M)$.

By using i), ii) and for any $X, Y \in \mathfrak{B}(M), N, \widetilde{N} \in \mathfrak{N}(M)$, and for all $\varphi \in \mathfrak{F}(M)$, we have

$$\nabla^{\perp}_X + \nabla^{\perp}_Y = \nabla^{\perp}_{X+Y}, \tag{3.1}$$

$$\nabla^{\perp}_{\varphi X} = \varphi.\nabla^{\perp}_X, \tag{3.2}$$

and

$$\nabla^{\perp}_X(N + \widetilde{N}) = \nabla^{\perp}_X N + \nabla^{\perp}_X \widetilde{N}. \tag{3.3}$$

Suppose that $X \in \mathfrak{B}(M), N \in \mathfrak{N}(M), \varphi \in \mathfrak{F}(M)$, we have

$$\begin{aligned} \nabla^{\perp}_X(\varphi.N) &= \left(\widetilde{\nabla}_X(\varphi.N)\right)^{\perp} = \left(X[\varphi].N + \varphi.\widetilde{\nabla}_X N\right)^{\perp} \\ &= X[\varphi].N + \varphi.\nabla^{\perp}_X N, \text{ for any } N \in \mathfrak{N}(M). \end{aligned} \tag{3.4}$$

Hence,

$$\nabla^{\perp}_X(\varphi.N) = X[\varphi].N + \varphi.\nabla^{\perp}_X N. \tag{3.5}$$

From Equations (3.3, 3.5), we easily get $\nabla^{\perp}_X$ the derivative on the module $\mathfrak{N}(M)$.

Theorem 3.1. *i) Suppose that $X, Y \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$. Then we have*

$$R^{\perp}(X, Y, N).N = \left(\left[\nabla^{\perp}_X, \nabla^{\perp}_Y\right](N)\right).N; \tag{3.6}$$

ii) K is a module on $\mathfrak{F}(M)$ with operations i) and ii), and the operation iii) is antisymmetric, satisfying Jacobi's identity.

Proof.

i) Suppose that $X, Y \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$, we have

$$\begin{aligned} & N^2 = 1 \\ & \Rightarrow [X, Y]\left[N^2\right] = 0, \forall X, Y \in \mathfrak{B}(M) \\ & \Rightarrow \left(\nabla^{\perp}_{[X,Y]} N\right).N = 0. \end{aligned} \tag{3.7}$$

Thus, using Equation (3.7), we obtain

$$\begin{aligned} R^{\perp}(X, Y, N).N &= \left(\nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N\right).N \\ &= \left(\left[\nabla^{\perp}_X, \nabla^{\perp}_Y\right](N)\right).N - \left(\nabla^{\perp}_{[X,Y]} N\right).N \\ &= \left(\left[\nabla^{\perp}_X, \nabla^{\perp}_Y\right](N)\right).N. \end{aligned} \tag{3.8}$$

ii) Note that operations i) and ii), K is a module on $\mathfrak{F}(M)$ and the operation iii) is antisymmetric. Now, we prove the operation iii) satisfying Jacobi's identity. Indeed, for every $X, Y, Z \in \mathfrak{B}(M)$, one has

$$\left[\left[\nabla^{\perp}_X, \nabla^{\perp}_Y\right], \nabla^{\perp}_Z\right] = \nabla^{\perp}_X \nabla^{\perp}_Y \nabla^{\perp}_Z - \nabla^{\perp}_Y \nabla^{\perp}_X \nabla^{\perp}_Z - \nabla^{\perp}_Z \nabla^{\perp}_X \nabla^{\perp}_Y + \nabla^{\perp}_Z \nabla^{\perp}_Y \nabla^{\perp}_X; \tag{3.9}$$

$$\left[\left[\nabla^{\perp}_Y, \nabla^{\perp}_Z\right], \nabla^{\perp}_X\right] = \nabla^{\perp}_Y \nabla^{\perp}_Z \nabla^{\perp}_X - \nabla^{\perp}_Z \nabla^{\perp}_Y \nabla^{\perp}_X - \nabla^{\perp}_X \nabla^{\perp}_Y \nabla^{\perp}_Z + \nabla^{\perp}_X \nabla^{\perp}_Z \nabla^{\perp}_Y; \tag{3.10}$$

$$\left[\left[\nabla^{\perp}_Z, \nabla^{\perp}_X\right], \nabla^{\perp}_Y\right] = \nabla^{\perp}_Z \nabla^{\perp}_X \nabla^{\perp}_Y - \nabla^{\perp}_X \nabla^{\perp}_Z \nabla^{\perp}_Y - \nabla^{\perp}_Y \nabla^{\perp}_Z \nabla^{\perp}_X + \nabla^{\perp}_Y \nabla^{\perp}_X \nabla^{\perp}_Z. \tag{3.11}$$

Using Equations (3.9, 3.10, 3.11), it is easy to obtain

$$\left[\left[\nabla^{\perp}_X, \nabla^{\perp}_Y\right], \nabla^{\perp}_Z\right] + \left[\left[\nabla^{\perp}_Y, \nabla^{\perp}_Z\right], \nabla^{\perp}_X\right] + \left[\left[\nabla^{\perp}_Z, \nabla^{\perp}_X\right], \nabla^{\perp}_Y\right] = 0.$$

□

In particular, if M is the hypersubface in $\widetilde{M} = \mathbb{R}^m$ and N is an unit normal vector of M, then $R^{\perp}(X, Y, N) = 0$, for any $X, Y, Z \in \mathfrak{B}(M)$ (Since $\nabla^{\perp}_X N = \nabla^{\perp}_Y N = 0$).

