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EXISTENCE OF MULTIPLE WEAK SOLUTIONS FOR A QUASI-LINEAR ELLIPTIC EQUATION CONTAINING $p(x)$-LAPLACIAN WITH MIXED BOUNDARY CONDITIONS

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Abstract

In this paper, we consider a mixed boundary value problem to a $p(x)$-Laplacian equation. More precisely we consider the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of at least two nontrivial weak solutions under some hypotheses on the data and parameters.

1 Introduction

In this paper, we consider the following problem

$$\begin{cases} -\mathrm{div}\,[S_t(x, |\nabla u|^2)\nabla u] + a(x)|u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ S_t(x, |\nabla u|^2)\frac{\partial u}{\partial \boldsymbol{n}} = \mu g(x,u) & \text{on } \Gamma_2, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a $C^{0,1}$-boundary Γ, $S(x,t)$ is a Carathéodory function satisfying some structure conditions, $a(x)$ is a measurable function on Ω satisfying

$$0 < a_* \leq a(x) \leq a^* < \infty \text{ for a.e } x \in \Omega, \tag{1.2}$$

Γ_1 and Γ_2 are disjoint open subsets of Γ such that

$$\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \tag{1.3}$$

and $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. Thus we impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The functions $f(x,t)$ and $g(x,t)$ are given real valued Carathéodory functions defined in $\Omega \times \mathbb{R}$ and $\Gamma_2 \times \mathbb{R}$, respectively, and λ and μ are parameters. When $S(x,t) = \frac{1}{p(x)} t^{p(x)}$, the operator $\operatorname{div}[S_t(x, |\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u]$ in the left-hand side of the first equation of (1.1) is the, so called, $p(x)$-Laplacian operator $\Delta_{p(x)} u = \operatorname{div}(|\boldsymbol{\nabla} u|^{p(x)-2}\boldsymbol{\nabla} u)$, where $p(x) > 1$.

The study of such type of differential equations with $p(x)$-growth conditions is a very interesting topic recently. Such problem stimulates its application in mathematical physics, in particular, in elastic mechanics (cf. Zhikov [25]), in electrorheological fluids (cf. Diening [7], Halsey [13], Mihăilescu and Rădulescu [17], Růžička [18]).

Over the last two decades, there are many articles on the existence of weak solutions for the Dirichlet boundary condition, that is, in the case $\Gamma_2 = \emptyset$ in problem (1.1), (for example, see Fan [9], Fan and Zhao [10], Avci [4], Yücedaĝ [21]). On the other hand, for the Steklov boundary condition, that is, in the case $\Gamma_1 = \emptyset$, for example, see Ji [14], Wei and Chen [19], Yücedaĝ [22], Allaoui et al [1], Ayoujil [5], Deng [6].

To the best of our knowledge, we can not find any problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1) except the case $p(x) = p = \text{const.}$ in Zeidler [23], so we are convinced of the reason for existence of this paper.

Under some assumptions on given functions f and g in problem (1.1), we show the existence of at least two nontrivial weak solutions according to the values of parameters λ and μ (cf. Theorem 3.1). In order to do so, we use the direct method of calculus of variation in this paper.

The paper is organized as follows. Section 2 consists of three subsections. In subsection 2.1, we recall some results on variable exponent Lebesgue-Sobolev spaces. In subsection 2.2, we introduce a Carathéodory function $S(x,t)$. In subsection 2.3, we set a problem rigorously and the properties of associated functionals. In Section 3, we state the main theorems on the existence of at least two nontrivial weak solutions for problem (1.1). Section 4 devotes the proof of the main theorem.

2 Preliminaries

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \ldots, a_N)$ and $\boldsymbol{b} = (b_1, \ldots, b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$.

2.1 Basic properties of variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$.

In this subsection, we recall some results on variable exponent Lebesgue-Sobolev spaces. See [10], Diening et al. [8], Kováčik and Rácosníc [15] and references therein for more detail. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ. Write $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}); p(x) > 1 \text{ for all } x \in \overline{\Omega}\}$, and let

$$p^+ = \max_{x \in \overline{\Omega}} p(x) \text{ and } p^- = \min_{x \in \overline{\Omega}} p(x) \ (> 1) \text{ for } p \in C_+(\overline{\Omega}).$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We introduce the Luxemburg norm on $L^{p(\cdot)}(\Omega)$ by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0; \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Then $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ becomes a Banach space. The conjugate space of $L^{p(\cdot)}(\Omega)$ becomes $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. A modular $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx \text{ for } u \in L^{p(\cdot)}(\Omega).$$

The following propositions are well known (see Fan et al. [12], [19], Fan and Zhao [11], Zhao et al. [24], [21]).

Proposition 2.1. *Let* $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \ldots)$. *Then we have*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

Hence $\min\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\} \leq \rho_{p(\cdot)}(u) \leq \max\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0 \Longleftrightarrow \{u_n\}$ *converges to* u *in measure in* Ω *and* $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u)$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

Let $q \in C_+(\Gamma) := \{q \in C(\Gamma); q(x) > 1 \text{ on } \Gamma\}$ and denote the surface measure on Γ by $d\sigma$. We define

$$Ł^{q(\cdot)}(\Gamma) = \Big\{u; u : \Gamma \to \mathbb{R} \text{ is measurable with respect to } d\sigma \quad \text{satisfying } \int_\Gamma |u(x)|^{q(x)} d\sigma < \infty\Big\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma \le 1\right\},$$

and we also define a modular $\rho_{q(\cdot),\Gamma}$ on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma.$$

Proposition 2.2. *We have the following.*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)} \ge 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

The following is a generalized Hölder inequality.

