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THE ADMISSIBLE MONOMIAL BASIS FOR THE POLYNOMIAL ALGEBRA IN DEGREE THIRTEEN

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Abstract

Let $\mathbf{P}(n) = \mathbb{F}_2[x_1, x_2, ..., x_n]$ be the polynomial algebra in n variables x_i, of degree one, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ according to well known rules. The hit problem, set up by F.Peterson, of determining $\mathcal{A}^+\mathbf{P}(n)$, the subspace of all polynomials in the image of the action of the mod-2 Steenrod algebra has been studied by several authors. We are interested in the related problem of determining a basis for the quotient vector space $\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n)$. In this paper, we give an explicit formula for the dimension of $\mathbf{Q}(n)$ in degree thirteen.

1 Introduction

Let $\mathbf{P}(n) = \mathbb{F}_2[x_1, x_2, ..., x_n]$ be the polynomial algebra in n variables x_i, of degree one, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0 \\ x_j^2, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

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and subject to the Cartan formula

$$Sq^n(uv) = \sum_{i=0}^{n} Sq^i(u)Sq^{n-i}(v)$$

for $u, v \in \mathbf{P}(n)$. A polynomial $f \in \mathbf{P}(n)$ is in the image of the the action of the Steenrod algebra if

$$f = \sum_{i>0} Sq^i(f_i)$$

for some polynomials $f_i \in \mathbf{P}(n)$. That means f belongs to $\mathcal{A}^+\mathbf{P}(n)$, the subspace of all hit polynomials. The problem of determining $\mathcal{A}^+\mathbf{P}(n)$ is called the hit problem and has been studied by several authors. In this paper, we are interested in the related problem of finding a basis for the quotient vector space

$$\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n).$$

This problem was first studied by Peterson [7], Wood[15], Singer[11], Priddy[9], Carlisle-Wood[1], who showed its relationship to several classical problems in homotopy theory. The quotient $\mathbf{Q}(n)$ has been calculated by Peterson[8] for $k = 2$ and by Kameko[3] for $k = 3$ in his thesis. For $n = 4$, the problem has explicitly been determined in Sum [13] for the cases of the degree $2^{s+t} + 2^s - 2$ and the other cases have explicitly been determined in Sum [14]. This problem has been completely solved in all degrees less than 13, Mothebe et al [4]. The results are used to study the Singer algebraic transfer which is a homomorphism from the homology of the mod-2 Steenrod algebra, $Tor^{\mathcal{A}}_{n,n+d}(\mathbb{F}_2, \mathbb{F}_2)$, to the subspace of $QP^d(n)$ consisting of all the GL_n-invariant classes of degree d. The Singer algebraic transfer is a useful tool in describing the homology groups of the Steenrod algebra.

The following result is useful for determining $\mathcal{A}$-generators for $\mathbf{P}(n)$. Let $\alpha(m)$ denote the number of digits 1 in the binary expansion of m. In [15], Wood proved the following.

Theorem 1.1. *Let $u \in \boldsymbol{P}(n)$ be a monomial of degree d. If $\alpha(n+d) > n$, then u is hit.*

Our main result is Theorem 1.2 below which is based on and expands the previous work of Mothebe et al [4]. We explicitly determine the dimension of $\mathbf{Q}^{13}(n)$ for all $n \geq 1$. We have:

Theorem 1.2. *For all $n \geq 1$:*

$$\dim(\boldsymbol{Q}^{13}(n)) = \sum_{3 \leq j \leq 13} \binom{n}{j} C_j$$

where C_j, $3 \leq j \leq 13$, are determined by the following table:

j	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*
C_j	*3*	*23*	*105*	*268*	*415*	*438*	*322*	*164*	*55*	*11*	*1*

The following tables shows the results obtained using Theorem 1.2

n	1	2	3	4	5	6	7	8	9	10
$\dim(\mathbf{Q}^{13}(n))$	0	0	3	35	250	1303	5406	18920	58096	160824

n	11	12	13
$\dim(\mathbf{Q}^{13}(n))$	409200	969068	4241740

We use the convention that $\binom{n}{i} = 0$ if $n < i$.

In Section 2, we recall some results on admissible monomials and hit monomials in $\mathbf{P}(n)$. Theorem 1.2 will be proved in Section 3.

2 Preliminaries

In this section, we recall some results in Kameko[3], Singer[12], Sum[13] and Mothebe [5] on admissible monomials and hit monomials in $\mathbf{P}(n)$. These results will be used in Section 3.

Let $\alpha_i(a)$ denote the i-th coefficient in the binary expansion of a nonnegative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + ...$, for $\alpha_i(a) = 0$ or 1 and $i \geq 0$. Let $x = x_1^{a_1} x_2^{a_2} ... x_n^{a_n} \in \mathbf{P}(n)$. We define two sequences associated with x by

$$\begin{aligned} \omega(x) = & \ (\omega_0(x), \omega_1(x), ..., \omega_i(x), ...), \\ \sigma(x) = & \ (a_1, a_2, ..., a_n), \end{aligned}$$

where $\omega_i(x) = \sum_{1 \leq j \leq n} \alpha_i(a_j)$. $\omega(x)$ is called the **weight vector** of the monomial x and $\sigma(x)$ is called the **exponent vector** of the monomial x.

Given two sequences $p = (u_0, u_1, \ldots, u_l, 0, \ldots)$, $q = (v_0, v_1, \ldots, v_l, 0, \ldots)$, we say $p < q$ if there is a positive integer k such that $u_i = v_i$ for all $i < k$ and $u_k < v_k$. We are now in a position to define an order relation on monomials.

Definition 2.1. Let a, b be monomials in $\mathbf{P}(n)$. We say that $a < b$ if and only if one of the following holds:

1. $\omega(a) < \omega(b)$,

2. $\omega(a) = \omega(b)$ and $\sigma(a) < \sigma(b)$.

The order on the set of sequences of nonnegative integers is the lexicographical one.