Theorem 3.2. *Let $X \in \mathfrak{B}(M)$ and $\{N_{n+1}, ..., N_m\}$ the orthonormal basis on $\mathfrak{N}(M)$. Then the matrix of $\nabla^{\perp}_X$ is the antisymmetric matrix.*

Proof. For any $X \in \mathfrak{B}(M)$ and for each $j = n+1, ..., m$, we have

$$\begin{aligned} & N_j^2 = 1 \Rightarrow X\left[N_j^2\right] = 0 \Rightarrow \left(\widetilde{\nabla}_X N_j\right).N_j = 0 \\ & \Rightarrow \left(\left(\widetilde{\nabla}_X N_j\right)^{\top} + \nabla^{\perp}_X N_j\right).N_j = 0 \Rightarrow \left(\nabla^{\perp}_X N_j\right).N_j = 0 \\ & \Rightarrow \left(\nabla^{\perp}_X N_j\right) \perp N_j. \end{aligned}$$

Thus

$$\nabla^{\perp}_{X} N_j = A_{1j}.N_{n+1} + ... + \widehat{A_{jj}.N_{n+j}} + ... + A_{m-n\,j}.N_m,$$

(Here $A_{1j}.N_{n+1} + ... + A_{j-1\,j}.N_{n+j-1} + A_{j+1\,j}.N_{n+j+1} + ... + A_{m-n\,j}.N_m$ is written as $A_{1j}.N_{n+1} + ... + \widehat{A_{jj}.N_{n+j}} + ... + A_{m-n\,j}.N_m$).
Hence, the matrix A_X of the normal connection $\nabla^{\perp}_{X}$ for the basis $\{N_{n+1}, ..., N_m\}$ may be written in the form

$$A = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1\,m-n} \\ A_{21} & 0 & \cdots & A_{2\,m-n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m-n\,1} & A_{m-n\,2} & \cdots & 0 \end{bmatrix}$$

Next, one has

$$\begin{aligned} & N_j.N_h = 0, \forall j \neq h \in \{n+1, ..., m\} \\ & \Rightarrow X\left[N_j.N_h\right] = 0 \\ & \Rightarrow \left(\widetilde{\nabla}_X N_j\right).N_h = -\left(\widetilde{\nabla}_X N_h\right).N_j \\ & \Rightarrow \left(\nabla^{\perp}_{X} N_j\right).N_h = -\left(\nabla^{\perp}_{X} N_h\right).N_j \\ & \Rightarrow A_{jh} = -A_{hj}, \forall j \neq h \in \{n+1, ..., m\}. \end{aligned}$$

Therefore

$$A = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1\,m-n} \\ -A_{12} & 0 & \cdots & A_{2\,m-n} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{1\,m-n} & -A_{2\,m-n} & \cdots & 0 \end{bmatrix}$$

So that the matrix of $\nabla^{\perp}_{X}$ is the antisymmetric matrix.
This proves the theorem. □

Let $\varphi : \mathfrak{B}(M) \to \mathfrak{N}(M)$ be a modular homomorphic. Then the conjugate derivative $d_{\nabla^{\perp}}\varphi$ with the normal connection $\nabla^{\perp}$ of φ is defined by

$$\left(d_{\nabla^{\perp}}\varphi\right)(X,Y) = \nabla^{\perp}_{X}\varphi(Y) - \nabla^{\perp}_{Y}\varphi(X) - \varphi([X,Y]), \forall\, X, Y, Z \in \mathfrak{B}(M). \quad (3.12)$$

Example 3.3. *Consider M is a subface S in $\widetilde{M} = \mathbb{R}^3$ determined by*

$$\begin{aligned} & r : \mathbb{R}^2 \to \mathbb{R}^3 \\ & (u,v) \mapsto r(u,v) \end{aligned}$$

and the unit normal vector of S is given by

$$N = \frac{R_u \wedge R_v}{\|R_u \wedge R_v\|},$$

where $R_u = \frac{\partial}{\partial u} r(u,v)$, $R_v = \frac{\partial}{\partial v} r(u,v)$. *We consider the mapping*

$$\varphi : \mathfrak{B}(S) \to \mathfrak{N}(S)$$
$$X = f_1.R_u + f_2.R_v \mapsto \varphi(X) = (f_1 + f_2).N$$