Proposition 2.3. *Let* $p \in C_+(\overline{\Omega})$. *For any* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p^+}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}. \tag{2.1}$$

Since $L^{p(\cdot)}(\Omega) \subset L^1_{\text{loc}}(\Omega)$, every function in $L^{p(\cdot)}(\Omega)$ has a distributional (weak) derivatives. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); \boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(\Omega)\},$$

where $\boldsymbol{\nabla}$ is the gradient operator, equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \int_\Omega \left(\left|\frac{u(x)}{\lambda}\right|^{p(x)} + \left|\frac{\boldsymbol{\nabla} u(x)}{\lambda}\right|^{p(x)}\right) dx \le 1\right\}.$$

Define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N \end{cases}$$

and

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.4. (i) *The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If $q(x) \in C_+(\overline{\Omega})$ satisfies $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$ is compact.*

(iii) *If $q(x) \in C_+(\Gamma)$ satisfies $q(x) < p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is compact. In particular, the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$ is compact and there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Gamma)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)} \text{ for } u \in W^{1,p(\cdot)}(\Omega).$$

Let Γ_1 and Γ_2 be satisfy (1.3). For $p \in C_+(\overline{\Omega})$, define

$$L^{p(\cdot)}(\Gamma_1) = \{v; v : \Gamma_1 \to \mathbb{R} \text{ is measurable with respect to } d\sigma \\ \text{and there exists } u \in L^{p(\cdot)}(\Gamma) \text{ such that } u = v \text{ on } \Gamma_1\}$$

with the norm

$$\|v\|_{L^{p(\cdot)}(\Gamma_1)} = \inf\{\|u\|_{L^{p(x)}(\Gamma)}; u \in L^{p(\cdot)}(\Gamma) \text{ and } u = v \text{ on } \Gamma_1\}.$$

Clearly, the restriction mapping $L^{p(\cdot)}(\Gamma) \to L^{p(\cdot)}(\Gamma_1)$ is continuous, so

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Gamma) \hookrightarrow L^{p(\cdot)}(\Gamma_1),$$

where the symbol $\hookrightarrow$ denotes that the embedding mapping is continuous, and there exists a constant $C > 0$ such that

$$\|v\|_{L^{p(\cdot)}(\Gamma_1)} \leq \|v\|_{L^{p(\cdot)}(\Gamma)} \leq C\|v\|_{W^{1,p(\cdot)}(\Omega)} \text{ for all } v \in W^{1,p(\cdot)}(\Omega).$$

Define a space

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.2}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable, uniformly convex Banach space. We define the norm

$$\|v\|_X = \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for } v \in X$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ according to the following Poincaré type inequality.

Proposition 2.5. *Let Ω be a bounded domain with the boundary Γ satisfying (1.3). Then there exists a constant $C > 0$ such that*

$$\|v\|_{L^{p(\cdot)}(\Omega)} \leq C\|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for all } v \in X.$$

Proof. If the conclusion does not hold, there exists a sequence $\{v_n\} \subset X$ such that $\|v_n\|_{L^{p(\cdot)}(\Omega)} = 1$ and $\|\boldsymbol{\nabla} v_n\|_{L^{p(\cdot)}(\Omega)} \leq 1/n$. Then $\{v_n\}$ is bounded in X. Since X is a reflexive Banach space, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $v \in X$ such that $v_{n_k} \to v$ weakly in X, $v_{n_k} \to v$ weakly in $L^{p(\cdot)}(\Omega)$ and $\boldsymbol{\nabla} v_{n_k} \to 0$ strongly in $\boldsymbol{L}^{p(\cdot)}(\Omega)$. Since $v_{n_k} \to v$ in $\mathcal{D}'(\Omega)$, we have $\boldsymbol{\nabla} v_{n_k} \to \boldsymbol{\nabla} v$ in $\mathcal{D}'(\Omega)$. Hence $\boldsymbol{\nabla} v = \mathbf{0}$ in $\mathcal{D}'(\Omega)$. Thereby $v =$ const.. Since v vanishes on $\Gamma_1 (\neq \emptyset)$, we have $v = 0$. Therefore, $v_{n_k} \to 0$ weakly in X. Since $p(x) < p^*(x)$, the embedding $X \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact. Hence $v_{n_k} \to 0$ strongly in $L^{p(\cdot)}(\Omega)$. This contradicts $\|v_{n_k}\|_{L^{p(\cdot)}(\Omega)} = 1$. □