Following Kameko [3] we define:

Definition 2.2. A monomial x is said to be **inadmissible** if there exists monomials $y_1, y_2, ..., y_k$ such that

$$x \equiv y_1 + y_2 + ... + y_k \mod \mathcal{A}^+\mathbf{P}(n) \text{ and } y_j < x, j = 1, 2, ..., k$$

x is said to be **admissible** if it is not inadmissible. The set of all the classes represented by the admissible monomials in $\mathbf{P}(n)$ is a basis for $\mathbf{Q}(n)$.

The following Theorem is our main tool.

Theorem 2.3. (Kameko[*3*], Sum[*13*]). *Let x, w be monomials in $\boldsymbol{P}(n)$ such that $\omega_i(x) = 0$ for $i > r > 0$. If w is inadmissible, then xw^{2^r} is also inadmissible.*

Proposition 1. *If $a = x_1^{m_1}...x_n^{m_n} \in \boldsymbol{P}(n)$ is an admissible monomial then $m_1 = 2^\lambda - 1$ for some $\lambda \geq 0$.*

Proof. The lemma is clearly true if $n = 1$. Suppose that $m_1 = 2^\lambda - 2$. Let $b = x_1^{e_1}...x_n^{e_n}$ be the monomial obtained from a by replacing m_1 by
$2^\lambda - 3$. Then $a = Sq^1(b) + x_1^{2^\lambda - 3}Sq^1(x_2^{m_2}...x_n^{m_n})$ and the fact that all terms in $x_1^{2^\lambda-3}Sq^1(x_2^{m_2}...x_n^{m_n})$ are of lower order than a shows that a is inadmissible. But every monomial with $m_1 \neq 2^\lambda - 1$ is of the form cd^{2r} for some monomial $d = x_1^{t_1}...x_n^{t_n}$ with $t_1 = 2^\lambda - 2$ so the general result follows from Theorem 2.3. □

Now, we recall a result of Singer[12] on hit monomials in $\mathbf{P}(n)$.

Definition 2.4. A monomial $z = x_1^{a_1}x_2^{a_2}...\,x_n^{a_n}$ is called a spike if $a_j = 2^{s_j} - 1$ for s_j a nonnegative integer and $j = 1, 2, ..., n$. For convenience we assume that $s_1 \geq s_2 \geq ...s_n \geq 0$. If z is the spike with $s_1 \geq s_2 \geq ... \geq s_{r-1} \geq s_r \geq 0$ and $s_{j-1} = s_j$ only if $j = r$ or $s_{j+1} = 0$, then it is called the **minimal spike**.

Note that the minimal spike is the spike of lowest weight order amongst all spikes in a given degree. Spikes are admissible monomials as a spike can never appear as term in a hit polynomial.

The following is a useful criterion for hit monomials in $\mathbf{P}(n)$.

Theorem 2.5. (Singer[*12*]). *Suppose $x \in \boldsymbol{P}(n)$ is a monomial of degree d, where $\alpha(d + n) \leq n$. Let z be the minimal spike of degree d. If $\omega(x) < \omega(z)$, then x is hit.*

We note the following stronger version of Theorem 2.5. Let b be a monomial of degree d. For $l > 0$ define $d_l(b)$ to be the integer $d_l(b) = \sum_{j \geq l} \omega_j(b)2^{j-l}$.

In [10] Silverman proved that:

Theorem 2.6. (Silverman[*10*]). *Let $b \in \boldsymbol{P}(n)$ be a monomial of degree d, where $\alpha(n + d) \leq n$. Let v be the minimal spike of degree d. If $d_l(b) > d_l(v)$ for some $l \geq 1$, then b is hit.*

We shall require the following result of Mothebe[5]:

Theorem 2.7. (Mothebe[5]). *If* $u = x_1^{m_1} \dots x_k^{m_k} \in \boldsymbol{P}^d(k)$ *and* $v = x_1^{e_1} \dots x_r^{e_r} \in \boldsymbol{P}^{d'}(r)$ *are admissible monomials, then for each permutation* $\sigma \in S_{k+r}$ *for which* $\sigma(i) < \sigma(j)$, $i < j \leq k$ *and* $\sigma(s) < \sigma(t)$, $k < s < t \leq k + r$, *the monomial*

$$x_{\sigma(1)}^{m_1} \dots x_{\sigma(k)}^{m_k} x_{\sigma(k+1)}^{e_1} \dots x_{\sigma(k+r)}^{e_r} \in \boldsymbol{P}^{d+d'}(k+r)$$

is admissible.

Theorem 2.7 is a generalization of the following result of Mothebe and Uys [6]:

Notation 2.8. Let $u = x_1^{m_1} \dots x_{n-1}^{m_{n-1}} \in \mathbf{P}(n-1)$ be a monomial of degree d'. Given any pair of integers (j, λ), $1 \leq j \leq n$, $\lambda \geq 0$, we write $h_j^\lambda(u)$ for the monomial $x_1^{m_1} \dots x_{j-1}^{m_{j-1}} x_j^{2^\lambda - 1} x_{j+1}^{m_j} \dots x_n^{m_{n-1}} \in \mathbf{P}^{d' + (2^\lambda - 1)}(n)$.

Theorem 2.9. (Mothebe, Uys[6]). *Let* $u \in \boldsymbol{P}(n-1)$ *be a monomial of degree* d', *where* $\alpha(d' + n - 1) \leq n - 1$. *If* u *is admissible, then for each pair of integers* (j, λ), $1 \leq j \leq n$, $\lambda \geq 0$, $h_j^\lambda(u)$ *is admissible.*

As a corollary to Theorem 2.9, suppose that λ is fixed and that d' is also fixed. Let $x_1^{2^{\lambda_1} - 1} \dots x_{n-2}^{2^{\lambda_n} - 2}$ be the minimal spike of degree d'. If $\lambda \geq \lambda_1$, then for all j, $1 \leq j \leq n$, we have

Corollary 2.10. $u = x_1^{m_1} \dots x_{n-1}^{m_{n-1}} \in \mathbf{P}^{d'}(n-1)$ is admissible if and only if $h_j^\lambda(u) \in \mathbf{P}^d(n)$ is admissible.