Then we have

$$(d_{\nabla^\perp}\varphi)(R_u, R_v) = \nabla^\perp_{R_u} N - \nabla^\perp_{R_v} N - \varphi([R_u, R_v]) = 0.$$

Proposition 3.4. *Let* $\varphi : \mathfrak{B}(M) \to \mathfrak{N}(M)$ *be a module homomorphic. Then the map* $d_{\nabla^\perp}\varphi : \mathfrak{B}(M) \times \mathfrak{B}(M) \to \mathfrak{N}(M)$ *is the bilinear antisymmetric mapping.*

Proof. We prove that $d_{\nabla^\perp}\varphi$ is a bilinear mapping for the first variable and the proof analogous for the second variable. Indeed, for every $X, X', Y \in \mathfrak{B}(M)$, we have

$$\begin{aligned}
(d_{\nabla^\perp}\varphi)(X + X', Y) &= \nabla^\perp_{X+X'}\varphi(Y) - \nabla^\perp_Y \varphi(X + X') - \varphi([X + X', Y]) \\
&= \nabla^\perp_X \varphi(Y) + \nabla^\perp_{X'}\varphi(X) - \nabla^\perp_Y \varphi(X) - \nabla^\perp_Y \varphi(X') - \varphi([X, Y]) - \varphi([X', Y]) \\
&= (d_{\nabla^\perp}\varphi)(X, Y) + (d_{\nabla^\perp}\varphi)(X', Y).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(d_{\nabla^\perp}\varphi)(fX, Y) &= \nabla^\perp_{f.X}\varphi(Y) - \nabla^\perp_Y \varphi(f.X) - \varphi([f.X, Y]) \\
&= f.\nabla^\perp_X \varphi(Y) - Y[f].\varphi(X) - f\nabla^\perp_Y \varphi(X) - f.\varphi([X, Y]) - Y[f].\varphi(X) \\
&= f\left(\nabla^\perp_X \varphi(Y) - \nabla^\perp_Y \varphi(X) - \varphi[X, Y]\right) \\
&= f.(d_{\nabla^\perp}\varphi)(X, Y).
\end{aligned}$$

Next, we prove $d_{\nabla^\perp}\varphi$ is the antisymmetric mapping. Indeed, $\forall X, Y \in \mathfrak{B}(M)$, we have

$$\begin{aligned}
(d_{\nabla^\perp}\varphi)(X, Y) &= \nabla^\perp_X \varphi(Y) - \nabla^\perp_Y \varphi(X) - \varphi([X, Y]) \\
&= -\left[\nabla^\perp_Y \varphi(X) - \nabla^\perp_X \varphi(Y) - \varphi([Y, X])\right] \\
&= -(d_{\nabla^\perp}\varphi)(Y, X).
\end{aligned}$$

This proves the proposition. □

Theorem 3.5. *Suppose that* $X, Y \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$. *Then we have*

$$\left(d_{\nabla^\perp} h^\perp_N\right)(X, Y) = R^\perp(X, Y, N). \tag{3.13}$$

Proof. For every $X, Y \in \mathfrak{B}(M), N \in \mathfrak{N}(M)$, we have

$$\begin{aligned}
\left(d_{\nabla^\perp} h^\perp_N\right)(X, Y) &= \nabla^\perp_X \left(h^\perp_N(Y)\right) - \nabla^\perp_Y \left(h^\perp_N(X)\right) - h^\perp_N([X, Y]) \\
&= \nabla^\perp_X \left(\nabla^\perp_Y N\right) - \nabla^\perp_Y \left(\nabla^\perp_X N\right) - \nabla^\perp_{[X,Y]} N \\
&= R^\perp(X, Y, N).
\end{aligned}$$

This proves the theorem. □

By using the directional derivative of Weingarten normal mapping $h_N^{\perp}$ along a vector field X, we get the following Theorem.

Theorem 3.6. *Let $X, Y \in \mathfrak{B}(M)$ and $N \in \mathfrak{N}(M)$. Then we have*

$$\left(d_{\nabla^{\perp}} h_N^{\perp}\right)(X,Y) = (\triangledown_X h_N^{\perp})(Y) - (\triangledown_Y h_N^{\perp})(X). \tag{3.14}$$

Proof. For all X, Y $\in \mathfrak{B}(M)$ and for each $N \in \mathfrak{N}(M)$, we have

$$\begin{aligned}\left(d_{\nabla^{\perp}} h_N^{\perp}\right)(X,Y) &= \triangledown_X^{\perp}(h_N^{\perp}(Y)) - \triangledown_Y^{\perp}(h_N^{\perp}(X)) - h_N^{\perp}([X,Y])\\ &= (\triangledown_X h_N^{\perp})(Y) + h_N^{\perp}(\triangledown_X Y) - (\triangledown_Y h_N^{\perp})(X) - h_N^{\perp}(\triangledown_Y X) - h_N^{\perp}([X,Y])\\ &= (\triangledown_X h_N^{\perp})(Y) - (\triangledown_Y h_N^{\perp})(X) + h_N^{\perp}(\triangledown_X Y - \triangledown_Y X - [X,Y])\\ &= (\triangledown_X h_N^{\perp})(Y) - (\triangledown_Y h_N^{\perp})(X).\end{aligned}$$

This proves the theorem. □

4 The Lie derivative of symmetric connections

In this section, we construct the Lie derivative of a linear connection on the Riemann manifold M and given some properties of the Lie derivative of symmetric connections on M.