2.2 A Carathéodory function

Let $p \in C_+(\overline{\Omega})$ be fixed. Let $S(x,t)$ be a Carathéodory function defined on $\Omega \times [0,\infty)$, and assume that for a.e. $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C([0,\infty))$ satisfies the following structure conditions: there exist positive constants $0 < s_* \leq s^* < \infty$ such that for a.e. $x \in \Omega$

$$S(x,0) = 0 \text{ and } s_* t^{(p(x)-2)/2} \leq S_t(x,t) \leq s^* t^{(p(x)-2)/2} \text{ for } t > 0. \tag{2.3a}$$

$$s_* t^{(p(x)-2)/2} \leq S_t(x,t) + 2tS_{tt}(x,t) \leq s^* t^{(p(x)-2)/2} \text{ for } t > 0. \tag{2.3b}$$

$$S_{tt}(x,t) < 0 \text{ when } 1 < p(x) < 2 \text{ and } S_{tt}(x,t) \geq 0 \text{ when } p(x) \geq 2 \text{ for } t > 0, \tag{2.3c}$$

where $S_t = \partial S/\partial t$ and $S_{tt} = \partial^2 S/\partial t^2$. We note that from (2.3a), we have

$$\frac{2}{p(x)} s_* t^{p(x)/2} \leq S(x,t) \leq \frac{2}{p(x)} s^* t^{p(x)/2} \text{ for } t \geq 0. \tag{2.4}$$

We introduce two examples. When $S(x,t) = \nu(x)\frac{1}{p(x)} t^{p(x)/2}$, where ν is a measurable function in Ω satisfying $0 < \nu_* \leq \nu(x) \leq \nu^* < \infty$ for a.e. in Ω, the function $S(x,t)$ satisfies (2.3a)-(2.3c). In particular case $\nu \equiv 1$, this example corresponds to the $p(x)$-Laplacian operator. As an another example, we can take

$$g(t) = \begin{cases} ae^{-1/t} + a & \text{for } t > 0, \\ a & \text{for } t = 0, \end{cases}$$

where $a > 0$ is a constant. Then we can see that $S(x,t) = \nu(x) g(t) \frac{1}{p(x)} t^{p(x)/2}$ satisfies (2.3a)-(2.3c) if $p(x) \geq 2$ for all $x \in \overline{\Omega}$, (cf. Aramaki [3]).

2.3 Setting of the problem

We consider problem (1.1). Throughout this paper, we suppose the following (f_0) and (g_0).

(f_0) Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying

$$|f(x,t)| \le C_1 + C_2|t|^{\alpha(x)-1} \text{ for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R},$$

where C_1 and C_2 are non-negative constants, $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

(g_0) Let $g : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying

$$|g(x,t)| \le D_1 + D_2|t|^{\beta(x)-1} \text{ for a.e. } x \in \Gamma_2 \text{ and all } t \in \mathbb{R},$$

where D_1 and D_2 are non-negative constants, $\beta \in C_+(\overline{\Gamma_2})$ and $\beta(x) < p^\partial(x)$ for all $x \in \overline{\Gamma_2}$.

Define

$$F(x,t) = \int_0^t f(x,s)ds \text{ for a.e. } x \in \Omega \text{ and } t \in \mathbb{R}, \tag{2.5}$$

$$G(x,t) = \int_0^t g(x,s)ds \text{ for a.e. } x \in \Gamma_2 \text{ and } t \in \mathbb{R}. \tag{2.6}$$

Now we give the notion of weak solutions for problem (1.1).

Definition 2.6. *We say $u \in X$ is a weak solution of problem* (1.1)*, if*

$$\int_\Omega S_t(x, |\nabla u|^2)\nabla u \cdot \nabla v dx + \int_\Omega a(x)|u|^{p(x)-2}uv dx$$
$$= \lambda \int_\Omega f(x,u)v dx + \mu \int_{\Gamma_2} g(x,u)v d\sigma \textit{ for all } v \in X. \tag{2.7}$$

We solve problem (1.1) by the direct method of calculus of variation. For this purpose, we consider the following functional on X defined by

$$I(u) = \Phi(u) - \lambda J(u) - \mu K(u), \tag{2.8}$$

where, for $u \in X$,

$$\Phi(u) = \frac{1}{2}\int_\Omega S(x, |\nabla u(x)|^2)dx + \int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx, \tag{2.9}$$

$$J(u) = \int_\Omega F(x, u(x))dx, \tag{2.10}$$

$$K(u) = \int_{\Gamma_2} G(x, u(x))d\sigma. \tag{2.11}$$

If $\Phi, J, K \in C^1(X,\mathbb{R})$, that is, Φ, J and K have continuous Gâteau derivatives, we can easily derive

$$\langle I'(u), v\rangle = \int_\Omega S_t(x,|\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u\cdot\boldsymbol{\nabla} v dx + \int_\Omega a(x)|u|^{p(x)-2}uvdx \\ - \lambda\int_\Omega f(x,u)vdx - \mu\int_{\Gamma_2} g(x,u)vd\sigma \text{ for all } u,v\in X, \quad (2.12)$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between X^* and X, and $I' : X \to X^*$ is the Fréchet derivative of I. Thus if $u \in X$ is a critical point of I, that is, $I'(u) = 0$, then u satisfies (2.7), so u is a weak solution of problem (1.1).

Now we give the properties of the functionals Φ, J and K defined by (2.9), (2.10) and (2.11).