In Corollary 2.10 the converse is an immediate consequence of Theorem 2.6 and the Cartan formula.

One of the main tools in the study of the hit problem is Kameko's squaring operation $\widetilde{Sq}_*^0 : \mathbf{Q}(n) \to \mathbf{Q}(n)$. This homomorphism is induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}_*^0 : \mathbf{P}(n) \to \mathbf{P}(n)$, given by

$$\widetilde{Sq}_*^0(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \dots x_n y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in \mathbf{P}(n)$.

For any nonnegative integer k, set $\mu(k) = \min\{m \in \mathbb{Z} : \alpha(m + k) \leq m\}$.

Theorem 2.11. (Kameko[3]). *Let* d' *be a positive integer. If* $\mu(2d' + n) = n$, *then* $(\widetilde{Sq}_*^0)_m : \boldsymbol{Q}^{2d'+n}(n) \to \boldsymbol{Q}^{d'}(n)$ *is an isomorphism of* $\mathbb{F}_2$*-vector spaces.*

Let $r,\ 1 \leq r \leq n$, be an integer. For latter use, we let $\mathbf{P}_0(r)$ be the subspace of $\mathbf{P}(r)$ generated by monomials $x = x_1^{a_1}x_2^{a_2}...x_r^{a_r}$ for which $a_1a_2...a_r = 0$ and $\mathbf{P}_+(r)$ be the subspace of $\mathbf{P}(r)$ generated by monomials $x = x_1^{a_1}x_2^{a_2}...x_r^{a_r}$ for which $a_1a_2...a_r \neq 0$. It is easy to see that $\mathbf{P}_0(r)$ and $\mathbf{P}_+(r)$ are $\mathcal{A}$-submodules of $\mathbf{P}(r)$. Furthermore, we have $\mathbf{Q}(r) = \mathbf{QP}_0(r) \bigoplus \mathbf{QP}_+(r)$ where

$$\mathbf{QP}_+(r) = \mathbf{P}_+(r)/\mathcal{A}^+\mathbf{P}_+(r)$$

and

$$\mathbf{QP}_0(r) = \mathbf{P}_0(r)/\mathcal{A}^+\mathbf{P}_0(r).$$

Then for each $n \geq 1$ we have a direct sum decomposition:

$$\mathbf{Q}(n) \cong \bigoplus_{r=1}^{n} \bigoplus_{k=1}^{\binom{n}{r}} \mathbf{QP}_+(r)$$

Thus for any integer $d > 0$ we have the following inexplicit formula for the dimension of $\mathbf{Q}^d(n)$

$$\dim(\mathbf{Q}^d(n)) = \sum_{r=1}^{n} \binom{n}{r} \dim(\mathbf{QP}_+^d(r)) \tag{1}$$

Our main result is obtained by evaluating Formula (1) explicitly when $d = 13$.

For any $I = (i_0, i_1, ..., i_r), 0 < i_0 < i_1 < ... < i_r \leq k, 0 \leq r < k$, we define the homomorphism $p_I : \mathbf{P}(n) \to \mathbf{P}(n-1)$ of algebras by substituting

$$p_I(x_j)) = \begin{cases} x_j, & \text{if } 1 \leq j < i_0, \\ \sum_{0 \leq s \leq r} x_{i_s - 1}, & \text{if } j = \ i_0, \\ x_{j-1}, & \text{if } i_0 < j \leq n. \end{cases}$$

Then p_I is a homomorphism of $\mathcal{A}$-modules. In particular, for $I = (i)$, we have $p_i(x_i) = 0$.

3 Proof of Theorem 1.2

The result of Theorem 1.2 is a consequence of Lemma 3.1 which we prove below.

Lemma 3.1.

1. $\dim(\mathbf{QP}_+^n(n)) = 1$ for all $n \geq 1$
2. $\dim(\mathbf{QP}_+^n(n-1)) = n-2$ for all $n \geq 3$
3. $\dim(\mathbf{QP}_+^n(n-2)) = \binom{n-2}{2}$ for all $n \geq 6$
4. $\dim(\mathbf{QP}_+^n(n-3)) = \binom{n-4}{3} + (n-3)(n-5)$ for all $n \geq 7$
5. $$\dim(\mathbf{QP}_+^n(n-4)) = \left(\binom{n-5}{4} - 1\right) + \binom{n-4}{2} + (n-4)\binom{n-6}{2} + \binom{n-5}{2} \quad \text{for all } n \geq 10$$
6. $$\begin{aligned}\dim(\mathbf{QP}_+^n(n-5)) = {} & n-6 + \frac{(n-5)!}{2(n-7)(n-9)!} + \binom{n-5}{2} + \binom{n-6}{2} \\ & + (n-5)\binom{n-7}{3} + 2\left(\sum_{i=2}^{n-8}\binom{i}{2}\right) + \binom{n-8}{2} \\ & + \left(\binom{n-7}{2} - 1\right) + \left(\binom{n-6}{5} - 3\right) \quad \text{for all } n \geq 13\end{aligned}$$
7. $$\begin{aligned}\dim(\mathbf{QP}_+^n(n-6)) \geq {} & n-6 + ((n-7)(n-8)-2) + \binom{n-6}{2} + \binom{n-6}{3} \\ & + (n-6)\binom{n-8}{2} + \frac{(n-7)!}{4(n-11)!} + (n-8)\binom{n-9}{2} \\ & + \binom{n-9}{2} + \left(\binom{n-6}{3} - 1\right) + 2\binom{n-7}{3} - (n-9) \\ & + (n-6)\left(\binom{n-8}{4} - 1\right) + (n-8)\binom{n-9}{3} \quad \text{for all } n \geq 13\end{aligned}$$
8. $\dim(\mathbf{QP}_+^{13}(6)) = 268$
9. $\dim(\mathbf{QP}_+^{13}(5)) = 105$
10. $\dim(\mathbf{QP}_+^{13}(4)) = 23$
11. $\dim(\mathbf{QP}_+^{13}(3)) = 3$

The cases $1-5$ are proved in [4], $6-7$ are partially dealt with in [4] and $10-11$ are known cases. Thus in this paper we shall justify the formulae in 6 and 7 and obtain results 8 and 9.