Definition 4.1. Suppose that $\triangledown$ be a linear connection on the manifold M. The mapping

$$L_X\triangledown : \mathfrak{B}(M) \times \mathfrak{B}(M) \to \mathfrak{B}(M)$$

satisfying the condition

$$(L_X\triangledown)(Y,Z) = L_X(\triangledown_Y Z) - \triangledown_{L_X Y} Z - \triangledown_Y(L_X Z), \tag{4.1}$$

for all $Y, Z \in \mathfrak{B}(M)$ is called the Lie derivative of the linear connection $\triangledown$ along a vector X.

Definition 4.2. *[24]* Let ∇ be a linear connection on M and T be a torsion tensor of ∇.

i) If $T = 0$, we will call ∇ *torsion free connection*, or *a symmetric connection*;

ii) The vector field $X \in \mathfrak{B}(M)$ is called *the parallel vector field* on M if

$$\nabla_Z X = 0, \text{for any } Z \in \mathfrak{B}(M).$$

Let ∇ be a linear connection on the manifold M. For every $X, Y \in \mathfrak{B}(M)$, we put $\widehat{\nabla}_X Y = \nabla_Y X + [X,Y]$. Then $\widehat{\nabla}$ is the linear connection on M.

Proposition 4.3. *If ∇ is a symmetric connection on the manifold M, then $\widehat{\nabla}$ is a symmetric connection on M.*

Proof. For every $X, Y \in \mathfrak{B}(M)$, we have

$$\begin{aligned}\widehat{T}(X,Y) &= \widehat{\nabla}_X Y - \widehat{\nabla}_Y X - [X,Y] \\ &= \nabla_Y X + [X,Y] - (\nabla_X Y + [Y,X]) - [X,Y] \\ &= \nabla_Y X + [X,Y] - \nabla_X Y + [X,Y] - [X,Y] \\ &= -\left(\nabla_X Y - \nabla_Y X - [X,Y]\right) = -T(X,Y).\end{aligned}$$

Since ∇ is a symmetric connection on M, thus $T(X,Y) = 0, \forall X, Y \in \mathfrak{B}(M)$. Hence, $\widehat{T} = 0$. So that $\widehat{\nabla}$ is a symmetric connection on M.
This proves the proposition. $\square$

Proposition 4.4. *Suppose that ∇^1, ∇^2 be symmetric connections on the manifold M. Then, we have $\varphi.\widehat{\nabla}^1 + (1-\varphi).\widehat{\nabla}^2$ is the symmetric connection on M, for every $\varphi \in \mathfrak{F}(M)$.*

Proof. Suppose that ∇^1, ∇^2 be symmetric connections on the manifold M. Applying Proposition 4.3, we obtain $\widehat{\nabla}^1, \widehat{\nabla}^2$ are symmetric connections on M. Hence, for every $X, Y \in \mathfrak{B}(M)$, we have

$$\begin{aligned}&\left(\varphi.\widehat{\nabla}^1 + (1-\varphi).\widehat{\nabla}^2\right)(X,Y) - \left(\varphi.\widehat{\nabla}^1 + (1-\varphi).\widehat{\nabla}^2\right)(Y,X) - [X,Y] \\ &= \varphi\left(\widehat{\nabla}^1_X Y - \widehat{\nabla}^1_Y X - [X,Y]\right) + (1-\varphi)\left(\widehat{\nabla}^2_X Y - \widehat{\nabla}^2_Y X - [X,Y]\right) = 0.\end{aligned}$$

Consequently, $\varphi.\widehat{\nabla}^1 + (1-\varphi).\widehat{\nabla}^2$ is the symmetric connection on M, for every $\varphi \in \mathfrak{F}(M)$. This proves the proposition. $\square$

Proposition 4.5. *Let $X \in \mathfrak{B}(M)$. Then we have*

$$(L_X\widehat{\nabla})(Y,Z) = (L_X\nabla)(Z,Y), \forall Y, Z \in \mathfrak{B}(M).$$