Proposition 2.7. *Let $p \in C_+(\overline{\Omega})$. Assume that (f_0) and (g_0) hold. Then we can see that the following properties are satisfied.*

(i) *We can see that $\Phi, J, K \in C^1(X,\mathbb{R})$.*

(ii) *The functional Φ is sequentially weakly lower semi-continuous. The mapping $\Phi' : X \to X^*$ is a strictly monotone, bounded on each bounded subset of X, homeomorphism, and of (S_+)-type, namely, if $u_n \to u$ weakly in X and $\limsup_{n\to\infty}\langle\Phi'(u_n), u_n - u\rangle \le 0$, then we have $u_n \to u$ strongly in X.*

(iii) *The mappings $J', K' : X \to X^*$ are sequentially weakly-strongly continuous, that is, if $u_n \to u$ weakly in X, then $J'(u_n) \to J'(u)$ and $K'(u_n) \to K'(u)$ strongly in X^*, so the functionals $J, K : X \to \mathbb{R}$ are sequentially weakly continuous,*

For the proof, see [10], [14, Proposition 2.5].

3 Statement of the main theorem

In this section, we state the main theorem on the existence of at least two nontrivial weak solutions to problem (1.1) rigorously. In order to do so, we assume the following.

(f_1) (f_0) holds with $\alpha \in C_+(\overline{\Omega})$ satisfying $p(x) < \alpha(x) < p^*(x)$ for $x \in \overline{\Omega}$ and $p^+ < \alpha_-$.

(g_1) There exists a positive function $g(x) \in L^\infty(\Gamma_2)$ on Γ_2 such that

$$g(x,t) = g(x)|t|^{\beta(x)-2}t \text{ for } x \in \overline{\Gamma_2} \text{ and all } t \in \mathbb{R},$$

where $\beta \in C_+(\overline{\Gamma_2}), \beta(x) < p^\partial(x)$ for $x \in \overline{\Gamma_2}$ and $\beta^+ < p^-$.

Clearly (f_1) and (g_1) are more stronger than (f_0) and (g_0), respectively.

(f_2) $\lim_{t\to0}\frac{f(x,t)}{|t|^{p(x)-1}} = 0$ uniformly in $x \in \Omega$.

(f_3) There exists $q > \max\left\{\frac{s^*}{s_*}, \frac{a^*}{a_*}\right\}p^+(> p^+)$ such that the inequality $qF(x,t) \le f(x,t)t$ holds for all $(x,t) \in \Omega\times\mathbb{R}$.

(f_4) $\inf_{x\in\Omega,|t|=1} F(x,t) > 0$.

We are in a position to state the main theorem.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ satisfying* (1.3) *and a function $a(x)$ a measurable function in Ω satisfying* (1.2)*. Moreover, we assume that (f_1)-(f_4) and (g_1) hold. Then for any $\lambda^* > 0$, there exists $\mu^* > 0$ such that for any $0 < \lambda < \lambda^*$ and $0 < \mu < \mu^*$, problem* (1.1) *has at least two nontrivial weak solutions.*

4 Proof of Theorem 3.1

According to (f_2), for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x,t)| \le \varepsilon |t|^{p(x)-1} \text{ for any } x \in \Omega \text{ and } |t| < \delta.$$

From (f_1), for $|t| \ge \delta$,

$$|f(x,t)| \le C_1 \left(\frac{|t|}{\delta}\right)^{\alpha(x)-1} + C_2|t|^{\alpha(x)-1} \le \left(\frac{C_1}{\delta^{\alpha(x)-1}} + C_2\right)|t|^{\alpha(x)-1}.$$

Summing up, there exists a constant $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon|t|^{p(x)-1} + C_\varepsilon|t|^{\alpha(x)-1} \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

Therefore,

$$\begin{aligned} |F(x,t)| = \left|\int_0^t f(x,s)ds\right| &\le \frac{\varepsilon}{p(x)}|t|^{p(x)} + \frac{C_\varepsilon}{\alpha(x)}|t|^{\alpha(x)} \\ &\le \frac{\varepsilon}{p^-}|t|^{p(x)} + \frac{C_\varepsilon}{\alpha^-}|t|^{\alpha(x)} \text{ for all } (x,t) \in \Omega \times \mathbb{R}. \quad (4.1) \end{aligned}$$

Hence we have, for $u \in X$,

$$\int_\Omega F(x,u)dx \le \frac{\varepsilon}{p^-}\int_\Omega |u|^{p(x)}dx + \frac{C_\varepsilon}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx. \quad (4.2)$$

Moreover, from (g_1), we have

$$\int_{\Gamma_2} G(x,u)d\sigma \le \frac{\|g\|_{L^\infty(\Gamma_2)}}{\beta^-}\int_{\Gamma_2} |u|^{\beta(x)}d\sigma. \quad (4.3)$$

Lemma 4.1. *For any $\lambda^* > 0$, there exist $\rho, c_0, \mu^* > 0$ depending on λ^* such that for any $0 < \lambda < \lambda^*$ and $0 < \mu < \mu^*$, we have*

$$I(u) \ge c_0 \textit{ for all } u \in X \textit{ with } \|u\|_X = \rho.$$

Proof. From (2.4), (1.2), (4.2) and (4.3), we can derive

$$I(u) \geq \frac{s_*}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{a_*}{p^+}\int_\Omega |u|^{p(x)}dx - \frac{\lambda^*\varepsilon}{p^-}\int_\Omega |u|^{p(x)}dx - \frac{\lambda^* C_\varepsilon}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx - \frac{\mu\|g\|_{L^\infty(\Gamma_2)}}{\beta^-}\int_{\Gamma_2}|u|^{\beta(x)}d\sigma. \quad (4.4)$$