3.1 Dimension of $\mathbf{QP}^{13}_{+}(5)$

In this section we show that $\dim(\mathbf{QP}^{13}_{+}(5)) = 105$. We first note that in $\mathbf{P}^{13}_{+}(5)$ the monomials that do not meet the hit criterion of Theorem 2.6 are

$$a = x_1x_2^3x_3^3x_4^3x_5^3 \quad b = x_1x_2x_3x_4^3x_5^7 \quad c = x_1x_2x_3^3x_4^5x_5^3 \quad d = x_1x_2^2x_3^3x_4^3x_5^4$$
$$e = x_1x_2x_3^6x_4^3x_5^2 \quad f = x_1x_2x_3^2x_4^2x_5^7 \quad g = x_1x_2^3x_3^5x_4^2x_5^2$$

and all their permutation representatives. We know that a, b and all their permutation representatives are admissible. By Theorem 2.7 the following permutation representatives of c, d, e, f and g are admissible:

$$a_1 = x_1x_2^3x_3^3x_4^5x_5 \quad a_2 = x_1^3x_2x_3^3x_4^5x_5 \quad a_3 = x_1^3x_2^3x_3x_4^5x_5 \quad a_4 = x_1^3x_2^3x_3^5x_4x_5$$
$$a_5 = x_1x_2^3x_3^5x_4^3x_5 \quad a_6 = x_1^3x_2x_3^5x_4^3x_5 \quad a_7 = x_1^3x_2^5x_3x_4^3x_5 \quad a_8 = x_1^3x_2^5x_3^3x_4x_5$$
$$a_9 = x_1x_2x_3^3x_4^3x_5^5 \quad a_{10} = x_1x_2^3x_3x_4^3x_5^5 \quad a_{11} = x_1x_2^3x_3^3x_4x_5^5 \quad a_{12} = x_1x_2x_3^3x_4^5x_5^3$$
$$a_{13} = x_1x_2^3x_3x_4^5x_5^3 \quad a_{14} = x_1x_2^3x_3^5x_4x_5^3 \quad a_{15} = x_1^3x_2x_3^5x_4x_5^3 \quad a_{16} = x_1^3x_2^5x_3x_4x_5^3$$
$$a_{17} = x_1^3x_2x_3x_4^3x_5^5 \quad a_{18} = x_1^3x_2x_3^3x_4x_5^5 \quad a_{19} = x_1^3x_2x_3x_4^5x_5^3 \quad a_{20} = x_1^3x_2^3x_3x_4x_5^5$$
$$a_{21} = x_1^3x_2x_3^3x_4^4x_5^2 \quad a_{22} = x_1x_2^3x_3^3x_4^4x_5^2 \quad a_{23} = x_1x_2^3x_3^4x_4^3x_5^2 \quad a_{24} = x_1x_2^3x_3^4x_4^2x_5^3$$
$$a_{25} = x_1^3x_2^3x_3x_4^4x_5^2 \quad a_{26} = x_1^3x_2x_3^4x_4^3x_5^2 \quad a_{27} = x_1^3x_2x_3^4x_4^2x_5^3 \quad a_{28} = x_1^3x_2x_3x_4^2x_5^6$$
$$a_{29} = x_1x_2^3x_3x_4^2x_5^6 \quad a_{30} = x_1x_2x_3^3x_4^2x_5^6 \quad a_{31} = x_1x_2x_3^2x_4^3x_5^6 \quad a_{32} = x_1x_2x_3^2x_4^6x_5^3$$
$$a_{33} = x_1^3x_2^3x_3^4x_4x_5^2 \quad a_{34} = x_1^3x_2^4x_3^3x_4x_5^2 \quad a_{35} = x_1^3x_2^4x_3x_4^3x_5^2 \quad a_{36} = x_1^3x_2^4x_3x_4^2x_5^3$$
$$a_{37} = x_1^3x_2x_3x_4^6x_5^2 \quad a_{38} = x_1x_2^3x_3x_4^6x_5^2 \quad a_{39} = x_1x_2x_3^3x_4^6x_5^2 \quad a_{40} = x_1x_2x_3^6x_4^3x_5^2$$
$$a_{41} = x_1x_2x_3^6x_4^2x_5^3 \quad a_{42} = x_1^3x_2x_3^2x_4x_5^6 \quad a_{43} = x_1x_2^3x_3^2x_4x_5^6 \quad a_{44} = x_1x_2^2x_3^3x_4x_5^6$$
$$a_{45} = x_1x_2^2x_3x_4^3x_5^6 \quad a_{46} = x_1x_2^2x_3x_4^6x_5^3 \quad a_{47} = x_1^3x_2x_3^6x_4x_5^2 \quad a_{48} = x_1x_2^3x_3^6x_4x_5^2$$
$$a_{49} = x_1x_2^6x_3^3x_4x_5^2 \quad a_{50} = x_1x_2^6x_3x_4^3x_5^2 \quad a_{51} = x_1x_2^6x_3x_4^2x_5^3 \quad a_{52} = x_1^3x_2x_3^2x_4^5x_5^2$$
$$a_{53} = x_1x_2^3x_3^2x_4^5x_5^2 \quad a_{54} = x_1x_2^2x_3^3x_4^5x_5^2 \quad a_{55} = x_1x_2^2x_3^5x_4^3x_5^2 \quad a_{56} = x_1x_2^2x_3^5x_4^2x_5^3$$
$$a_{57} = x_1^3x_2^3x_3x_4^2x_5^4 \quad a_{58} = x_1^3x_2x_3^3x_4^2x_5^4 \quad a_{59} = x_1^3x_2x_3^2x_4^3x_5^4 \quad a_{60} = x_1^3x_2x_3^2x_4^4x_5^3$$
$$a_{61} = x_1x_2^3x_3^3x_4^2x_5^4 \quad a_{62} = x_1x_2^3x_3^2x_4^3x_5^4 \quad a_{63} = x_1x_2^3x_3^2x_4^4x_5^3 \quad a_{64} = x_1x_2^2x_3^3x_4^3x_5^4$$
$$a_{65} = x_1x_2^2x_3^3x_4^4x_5^3 \quad a_{66} = x_1x_2^2x_3^4x_4^3x_5^3 \quad a_{67} = x_1^7x_2x_3x_4^2x_5^2 \quad a_{68} = x_1x_2^7x_3x_4^2x_5^2$$
$$a_{69} = x_1x_2x_3^7x_4^2x_5^2 \quad a_{70} = x_1x_2x_3^2x_4^7x_5^2 \quad a_{71} = x_1x_2x_3^2x_4^2x_5^7 \quad a_{72} = x_1^7x_2x_3^2x_4x_5^2$$
$$a_{73} = x_1x_2^7x_3^2x_4x_5^2 \quad a_{74} = x_1x_2^2x_3^7x_4x_5^2 \quad a_{75} = x_1x_2^2x_3x_4^7x_5^2 \quad a_{76} = x_1x_2^2x_3x_4^2x_5^7$$
$$a_{77} = x_1x_2^3x_3^5x_4^2x_5^2 \quad a_{78} = x_1^3x_2x_3^5x_4^2x_5^2 \quad a_{79} = x_1^3x_2^5x_3x_4^2x_5^2 \quad a_{80} = x_1^3x_2^5x_3^2x_4x_5^2$$