Proof. For every $X, Y, Z \in \mathfrak{B}(M)$, we have

$$\begin{aligned}(L_X\widehat{\nabla})(Y,Z) &= L_X(\widehat{\nabla})(Y,Z) - \widehat{\nabla}_{L_X Y} Z - \widehat{\nabla}_Y L_X Z \\ &= L_X\left(\nabla_Z Y + [Y,Z]\right) - \nabla_Z L_X Y - [L_X Y, Z] - \nabla_{L_X Z} Y - [Y, L_X Z] \\ &= L_X\left(\nabla_Z Y\right) + L_X\left([Y,Z]\right) - \nabla_Z L_X Y - [[X,Y],Z] - \nabla_{L_X Z} Y - [Y,[X,Z]] \\ &= \left(L_X\left(\nabla_Z Y\right) - \nabla_{L_X Z} Y - \nabla_Z L_X Y\right) + [X,[Y,Z]] - [[X,Y],Z] - [Y,[X,Z]] \\ &= (L_X\nabla)(Z,Y) + [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] \\ &= (L_X\nabla)(Z,Y), \forall Y, Z \in \mathfrak{B}(M) \text{ (since Jacobian equation).}\end{aligned}$$

Hence, $(L_X\widehat{\nabla})(Y,Z) = (L_X\nabla)(Z,Y), \forall Y, Z \in \mathfrak{B}(M)$.
This proves the proposition. $\square$

Theorem 4.6. *If ∇ is a symmetric connection on M, then $L_X\widehat{\nabla} = L_X\nabla$,* for every $X \in \mathfrak{B}(M)$.

Proof. For every $X, Y, Z \in \mathfrak{B}(M)$, we have

$$\begin{aligned}
&(L_X\nabla)(Y,Z) - (L_X\nabla)(Z,Y)\\
&= [X, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y [X,Z] - \big([X, \nabla_Z Y] - \nabla_{[X,Z]}Y - \nabla_Z [X,Y]\big)\\
&= [X, \nabla_Y Z - \nabla_Z Y] - \big(\nabla_{[X,Y]}Z - \nabla_Z [X,Y]\big) - \big(\nabla_Y [X,Z] - \nabla_{[X,Z]}Y\big)\\
&= [X,[Y,Z]] - [[X,Y],Z] - [Y,[X,Z]] \ \text{(since T=0)}\\
&= [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \ \text{(since Jacobian equation).}
\end{aligned}$$

Thus, $(L_X\nabla)(Y,Z) = (L_X\nabla)(Z,Y), \forall Y, Z \in \mathfrak{B}(M)$.

Hence, applying Proposition 4.5, we obtain

$$(L_X\widehat{\nabla})(Y,Z) = (L_X\nabla)(Y,Z), \forall Y, Z \in \mathfrak{B}(M).$$

So that $L_X\widehat{\nabla} = L_X\nabla, \forall X \in \mathfrak{B}(M)$. This proves the proposition. $\square$

Theorem 4.7. *Suppose that $X \in \mathfrak{B}(M)$ and ∇ be a symmetric connection on the manifold M. Then, $L_X\widehat{\nabla} + \widehat{\nabla}$ is the symmetric connection on M.*

Proof. Since ∇ is a symmetric connection on M, thus, applying Proposition 4.3, we obtain $\widehat{\nabla}$ is the symmetric connection on M. Hence, we have

$$\begin{aligned}
&(L_X\widehat{\nabla} + \widehat{\nabla})(Y,Z) - (L_X\widehat{\nabla} + \widehat{\nabla})(Z,Y) - [Y,Z]\\
&= (L_X\widehat{\nabla})(Y,Z) + \widehat{\nabla}_Y Z - (L_X\widehat{\nabla})(Z,Y) - \widehat{\nabla}_Z Y - [Y,Z]\\
&= \Big(\widehat{\nabla}_Y Z - \widehat{\nabla}_Z Y - [Y,Z]\Big) + (L_X\widehat{\nabla})(Y,Z) - (L_X\widehat{\nabla})(Z,Y)\\
&= (L_X\widehat{\nabla})(Y,Z) - (L_X\widehat{\nabla})(Z,Y).
\end{aligned}$$

On the other hand, for every $X, Y, Z \in \mathfrak{B}(M)$, we have

$$\begin{aligned}
&(L_X\nabla)(Y,Z) - (L_X\nabla)(Z,Y)\\
&= [X, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y [X,Z] - \big([X, \nabla_Z Y] - \nabla_{[X,Z]}Y - \nabla_Z [X,Y]\big)\\
&= [X, \nabla_Y Z - \nabla_Z Y] - \big(\nabla_{[X,Y]}Z - \nabla_Z [X,Y]\big) - \big(\nabla_Y [X,Z] - \nabla_{[X,Z]}Y\big)\\
&= [X,[Y,Z]] - [[X,Y],Z] - [Y,[X,Z]] \ \text{(since T=0)}\\
&= [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \ \text{(since Jacobian equation).}
\end{aligned}$$

Thus, $(L_X\nabla)(Y,Z)=(L_X\nabla)(Z,Y), \forall Y,Z\in\mathfrak{B}(M)$.
Hence, by using Theorem 4.6, we obtain $(L_X\widehat{\nabla})(Y,Z)=(L_X\widehat{\nabla})(Z,Y)$, for every $Y,Z\in\mathfrak{B}(M)$. So that $(L_X\widehat{\nabla}+\widehat{\nabla})(Y,Z)-(L_X\widehat{\nabla}+\widehat{\nabla})(Z,Y)-[Y,Z]=0$. Thus, the torsion tensor of the connection $(L_X\widehat{\nabla}+\widehat{\nabla})$ is null. Hence, $L_X\widehat{\nabla}+\widehat{\nabla}$ is the symmetric connection on M. This proves the theorem. □

From the Theorem 4.7, thus, the symmetric connection isn't unique on M.