Choose $\varepsilon > 0$ so that $a_*/p^+ = \lambda^*\varepsilon/p^-$. Since $X \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ and $X \hookrightarrow L^{\beta(\cdot)}(\Gamma_2)$, there exists a constant $C_1 > 1$ such that

$$\|u\|_{L^{\alpha(\cdot)}(\Omega)} \leq C_1\|u\|_X \text{ and } \|u\|_{L^{\beta(\cdot)}(\Gamma_2)} \leq C_1\|u\|_X.$$

Let us choose $\|u\|_X \leq 1/C_1 (< 1)$. Then by Proposition 2.1 and 2.2,

$$\int_\Omega |u|^{\alpha(x)}dx \leq \|u\|^{\alpha^-}_{L^{\alpha(\cdot)}(\Omega)} \leq C_1^{\alpha^-}\|u\|_X^{\alpha^-},$$
$$\int_{\Gamma_2} |u|^{\beta(x)}d\sigma \leq \|u\|^{\beta^-}_{L^{\beta(\cdot)}(\Gamma_2)} \leq C_1^{\beta^-}\|u\|_X^{\beta^-}.$$

Thus it follows from (4.4) that

$$I(u) \geq \frac{s_*}{p^+}\|u\|_X^{p^+} - \frac{\lambda^* C_\varepsilon C_1^{\alpha^-}}{\alpha^-}\|u\|_X^{\alpha^-} - \frac{\mu\|g\|_{L^\infty(\Gamma_2)}C_1^{\beta^-}}{\beta^-}\|u\|_X^{\beta^-}. \quad (4.5)$$

We define

$$\rho = \min\left\{\left(\frac{s_*}{2p^+}\cdot\frac{\alpha^-}{\lambda^* C_\varepsilon C_1^{\alpha^-}}\right)^{1/(\alpha^- - p^+)}, \frac{1}{C_1}\right\}.$$

We note that

$$\rho^{\alpha^- - p^+} \leq \frac{s_*}{2p^+}\cdot\frac{\alpha^-}{\lambda^* C_\varepsilon C_1^{\alpha^-}}.$$

When $\|u\|_X = \rho$, from (4.5), we can derive

$$I(u) \geq \frac{s_*}{2p^+}\rho^{p^+} - \frac{\mu\|g\|_{L^\infty(\Gamma_2)}C_1^{\beta^-}}{\beta^-}\rho^{\beta^-}.$$

Define

$$\mu^* = \frac{s_*}{2p^+}\cdot\frac{\beta^-}{2\|g\|_{L^\infty(\Gamma_2)}C_1^{\beta^-}}\rho^{p^+ - \beta^-}.$$

Then for $0 < \mu < \mu^*$, we have

$$I(u) \geq c_0 := \frac{s_*}{4p^+}\rho^{p^+}.$$

Thus the lemma is proved. □

Lemma 4.2. *There exists $e \in X$ such that $\|e\|_X > \rho$ and $I(e) < 0$.*

Proof. For each $(x,t) \in \Omega \times \mathbb{R}$, define

$$\gamma_1(\tau) = \tau^{-q}F(x,\tau t) - F(x,t) \text{ for } \tau \geq 1.$$

It follows from (f_3) that

$$\gamma_1'(\tau) = \tau^{-q-1}(f(x,\tau t)\tau t - qF(x,\tau t)) \geq 0 \text{ for } \tau \geq 1.$$

Thus $\gamma_1(\tau)$ is increasing on $[1,\infty)$, so $\gamma_1(\tau) \geq \gamma_1(1)$ for $1 \leq \tau < \infty$. Therefore, we have

$$F(x,\tau t) \geq \tau^q F(x,t) \text{ for all } (x,t) \in \Omega \times \mathbb{R} \text{ and } \tau \geq 1. \tag{4.6}$$

By (f_4), there exists a non-negative function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \not\equiv 0$ and

$$\int_\Omega F(x,\varphi)dx > 0.$$

For $\tau \geq 1$, it follows from (4.6) that

$$\begin{aligned} I(\tau\varphi) &= \int_\Omega \left(\frac{1}{2}S(x,|\tau\boldsymbol{\nabla}\varphi|^2) + \frac{a(x)}{p(x)}|\tau\varphi|^{p(x)}\right)dx \\ &\qquad\qquad -\lambda\int_\Omega F(x,\tau\varphi)dx - \mu\int_{\Gamma_2} G(x,\tau\varphi)d\sigma \\ &\leq \frac{s^*}{p^-}\tau^{p^+}\int_\Omega |\boldsymbol{\nabla}\varphi|^{p(x)}dx + \frac{a^*}{p^-}\tau^{p^+}\int_\Omega |\varphi|^{p(x)}dx - \lambda\tau^q\int_\Omega F(x,\varphi)dx. \end{aligned}$$

Since $q > p^+$, $\lambda > 0$ and $\int_\Omega F(x,\varphi)dx > 0$, we see that $I(\tau\varphi) \to -\infty$ as $\tau \to \infty$. Hence there exists $\tau_0 = \tau(\lambda) > 0$ such that $\|\tau_0\varphi\|_X > \rho$ and $I(\tau_0\varphi) < 0$. It suffices to put $e = \tau_0\varphi$. □

Lemma 4.3. *The functional I satisfies the Palais-Smale condition, that is, if $\{u_n\} \subset X$ satisfies $I(u_n) \to c \in \mathbb{R}$ and $I'(u_n) \to 0$ in X^* as $n \to \infty$, then $\{u_n\}$ has a strongly convergent subsequence in X.*

Proof. Let $\{u_n\} \subset X$ be a sequence in X satisfying $I(u_n) \to c \in \mathbb{R}$ and $I'(u_n) \to 0$ in X^*.