By the Kameko homomorphism the monomials a_1 to a_{20} are the only permutation representatives of c that are admissible. By Corollary 2.10 a permutation representative of f is admissible if and only if a permutation representative of $x_1x_2x_3^2x_4^2$ is admissible in $\mathbf{P}^6(4)$. Thus the monomials a_{67} to a_{76} are the only permutation representatives of f that are admissible. The monomials d, e, g and their permutation representatives are of the form ux_j^3, $1 \leq j \leq 5$, for some $u \in \mathbf{P}^{10}(4)$. In the same vein as for f every permutation representative of the monomial e is admissible if and only if the monomial $u \in \mathbf{P}^{10}(4)$ is admissible. Thus the above listed permutation representatives of e are the only ones that are admissible. It is easy to see that all the permutation representatives of d and g which are not in the list above are in inadmissible . For example, $x_1^3x_2^5x_3^2x_4^2x_5$ is inadmissible since

$$x_1^3x_2^5x_3^2x_4^2x_5 \equiv x_1^3x_2^5x_3^2x_4x_5^2 + x_1^3x_2^5x_3x_4^2x_5^2 \mod \mathcal{A}^+\mathbf{P}(5)$$

The monomials a and b have 5 and 20 permutation representatives respectively. Thus the number of monomials in a basis for $\mathbf{QP}_+^{13}(5)$ is $80 + 20 + 5 = 105$.That is, $\dim(\mathbf{QP}_+^{13}(5)) = 105$.

3.2 Dimension of $\mathbf{QP}_+^{13}(6)$

We now show that $\dim(\mathbf{QP}_+^{13}(6)) = 268$.

If $n \geq 12$, then the monomials in the basis of $\mathbf{P}_+^n(n-7)$ with $\omega_0(-) = n-8$ and that do not meet the hit criterion of Theorem 2.6 are

$$\begin{aligned}
a_{n-7} &= x_1 ...\, x_{n-9}x_{n-8}^2x_{n-7}^7 \\
b_{n-7} &= x_1 ...\, x_{n-9}x_{n-8}^3x_{n-7}^6 \\
c_{n-7} &= x_1 ...\, x_{n-10}x_{n-9}^3x_{n-8}^3x_{n-7}^4 \\
d_{n-7} &= x_1 ...\, x_{n-11}x_{n-10}^2x_{n-9}^3x_{n-8}^3x_{n-7}^3 \\
e_{n-7} &= x_1 ...\, x_{n-10}x_{n-9}^2x_{n-8}^3x_{n-7}^5
\end{aligned}$$

and their permutation representatives.

If $n \geq 13$, then the monomial in the basis of $\mathbf{P}_+^n(n-7)$ with weight order $(n-10, 5)$ and that does not meet the hit criterion of Theorem 2.6 is

$$g_{n-7} = x_1 ...\, x_{n-12}x_{n-11}^2x_{n-10}^2x_{n-9}^2x_{n-8}^3x_{n-7}^3$$

and its permutation representatives. If $n = 13$, then the monomials in the basis for $\mathbf{P}_+^{13}(6)$ with weight order $(3, 3, 1)$ are $x_1x_2x_3^2x_4^2x_5^2x_6^5$, $x_1x_2x_3^3x_4^2x_5^2x_6^4$, $x_1x_2x_3x_4^2x_5^2x_6^6$ and their permutation representatives.

We first determine the number of monomials in $\mathbf{P}_+^n(n-7)$ with $\omega_0(-) = n-8$

that are admissible. If $n = 12$, it is known that $\mathbf{QP}^{12}_{+}(5)$ is generated by