Let $M=\mathbb{R}^n$. Then the usual directional derivative give rise to a linear connection. More precisely, if $X=X^i\partial_i$ and $Y=Y^j\partial_j$, then we define

$$\nabla_X Y=\nabla_{X^i\partial_i}Y^j\partial_j=X^i\partial_i(Y^j)\partial_j.$$

Then, ∇ is called the canonical connection on $\mathbb{R}^n$.

Proposition 4.8. *Suppose that ∇ be a canonical connection on $\mathbb{R}^n$ and X be a parallel vector field on $\mathbb{R}^n$. Then, we have $L_X\widehat{\nabla}=0$.*

Proof. We have

$$\begin{aligned}(L_X\widehat{\nabla})(Y,Z)&=L_X(\widehat{\nabla}_Y Z)-\widehat{\nabla}_{[X,Y]}Z-\widehat{\nabla}_Y\,[X,Z]\\
&=L_X(\nabla_Z Y+[Y,Z])-\nabla_Z[X,Y]-[[X,Y],Z]-\nabla_{[X,Z]}Y-[Y,[X,Z]]\\
&=L_X(\nabla_Z Y)+L_X([Y,Z])-\nabla_Z[X,Y]-[[X,Y],Z]-\nabla_{[X,Z]}Y-[Y,[X,Z]]\\
&=L_X(\nabla_Z Y)-\nabla_Z[X,Y]-\nabla_{[X,Z]}Y+[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]\\
&=L_X(\nabla_Z Y)-\nabla_Z[X,Y]-\nabla_{[X,Z]}Y\\
&=[X,\nabla_Z Y]-\nabla_Z[X,Y]-\nabla_{[X,Z]}Y\\
&=\nabla_X\nabla_Z Y-\nabla_{\nabla_Z Y}X-\nabla_Z\nabla_X Y-\nabla_Z\nabla_Y X-\nabla_{[X,Z]}Y\\
&=\nabla_X\nabla_Z Y-\nabla_Z\nabla_X Y-\nabla_{[X,Z]}Y \text{ (Since } X \text{ is a parallel vector field)}\\
&=R(X,Y,Z)=0.\end{aligned}$$

So that $L_X\widehat{\nabla}=0$. This proves the proposition. □

Definition 4.9. i) Suppose that $\theta:\mathfrak{B}(M)\to\mathfrak{B}(M)$ be a modular homomorphic. The derivative direction $\nabla_X\theta$ of θ along a vector field X is given by

$$(\nabla_X\theta)(Y)=\nabla_X(\theta(Y))-\theta(\nabla_X Y),\forall Y\in\mathfrak{B}(M).$$

ii) The Lie derivative $L_X\theta$ of θ along a vector field X is given by

$$(L_X\theta)(Y)=[X,\theta(Y)]-\theta([X,Y]),\forall Y\in\mathfrak{B}(M).$$

iii) The Lie product $[L_X,\nabla_Y]$ of L_X and ∇_Y is given by

$$[L_X,\nabla_Y](Z)=[X,\nabla_Y Z]-\nabla_Y[X,Z],\forall Z\in\mathfrak{B}(M).$$

Let $I : \mathfrak{B}(M) \to \mathfrak{B}(M)$ be an identity mapping, by using Definition 4.9, we have $\nabla_X I = 0$ and $L_X I = 0$. Indeed, we have

$$(\nabla_X I)(Y) = \nabla_X(I(Y)) - I(\nabla_X Y) = \nabla_X Y - \nabla_X Y = 0.$$

On the other hand, we have

$$(L_X I)(Y) = [X, I(Y)] - I([X,Y]) = [X,Y] - [X,Y] = 0.$$

Proposition 4.10. *Suppose that ∇ be a symmetric connection and X, Y be parallel vector fields on M. Then, we have $[L_X, \widehat{\nabla}_Y](Z) = R(X,Y,Z)$, for every $Z \in \mathfrak{B}(M)$.*

Proof. Since ∇ is a symmetric connection on M, thus by using Theorem 4.6, we have $L_Y \widehat{\nabla} = L_Y \nabla, \forall Y \in \mathfrak{B}(M)$. Hence, for every $Z \in \mathfrak{B}(M)$, applying Definition 4.9, we obtain

$$\begin{aligned}\left[L_X, \widehat{\nabla}_Y\right](Z) = [L_X, \nabla_Y](Z) &= [X, \nabla_Y Z] - \nabla_Y[X,Z] \\ &= \nabla_X\nabla_Y Z - \nabla_{\nabla_Y Z}X - \nabla_Y\nabla_X Z - \nabla_Y\nabla_Z X \cdot \\ &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z\end{aligned}$$