Step 1. We show that $\{u_n\}$ is bounded in X.

If $\{u_n\}$ is unbounded, there exists a subsequence (still denoted by $\{u_n\}$) such that $\|u_n\|_X \to \infty$ as $n \to \infty$. Since $|\langle I'(u_n), u_n\rangle| \leq \|I'(u_n)\|_{X^*}\|u_n\|_X$

and $I'(u_n) \to 0$ in X^*, for large n, using (f_3),

$$\begin{aligned} c + 1 + \|u_n\|_X &\geq I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\ &= \int_\Omega \left(\frac{1}{2}S(x, |\nabla u_n|^2) + \frac{a(x)}{p(x)}|u_n|^{p(x)}\right)dx - \lambda \int_\Omega F(x, u_n)dx \\ &\quad -\mu \int_{\Gamma_2} G(x, u_n)d\sigma - \frac{1}{q}\int_\Omega S_t(x, |\nabla u_n|^2)|\nabla u_n|^2 dx \\ &\quad -\frac{1}{q}\int_\Omega a(x)|u_n|^{p(x)}dx + \lambda \int_\Omega \frac{1}{q} f(x, u_n)u_n dx \\ &\qquad\qquad + \mu \int_{\Gamma_2} \frac{1}{q} g(x, u_n)u_n d\sigma \\ &\geq \left(\frac{s_*}{p^+} - \frac{s^*}{q}\right)\int_\Omega |\nabla u_n|^{p(x)}dx + \left(\frac{a_*}{p^+} - \frac{a^*}{q}\right)\int_\Omega |u_n|^{p(x)}dx \\ &\quad -\mu \int_{\Gamma_2} G(x, u_n)d\sigma + \mu \int_{\Gamma_2} \frac{1}{q} g(x, u_n)u_n d\sigma \\ &\geq \min\left\{\frac{s_*}{p^+} - \frac{s^*}{q}, \frac{a_*}{p^+} - \frac{a^*}{q}\right\}\int_\Omega (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx \\ &\quad -\mu \int_{\Gamma_2} G(x, u_n)d\sigma + \mu \int_{\Gamma_2} \frac{1}{q} g(x, u_n)u_n d\sigma. \end{aligned}$$

Here we have

$$\begin{aligned} \int_{\Gamma_2} |G(x, u_n)|d\sigma &\leq \frac{1}{\beta^-}\|g\|_{L^\infty(\Gamma_2)}\int_{\Gamma_2} |u_n|^{\beta(x)}d\sigma \\ &\leq \frac{1}{\beta^-}\|g\|_{L^\infty(\Gamma_2)} \max\{\|u_n\|^{\beta^-}_{L^{\beta(\cdot)}(\Gamma_2)}, \|u_n\|^{\beta^+}_{L^{\beta(\cdot)}(\Gamma_2)}\} \\ &\leq \frac{1}{\beta^-}\|g\|_{L^\infty(\Gamma_2)} C_1^{\beta^+}\|u_n\|_X^{\beta^+}. \end{aligned}$$

Similarly, we have

$$\frac{1}{q}\int_{\Gamma_2} |g(x, u_n)u_n|d\sigma \leq \frac{1}{q}\|g\|_{L^\infty(\Gamma_2)} C_1^{\beta^+}\|u_n\|_X^{\beta^+}.$$

Therefore, we have

$$c + 1 + \|u_n\|_X \geq \min\left\{\frac{s_*}{p^+} - \frac{s^*}{q}, \frac{a_*}{p^+} - \frac{a^*}{q}\right\}\|u_n\|_X^{p^-} - \mu C_2\|u_n\|_X^{\beta^+}.$$

Since $\beta^+ < p^-$ and (f_3) holds, dividing both-hand side by $\|u_n\|_X^{\beta^+}$ and letting $n \to \infty$, this leads a contradiction.

Step 2. Since X is a reflexive Banach space, passing to a subsequence, we can assume that $u_n \to u$ weakly in X for some $u \in X$, strongly in $L^{\alpha(\cdot)}(\Omega) \cap L^{\beta(\cdot)}(\Gamma_2)$. We remember

$$\langle I'(u), v\rangle = \langle \Phi'(u), v\rangle - \lambda\langle J'(u), v\rangle - \mu\langle K'(u), v\rangle \text{ for } u, v \in X.$$