$$
\begin{array}{llll}
b_1 = x_1x_2x_3x_4^7x_5^2 & b_2 = x_1x_2x_3^7x_4x_5^2 & b_3 = x_1x_2x_3^7x_4^2x_5 & b_4 = x_1x_2x_3x_4^2x_5^7 \\
b_5 = x_1x_2x_3^2x_4x_5^7 & b_6 = x_1x_2x_3^2x_4^7x_5 & b_7 = x_1x_2^2x_3x_4x_5^7 & b_8 = x_1x_2^2x_3x_4^7x_5 \\
b_9 = x_1x_2^7x_3x_4x_5^2 & b_{10} = x_1^7x_2x_3x_4x_5^2 & b_{11} = x_1^7x_2x_3x_4^2x_5 & b_{12} = x_1x_2^7x_3x_4^2x_5 \\
b_{13} = x_1^7x_2x_3^2x_4x_5 & b_{14} = x_1x_2^7x_3^2x_4x_5 & b_{15} = x_1x_2^2x_3^7x_4x_5 & b_{16} = x_1x_2x_3x_4^3x_5^6 \\
b_{17} = x_1x_2x_3^3x_4x_5^6 & b_{18} = x_1x_2x_3^3x_4^6x_5 & b_{19} = x_1x_2x_3x_4^6x_5^3 & b_{20} = x_1x_2x_3^6x_4x_5^3 \\
b_{21} = x_1x_2x_3^6x_4^3x_5 & b_{22} = x_1x_2^6x_3x_4x_5^3 & b_{23} = x_1x_2^6x_3x_4^3x_5 & b_{24} = x_1x_2^3x_3x_4x_5^6 \\
b_{25} = x_1^3x_2x_3x_4x_5^6 & b_{26} = x_1^3x_2x_3x_4^6x_5 & b_{27} = x_1x_2^3x_3x_4^6x_5 & b_{28} = x_1^3x_2x_3^6x_4x_5 \\
b_{29} = x_1x_2^3x_3^6x_4x_5 & b_{30} = x_1x_2^6x_3^3x_4x_5 & b_{31} = x_1x_2x_3^2x_4^3x_5^5 & b_{32} = x_1x_2^2x_3x_4^3x_5^5 \\
b_{33} = x_1x_2^2x_3^3x_4x_5^5 & b_{34} = x_1x_2^2x_3^3x_4^5x_5 & b_{35} = x_1x_2x_3^2x_4^5x_5^3 & b_{36} = x_1x_2^2x_3x_4^5x_5^3 \\
b_{37} = x_1x_2^2x_3^5x_4x_5^3 & b_{38} = x_1x_2^2x_3^5x_4^3x_5 & b_{39} = x_1x_2x_3^3x_4^2x_5^5 & b_{40} = x_1x_2^3x_3x_4^2x_5^5 \\
b_{41} = x_1x_2^3x_3^2x_4x_5^5 & b_{42} = x_1x_2^3x_3^2x_4^5x_5 & b_{43} = x_1x_2x_3^3x_4^5x_5^2 & b_{44} = x_1x_2^3x_3x_4^5x_5^2 \\
b_{45} = x_1x_2^3x_3^5x_4x_5^2 & b_{46} = x_1x_2^3x_3^5x_4^2x_5 & b_{47} = x_1^3x_2x_3x_4^2x_5^5 & b_{48} = x_1^3x_2x_3^2x_4x_5^5 \\
b_{49} = x_1^3x_2x_3^2x_4^5x_5 & b_{50} = x_1^3x_2x_3x_4^5x_5^2 & b_{51} = x_1^3x_2x_3^5x_4x_5^2 & b_{52} = x_1^3x_2x_3^5x_4x_5^2 \\
b_{53} = x_1^3x_2^5x_3x_4x_5^2 & b_{54} = x_1^3x_2^5x_3x_4^2x_5 & b_{55} = x_1^3x_2^5x_3^2x_4x_5 & b_{56} = x_1x_2x_3^3x_4^3x_5^4 \\
b_{57} = x_1x_2^3x_3x_4^3x_5^4 & b_{58} = x_1x_2^3x_3^3x_4x_5^4 & b_{59} = x_1x_2^3x_3^3x_4^4x_5 & b_{60} = x_1x_2x_3^3x_4^4x_5^3 \\
b_{61} = x_1x_2^3x_3x_4^4x_5^3 & b_{62} = x_1x_2^3x_3^4x_4x_5^3 & b_{63} = x_1x_2^3x_3^4x_4^3x_5 & b_{64} = x_1^3x_2x_3x_4^3x_5^4 \\
b_{65} = x_1^3x_2x_3^3x_4x_5^4 & b_{66} = x_1^3x_2x_3^3x_4^4x_5 & b_{67} = x_1^3x_2x_3x_4^4x_5^3 & b_{68} = x_1^3x_2x_3^4x_4x_5^3 \\
b_{69} = x_1^3x_2x_3^4x_4^3x_5 & b_{70} = x_1^3x_2^3x_3x_4x_5^4 & b_{71} = x_1^3x_2^3x_3x_4^4x_5 & b_{72} = x_1^3x_2^3x_3^4x_4x_5 \\
b_{73} = x_1^3x_2^4x_3x_4x_5^3 & b_{74} = x_1^3x_2^4x_3x_4^3x_5 & b_{75} = x_1^3x_2^4x_3^3x_4x_5 & b_{76} = x_1^3x_2x_3^2x_4^3x_5^3 \\
b_{77} = x_1x_2^3x_3^2x_4^3x_5^3 & b_{78} = x_1x_2^2x_3^3x_4^3x_5^3 & b_{79} = x_1^3x_2x_3^3x_4^2x_5^3 & b_{80} = x_1x_2^3x_3^3x_4^2x_5^3 \\
b_{81} = x_1^3x_2x_3^3x_4^3x_5^2 & b_{82} = x_1x_2^3x_3^3x_4^3x_5^2 & b_{83} = x_1^3x_2^3x_3x_4^2x_5^3 & b_{84} = x_1^3x_2^3x_3x_4^3x_5^2 \\
b_{85} = x_1^3x_2^3x_3^3x_4x_5^2 & & &
\end{array}
$$

For all $n \geq 12$, let $\mathbf{Q}^*\mathbf{P}^n_+(n-7)$ denote the set of all admissible monomials in $\mathbf{P}^n_+(n-7)$ with $\omega_0(-) = n-8$. Note that $\mathbf{Q}^*\mathbf{P}^{12}_+(5) = \mathbf{QP}^{12}_+(5)$. Then we claim that:

$$
\mathbf{Q}^*\mathbf{P}^{n+1}_+(n-6) = \bigcup_{j=1}^{n-6} h^1_j(\mathbf{Q}^*\mathbf{P}^n_+(n-7)). \qquad (2)
$$

The only permutation representatives of a_{n-7} that may not be obtained from $\mathbf{QP}^{12}_+(5)$ by inductively applying Formula (2) are those of the form $x_1^2x_2^{m_2}...\,x_{n-7}^{m_{n-7}}$ as well as the monomial $x_1^7x_2^2x_3...\,x_{n-7}$ all of which are clearly inadmissible. Hence the number of permutation representatives of a_{n-7} that are admissible is $(n-7)(n-9)$.