On the other hand, since ∇ is a symmetric connection and X, Y are parallel vector fields on M, thus $[X,Y] = 0$.
Consequently, $\left[L_X, \widehat{\nabla}_Y\right](Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = R(X,Y,Z)$.
This proves the proposition. □

Corollary 4.11. *Suppose that ∇ be a symmetric connection and X, Y be parallel vector fields on M. Then, we have $R(X,Y,Z) = 0, \forall Z \in \mathfrak{B}(M)$.*

Proof. For every $Z \in \mathfrak{B}(M)$, applying Definition 4.9, we obtain

$$\begin{aligned}\left[L_X, \widehat{\nabla}_Y\right](Z) &= [X, \widehat{\nabla}_Y Z] - \widehat{\nabla}_Y[X,Z] \\ &= [X, \nabla_Z Y + [Y,Z]] - \nabla_{[X,Z]}Y - [Y,[X,Z]] \cdot \\ &= [X,[Y,Z]] + [Y,[Z,X]] = -[Z,[X,Y]]\end{aligned}$$

On the other hand, since ∇ is a symmetric connection and X, Y are parallel vector fields on M, thus $[X,Y] = 0$.
Consequently, $\left[L_X, \widehat{\nabla}_Y\right](Z) = 0, \forall Z \in \mathfrak{B}(M)$. Hence, applying Proposition 4.10, we have $R(X,Y,Z) = 0, \forall Z \in \mathfrak{B}(M)$. This proves the corollary. □

Now, let M, N be Riemannian manifolds and $f : M \to N$ be a diffeomorphism and f_* be the push-forward of f. The mapping $f_* : \mathfrak{B}(M) \to \mathfrak{B}(N)$; f_* is the modular isomorphism. The mapping $\widehat{\nabla}^* : \mathfrak{B}(N) \times \mathfrak{B}(N) \to \mathfrak{B}(N)$ defined by

$$\widehat{\nabla}^*(f_*X, f_*Y) = f_*(\widehat{\nabla}_X Y), \forall X, Y \in \mathfrak{B}(M).$$

Then, $\widehat{\nabla}^*$ is the linear connection on the manifold N.

Proposition 4.12. *i) Let ∇ be a symmetric connection on M. Then $\widehat{\nabla}^*$ is the symmetric connection on N;*

ii) $\left[L_{f_*X}, \widehat{\nabla}^*_{f_*Y}\right](f_*Z) = f_*\left(\left[L_X, \widehat{\nabla}_Y\right](Z)\right)$, for all $X, Y, Z \in \mathfrak{B}(M)$.

Proof.

i) Let ∇ be a symmetric connection on M. Applying Proposition 4.3, we have $\widehat{\nabla}$ is the symmetric connection on M. Suppose that $\widehat{T}^*$ be the sorsion tensor of the connection $\widehat{\nabla}^*$ on the manifold N. Then, we have

$$\begin{aligned}\widehat{T}^*(f_*X, f_*Y) &= \widehat{\nabla}^*_{f_*X}(f_*Y) - \widehat{\nabla}^*_{f_*Y}(f_*X) - [f_*X, f_*Y] \\ &= f_*(\widehat{\nabla}_X Y) - f_*(\widehat{\nabla}_Y X) - f_*[X, Y] \\ &= f_*\left(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X - [X, Y]\right) = f_*(\widehat{T}(X, Y)) = 0.\end{aligned}$$

Thus, $\widehat{T}^* = 0$. Hence, $\widehat{\nabla}^*$ is the symmetric connection on N.

ii) For every $X, Y, Z \in \mathfrak{B}(M)$, we have

$$\begin{aligned}\left[L_{f_*X}, \widehat{\nabla}^*_{f_*Y}\right](f_*Z) &= \left[f_*X, \widehat{\nabla}^*_{f_*Y}(f_*Z)\right] - \widehat{\nabla}^*_{f_*Y}\left[f_*X, f_*Z\right] \\ &= \left[f_*X, f_*\left(\widehat{\nabla}_Y Z\right)\right] - \widehat{\nabla}^*_{f_*Y}\left(f_*\left[X, Z\right]\right) \\ &= f_*\left(\left[X, \widehat{\nabla}_Y Z\right]\right) - f_*\left(\widehat{\nabla}_Y\left[X, Z\right]\right) \\ &= f_*\left(\left[X, \widehat{\nabla}_Y Z\right] - \widehat{\nabla}_Y\left[X, Z\right]\right) = f_*\left(\left[L_X, \widehat{\nabla}_Y\right](Z)\right).\end{aligned}$$

This proves the proposition. □

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References

[1] R. Abraham, J.E. Marsden and T. Ratiu (2002), *Manifolds, Tensor Analysis and Applications.* Springer.

[2] E. Cartan (1962), *Afine, Projective, and Conformal Connection Spaces [Russian translation],* Kazan University, Kazan.