By the Hölder inequality (Proposition 2.3), we have

$$\begin{aligned}|\langle J'(u_n), u_n - u\rangle| &\le \int_\Omega |f(x,u_n)(u_n-u)|dx \le \int_\Omega (C_1 + C_2|u_n|^{\alpha(x)-1})|u_n - u|dx \\ &\le 2\|C_1 + C_2|u_n|^{\alpha(x)-1}\|_{L^{\alpha'(\cdot)}(\Omega)}\|u_n - u\|_{L^{\alpha(\cdot)}(\Omega)} \to 0 \text{ as } n \to \infty,\end{aligned}$$

and

$$\begin{aligned}|\langle K'(u_n), u_n - u\rangle| &\le \int_{\Gamma_2} |g(x,u_n)(u_n-u)|d\sigma \le \int_{\Gamma_2} g(x)|u_n|^{\beta(x)-1}|u_n - u|d\sigma \\ &\le \|g\|_{L^\infty(\Gamma_2)}\||u_n|^{\beta(x)-1}\|_{L^{\beta'(\cdot)}(\Gamma_2)}\|u_n - u\|_{L^{\beta(\cdot)}(\Gamma_2)} \to 0 \text{ as } n \to \infty,\end{aligned}$$

Here we used the boundedness of $\{u_n\}$ in X by Step 1. Since $\{u_n - u\}$ is bounded and $I'(u_n) \to 0$ in X^*, $\langle I'(u_n), u_n - u\rangle \to 0$ as $n \to \infty$. Thus $\langle \Phi'(u_n), u_n - u\rangle \to 0$ as $n \to \infty$. Since Φ' is (S_+)-type from Proposition 2.7, we can see that $u_n \to u$ strongly in X. □

Lemma 4.4. *There exists $w \in X$ such that $w \not\equiv 0$ and $I(\tau w) < 0$ for $\tau > 0$ small enough.*

Proof. For $(x,t) \in \Omega \times \mathbb{R}$, define $\gamma_2(\tau) = F(x, \tau^{-1}t)\tau^q$ for $\tau \ge 1$. Then from (f_3), we have

$$\begin{aligned}\gamma_2'(\tau) &= qF(x,\tau^{-1}t)\tau^{q-1} + f(x,\tau^{-1}t)\big(-\frac{1}{\tau^2}t\big)\tau^q \\ &= \tau^{q-1}\big(qF(x,\tau^{-1}t) - f(x,\tau^{-1}t)\tau^{-1}t\big) \le 0.\end{aligned}$$

Hence $\gamma_2(\tau)$ is non-increasing on $[1,\infty)$. For $|t| \ge 1$, $\gamma_2(1) \ge \gamma_2(|t|)$, that is,

$$F(x,t) \ge F(x, |t|^{-1}t)|t|^q \ge c|t|^q \text{ for some } c > 0.$$

Here we used (f_4). From (f_2), there exists $\eta > 0$ such that

$$\frac{|f(x,t)t|}{|t|^{p(x)}} = \frac{|f(x,t)|}{|t|^{p(x)-1}} \le 1 \text{ for all } x \in \Omega \text{ and } 0 < |t| < \eta.$$

From (f_1), there exists $C_\eta > 0$ such that

$$\frac{|f(x,t)t|}{|t|^{p(x)}} \le \frac{C_1 + C_2|t|^{\alpha(x)-1})|t|}{|t|^{p(x)}} \le C_\eta$$

for $x \in \Omega$ and $\eta \le |t| \le 1$. Hence

$$f(x,t)t \ge -(1+C_\eta)|t|^{p(x)} \text{ for } x \in \Omega \text{ and } |t| \le 1.$$

Therefore, by $F(x,0)=0$ and the mean value theorem,

$$F(x,t) = \int_0^1 f(x,\tau t)td\tau \ge -(1+C_\eta)\int_0^1 |\tau t|^{p(x)}\tau^{-1}d\tau \ge -\frac{1+C_\eta}{p^-}|t|^{p(x)}$$

for all $x \in \Omega$ and $|t| \le 1$. Summing up the above, we have

$$F(x,t) \ge c|t|^q - C_3|t|^{p(x)} \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

where $C_3 = (1+C_\eta)/p^- + c$. Since

$$G(x,t) = \int_0^1 g(x,\tau t)td\tau = g(x)\int_0^1 |\tau t|^{\beta(x)-2}(\tau t)td\tau = \frac{g(x)}{\beta(x)}|t|^{\beta(x)}$$

and $g(x) > 0$ on Γ_2, if we choose $w \in (C^\infty(\overline{\Omega}) \cap X) \setminus W_0^{1,p(\cdot)}(\Omega)$, where

$$W_0^{1,p(\cdot)}(\Omega) = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma\},$$

then we have $\int_{\Gamma_2} G(x,w)d\sigma > 0$. For $0 < \tau \le 1$, we have

$$\begin{aligned} I(\tau w) &\le \frac{s^*}{p^-}\tau^{p^-}\int_\Omega |\nabla w|^{p(x)}dx + \frac{a^*}{p^-}\tau^{p^-}\int_\Omega |w|^{p(x)}dx \\ &\quad -c\lambda\tau^q\int_\Omega |w|^q dx + \lambda C_3\tau^{p^-}\int_\Omega |w|^{p(x)}dx - \mu\tau^{\beta^+}\int_{\Gamma_2} G(x,w)d\sigma \\ &\le \frac{s^*}{p^-}\tau^{p^-}\int_\Omega |\nabla w|^{p(x)}dx + \left(\frac{a^*}{p^-} + \lambda C_3\right)\tau^{p^-}\int_\Omega |w|^{p(x)}dx \\ &\quad -\mu\tau^{\beta^+}\int_{\Gamma_2} G(x,w)d\sigma. \end{aligned}$$