The only permutations representatives of b_{n-7} that may not be obtained from $\mathbf{QP}^{12}_{+}(5)$ by inductively applying Formula (2) are those of the form $x_1^6x_2^{m_2}...\,x_{n-7}^{m_{n-7}}$ as well as the monomial $x_1^3x_2^6x_3...\,x_{n-7}$ all of which are clearly inadmissible. Hence the number of permutation representatives of b_{n-7} that are admissible is $(n-7)(n-9)$.

The only permutations representatives of c_{n-7} that may not be obtained from $\mathbf{QP}^{12}_{+}(5)$ by inductively applying Formula (2) are those of the form $x_1^4x_2^{m_2}...\,x_{n-7}^{m_{n-7}}$, $x_1x_2^4x_3^{m_3}...\,x_{n-7}^{m_{n-7}}$ as well as those of the form $x_1x_2x_3^4x_4...\,x_{n-7}^{m_{n-7}}$ all of which are clearly inadmissible. Hence the number of permutation representatives of c_{n-7} that are admissible is $2\binom{n-7}{3}$.

The only permutations representatives of d_{n-7} that may not be obtained from $\mathbf{QP}^{12}_{+}(5)$ by inductively applying Formula (2) are those of the form $x_1^2x_2^{m_2}...\,x_{n-7}^{m_{n-7}}$, $x_1^3x_2^2x_3^{m_3}...\,x_{n-7}^{m_{n-7}}$, $x_1^3x_2^3x_3^2x_4^{m_3}...\,x_{n-7}^{m_{n-7}}$ as well as the monomial $x_1^3x_2^3x_3^3x_4^2x_5...\,x_{n-7}$ all of which are clearly inadmissible. Hence the number of permutation representatives of d_{n-7} that are admissible is $(n-11)\binom{n-7}{3}$.

The only permutations representatives of e_{n-7} that may not be obtained from $\mathbf{QP}^{12}_{+}(5)$ by inductively applying Formula (2) are those with a factor of the form $x_i^5x_j^3x_k^2, i<j<k$, or of the form $x_t^5x_r^2x_s^3, t<r<s$, or those of the form $x_1^2x_2^{m_2}...\,x_{n-7}^{m_{n-7}}$ or of the form $x_1^3x_2^2x_3^{m_3}...\,x_{n-7}^{m_{n-7}}$, all of which are clearly inadmissible. Hence the number of permutation representatives of e_{n-7} that are admissible is $\binom{n-7}{3}+3\binom{n-8}{3}+\binom{n-9}{2}$.

Thus for all $n\geq 12$ the number of admissible monomials in $\mathbf{P}^{n}_{+}(n-7)$ with $\omega_0(-)=n-8$ is equal to

$$2(n-7)(n-9)+2\binom{n-7}{3}+(n-11)\binom{n-7}{3}$$
$$+\binom{n-7}{3}+3\binom{n-8}{3}+\binom{n-9}{2}.$$

If $n=13$ this formula yields 184 admissible monomials with $\omega_0(-)=5$.

If $n=13$, then $g_{n-7}=x_1x_2^2x_3^2x_4^2x_5^3x_6^3$. By Theorem 2.7, the following permutation representative of $x_1x_2^2x_3^2x_4^2x_5^3x_6^3$ are admissible in $\mathbf{P}^{13}_{+}(6)$

$$\begin{array}{lll}
f_1=x_1^3x_2^3x_3x_4^2x_5^2x_6^2 & f_2=x_1^3x_2x_3^3x_4^2x_5^2x_6^2 & f_3=x_1^3x_2x_3^2x_4^3x_5^2x_6^2\\
f_4=x_1^3x_2x_3^2x_4^3x_5^2x_6^2 & f_5=x_1^3x_2x_3^2x_4^2x_5^2x_6^3 & f_6=x_1x_2^3x_3^3x_4^2x_5^2x_6^2\\
f_7=x_1x_2^3x_3^2x_4^3x_5^2x_6^2 & f_8=x_1x_2^3x_3^2x_4^2x_5^3x_6^2 & f_9=x_1x_2^3x_3^2x_4^2x_5^2x_6^3\\
f_{10}=x_1x_2^2x_3^3x_4^3x_5^2x_6^2 & f_{11}=x_1x_2^2x_3^3x_4^2x_5^3x_6^2 & f_{12}=x_1x_2^2x_3^3x_4^2x_5^2x_6^3\\
f_{13}=x_1x_2^2x_3^2x_4^3x_5^3x_6^2 & f_{14}=x_1x_2^2x_3^2x_4^3x_5^2x_6^3 & f_{15}=x_1x_2^2x_3^2x_4^2x_5^3x_6^3
\end{array}$$

We see that if $x\neq f_t, \forall t, 1\leq t\leq 15$, then x is of the form $x_1^2x_2^{m_2}...\,x_{n-7}^{m_{n-7}}$, $x_1^3x_2^2x_3^{m_3}...\,x_{n-7}^{m_{n-7}}$ or of the form $x_1^3x_2^3x_3^2x_4^{m_3}...\,x_{n-7}^{m_{n-7}}$, all of which are clearly inadmissible. For example

$$x_1^3x_2^3x_3^2x_4x_5^2x_6^2 \equiv x_1^3x_2^3x_3x_4^2x_5^2x_6^2 + x_1^4x_2^3x_3x_4x_5^2x_6^2 + x_1^3x_2^4x_3x_4x_5^2x_6^2 \mod \mathcal{A}^+\mathbf{P}(6).$$

Thus there are 15 permutation representatives of $x_1x_2^2x_3^2x_4^2x_5^3x_6^3$ which are admissible.