[3] J. L. Cabrerizo, L. M. Fernandez, and M. Fernandez (1993), *The curvature of submanifolds of an S-space form,* Acta Math. Hungar. **3-4**(62), 373 - 383.

[4] B. Y. Chen (1999), *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions,* Glasg. Math. J. **1**(41), 33 - 41.

[5] K. P. Chi, N. H. Quang and B. C. Van (2012), *The Lie derivative of currents on Lie group*, Lobachevskyi Journal of Mathmatics. **33**(1), 10 - 21.

[6] Dinh Tien Cuong anh Nessim Sibony (2005), *Introduction to the theory of currents*, American Mathematical Society-Providence-Rhode Island. Volume 84.

[7] L. Falach and R. Segev (2015), Reynolds transport theorem for smooth deformations of currents on manifolds, *Mathematics and Mechanics of Solids*, **20**(6), 770 - 786.

[8] T. Frankel (2004), The Geometry of Physics, An Introduction. Cambridge.

[9] Jean Gallier (2012), *Notes on differential geometry and Lie groups,* Department of Computer and Information Science, University of Pennsylvania, Philadelphia, PA 19104, USA.

[10] Katharina Habermann, Andreas Klein (2003), *Lie derivative of symplectic spinor fields, metaplectic representation, and quantization.* Math. Kolloq. 57, 711 .

[11] Thomas Hochrainer and Michael Zaiser (2005), *Fundamentals of a continuum theory of dislocations,* Proceedings of science. SMPRI2005.

[12] S. P. Hong and M. M. Tripathi (2005), *On Ricci curvature of submanifolds,* Int. J. Pure Appl. Math. Sci. **2**(2), 227-245.

[13] Jeong-Sik Kim, Mohit Kumar Dwivedi, and Mukut Mani Tripathi (2007), *Ricci curvature of integral submanifold of an s-space form,* Bull. Korean Math. Soc. **3**(44), 395-406.

[14] Jung-Hwan Kwon and Young Jin Suh (1997), *Lie derivatives on Homogeneous real hypersurfaces of type a in complex space forms.* Bull. Korean Math. Soc. **34**(3), 459-468.

[15] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Inter- sience Publishers. New York - London. Vol. 1, 1963; Vol. 2, 1969.

[16] Katharina Habermann, Andreas Klein (2003), *Lie derivative of symplectic spinor fields, metaplectic representation, and quantization,* Rostock. Math. Kolloq. **57**, 71-91.

[17] B. L. Laptev (1967), Lie differentiations, In: Progress in Science and Technology, Series on Algebra, Topology and Geometry [in Russian], All-Union Institute for Scientic and Technical Information, USSR Academy of Sciences, Moscow, pp. 429465.12.7671[math-ph].

[18] John M. Lee (2006), *Riemannian Manifolds: An Introduction to Curvature,* University of Washington Seattle, WA 98195-4350 USA.

[19] Dr. Felipe Leitner (2007), *Applications of Cartan and Tractor Calculus to Conformal and CR-Geometry,* Von der Fakultaät Mathematik und Physik der Mathematics Universität Stuttgart als Habilitationsschrift genehmigte Abhandlung. Fachbereich Mathematik, Stuttgart.

[20] A. D. Nicola, I. Yudin (2015), Covariant Lie derivatives and Frölicher-Nijenhuis bracket on Lie algebroids, *International Journal of Geometric Methods in Modern Physics,* **12**(9), 1560018 (8 pages).

[21] R. P. Singh and S. D. Singh (2010), *Lie Derivatives and Almost Analytic Vector Fields in a Generalised Structure Manifold,* Int. J. Contemp. Math. Sciences, **5**(2), 81 - 90.

[22] Paolo Piccione and Daniel V.Tausk(2006), *connections compatible with tensors. A characterization of left-invariant Levi-Civita connections on Lie groups,* Revista de la Unió Matemática Argentina. **1**(47), 125-133.

[23] B.N. Shapukov, *Lie derivatives on fiber manifolds,* J. Math. Sci. **108**(2), 211-231, 2002.

[24] A. Ya. Sultanov, *Derivations of linear algebras and linear connections,* J. Math. Sci. **169**(3), 362-412, 2010.

[25] R. Thom (1964), *" Local topological properties of differentiable mappings"*, Differential analysis, Bombay colloquium.

[26] Dao Trong Thi and A. T. Fomenko (1991), *Minimal surfaces stratified multivarifolds, and the Plateau problem,* American Mathematical Society-Providence-Rhode Island. Volume 84.

[27] B. C. Van (2016), On the Lie derivative of forms of bidegre, *Bulletin of Mathematical Analysis and Applications*, **8**(4), 33 - 42.

[28] B. C. Van and T. T. K. Ha (2015), *Some properties on the Lie derivative of linear connections on* $\mathbb{R}^n$, East-West Journal of Mathematics, **17**(2), 113 - 124.

[29] K. Yano (1957), *The theory of Lie derivatives and its applications,* North-Holland Publishing Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publishers Inc., New York.