Since $\int_{\Gamma_2} G(x,w)d\sigma > 0$ and $\beta^+ < p^-$, there exists $\tau_0 = \tau_0(\lambda) > 0$ such that for $0 < \tau < \tau_0$, $I(\tau w) < 0$. □

Proof of Theorem 3.1

From Lemma 4.1, for any $\lambda^* > 0$, there exist $\rho, c_0, \mu^* > 0$ such that if $0 < \lambda < \lambda^*, 0 < \mu < \mu^*$, then $I(u) \ge c_0$ for all $u \in X$ with $\|u\|_X = \rho$. From Lemma 4.2, there exists $e \in X$ with $\|e\|_X > \rho$ such that $I(e) < 0 = I(0)$. Moreover, from Lemma 4.3, the functional I satisfies Palais-Smale condition. Hence all assumptions of the mountain pass theorem by Ambrosetti-Rabinowitz hold (cf. [2] and Willem [20, Theorem 1.17]). Let

$$C = \{c \in C([0,1];X) : c(0) = 0, c(1) = e\}$$

and $\overline{c} = \inf_{c\in C}\sup_{t\in[0,1]} I(c(t))$. Then there exists a critical point $u_1 \in X$ of I, that is, $I'(u_1) = 0$ and $I(u_1) = \overline{c} \geq c_0 > 0$. So u_1 is a nontrivial weak solution of problem (1.1).

We show that the existence of the second nontrivial weak solution $u_2 \in X$ such that $u_2 \neq u_1$ by the Ekeland variational principle (cf. Mawhin and Willem [16, Theorem 4.2]). If we put $B_\rho(0) = \{v \in X; \|v\|_X < \rho\}$, then it follows from Lemma 4.1 that $\inf_{v\in\partial B_\rho(0)} I(v) > 0$. From the proof of Lemma 4.1, I is bounded from below in $\overline{B}_\rho(0)$ and from Lemma 4.4, there exists $w \in X$ such that $I(\tau w) < 0$ for $\tau > 0$ small enough. Thus

$$-\infty < \underline{c} := \inf_{v\in\overline{B}_\rho(0)} I(v) < 0.$$

Choose $\varepsilon > 0$ so that $0 < \varepsilon < \inf_{v\in\partial B_\rho(0)} I(v) - \inf_{v\in\overline{B}_\rho(0)} I(v)$. Applying the Ekeland variational principle to $I : \overline{B}_\rho(0) \to \mathbb{R}$, there exists $u_\varepsilon \in \overline{B}_\rho(0)$ such that

$$I(u_\varepsilon) < \inf_{v\in\overline{B}_\rho(0)} I(v) + \varepsilon, \tag{4.7}$$

$$I(u_\varepsilon) \leq I(u) + \varepsilon\|u - u_\varepsilon\|_X \text{ for all } u \in \overline{B}_\rho(0) \text{ with } u \neq u_\varepsilon. \tag{4.8}$$

Since $I(u_\varepsilon) < \inf_{v\in\partial B_\rho(0)} I(v)$, we see that $u_\varepsilon \in B_\rho(0)$. Hence

$$\inf_{v\in B_\rho(0)} I(v) \leq I(u_\varepsilon) < \inf_{v\in\overline{B}_\rho(0)} I(v) + \varepsilon. \tag{4.9}$$

Define a functional $\widetilde{I} : \overline{B}_\rho(0) \to \mathbb{R}$ by $\widetilde{I}(u) = I(u) + \varepsilon\|u - u_\varepsilon\|_X$. Since $\widetilde{I}(u) \geq I(u_\varepsilon) = \widetilde{I}(u_\varepsilon)$ for $u \in \overline{B}_\rho(0)$, u_ε is a minimum of $\widetilde{I}$ in $\overline{B}_\rho(0)$. Hence

$$\frac{\widetilde{I}(u_\varepsilon + \tau v) - \widetilde{I}(u_\varepsilon)}{\tau} \geq 0 \text{ for all } \tau > 0 \text{ small enough and for all } v \in B_\rho(0).$$

Tha is,

$$\frac{I(u_\varepsilon + \tau v) - I(u_\varepsilon)}{\tau} + \varepsilon\|v\|_X \geq 0 \text{ for all } v \in B_\rho(0).$$

Letting $\tau \to +0$, we have $\langle I'(u_\varepsilon), v\rangle \geq -\varepsilon\|v\|_X$. If we replace v with $-v \in B_\rho(0)$, then we have

$$\langle I'(u_\varepsilon), v\rangle \leq \varepsilon\|v\|_X \text{ for all } v \in B_\rho(0).$$

This implies that $\|I'(u_\varepsilon)\|_{X^*} \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, there exists $u_n \in B_\rho(0)$ such that using (4.9),

$$I(u_n) \to \underline{c} < 0 \text{ and } I'(u_n) \to 0 \text{ in } X^*.$$

By Lemma 4.3, there exists $u_2 \in X$ such that $u_n \to u_2$ strongly in X. Since $I \in C^1(X;\mathbb{R})$, $I'(u_2) = 0$ and $I(u_2) = \underline{c} < 0$, that is, u_2 is a nontrivial weak solution of problem (1.1). Since $I(u_1) = \overline{c} \geq c_0 > 0$ and $I(u_2) = \underline{c} < 0$, we have $u_1 \neq u_2$. This completes the proof of Theorem 3.1.

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