We now determine the admissible permutation representatives of the monomials of weight order $(3,3,1)$. By Theorem 2.7, the following permutation representative of $x_1x_2x_3^2x_4^2x_5^3x_6^4$ are admissible in $\mathbf{P}_+^{13}(6)$

$$\begin{array}{lll}
c_1 = x_1^3x_2x_3^2x_4x_5^2x_6^4 & c_2 = x_1x_2^3x_3^2x_4x_5^2x_6^4 & c_3 = x_1x_2^2x_3^3x_4x_5^2x_6^4 \\
c_4 = x_1x_2^2x_3x_4^3x_5^2x_6^4 & c_5 = x_1x_2^2x_3x_4^2x_5^3x_6^4 & c_6 = x_1x_2^2x_3x_4^2x_5^4x_6^3 \\
c_7 = x_1^3x_2x_3x_4^2x_5^2x_6^4 & c_8 = x_1x_2^3x_3x_4^2x_5^2x_6^4 & c_9 = x_1x_2x_3^3x_4^2x_5^2x_6^4 \\
c_{10} = x_1x_2x_3^2x_4^3x_5^2x_6^4 & c_{11} = x_1x_2x_3^2x_4^2x_5^3x_6^4 & c_{12} = x_1x_2x_3^2x_4^2x_5^4x_6^3 \\
c_{13} = x_1^3x_2x_3x_4^2x_5^4x_6^2 & c_{14} = x_1x_2^3x_3x_4^2x_5^4x_6^2 & c_{15} = x_1x_2x_3^3x_4^2x_5^4x_6^2 \\
c_{16} = x_1x_2x_3^2x_4^3x_5^4x_6^2 & c_{17} = x_1x_2x_3^2x_4^4x_5^3x_6^2 & c_{18} = x_1x_2x_3^2x_4^4x_5^2x_6^3 \\
c_{19} = x_1^3x_2x_3^2x_4x_5^4x_6^2 & c_{20} = x_1x_2^3x_3^2x_4x_5^4x_6^2 & c_{21} = x_1x_2^2x_3^3x_4x_5^4x_6^2 \\
c_{22} = x_1x_2^2x_3x_4^3x_5^4x_6^2 & c_{23} = x_1x_2^2x_3x_4^4x_5^3x_6^2 & c_{24} = x_1x_2^2x_3x_4^4x_5^2x_6^3 \\
c_{25} = x_1^3x_2x_3^2x_4^4x_5x_6^2 & c_{26} = x_1x_2^3x_3^2x_4^4x_5x_6^2 & c_{27} = x_1x_2^2x_3^3x_4^4x_5x_6^2 \\
c_{28} = x_1x_2^2x_3^4x_4^3x_5x_6^2 & c_{29} = x_1x_2^2x_3^4x_4x_5^3x_6^2 & c_{30} = x_1x_2^2x_3^4x_4x_5^2x_6^3 \\
c_{31} = x_1x_2x_3^3x_4^4x_5^2x_6^2 & c_{32} = x_1x_2^3x_3^4x_4x_5^2x_6^2 & c_{33} = x_1x_2^3x_3^4x_4^2x_5x_6^2 \\
c_{34} = x_1x_2^3x_3x_4^4x_5^2x_6^2 & c_{35} = x_1^3x_2x_3x_4^4x_5^2x_6^2 & c_{36} = x_1^3x_2x_3^4x_4x_5^2x_6^2 \\
c_{37} = x_1^3x_2x_3^4x_4^2x_5x_6^2 & c_{38} = x_1^3x_2^4x_3x_4x_5^2x_6^2 & c_{39} = x_1^3x_2^4x_3x_4^2x_5x_6^2
\end{array}$$

We see that if $x \neq c_t, \forall t, 1 \leq t \leq 39$, then x is of one of the following forms:

(i) $x_1^2x_2^{m_2}...x_6^{m_6}$,

(ii) $x_1^4x_2^{m_2}...x_6^{m_6}$,

(iii) $x_1^3x_2^2x_3^{m_3}...x_6^{m_6}$,

(iv) $x_1x_2^4x_3^{m_3}...x_6^{m_6}$,

(v) $x_1x_2x_3^4x_4^{m_4}...x_6^{m_6}$,

(vi) $x_1x_2^2x_3^2x_4^{m_4}...x_6^{m_6}$,

(vii) $x_1^3x_2^4x_3^2x_4^{m_4}...x_6^{m_6}$,

(viii) $x_1^3x_2^4x_3x_4^2x_5^2x_6$

(ix) $x_1^3x_2x_3^2x_4^2x_5^{m_5}x_6^{m_6}$,

(x) $x_1^3x_2x_3^2x_4^4x_5^2x_6$,

(xi) $x_1x_2^3x_3^2x_4^2x_5^{m_5}x_6^{m_6}$,

(xii) $x_1x_2^3x_3^2x_4^4x_5^2x_6$,

(xiii) $x_1x_2^2x_3^3x_4^2x_5^{m_5}x_6^{m_6}$,

(xiv) $x_1x_2^3x_3^4x_4^2x_5^2x_6$,

(xv) $x_1x_2^2x_3^3x_4^4x_5^2x_6$,

(xvi) $x_1x_2^2x_3^4x_4^2x_5^{m_5}x_6^{m_6}$,

(xvii) $x_1x_2^2x_3^4x_4^3x_5^2x_6$,

(xviii) $x_1^3x_2x_3^4x_4^2x_5^2x_6$,

all of which are clearly inadmissible. For example

$$x_1^3x_2x_3^2x_4^2x_5x_6^4 \equiv x_1^3x_2x_3^2x_4x_5^2x_6^4 + x_1^3x_2x_3x_4^2x_5^2x_6^4 \mod \mathcal{A}^+\mathbf{P}(6).$$

We claim that the following permutation representative of $x_1x_2x_3x_4^2x_5^2x_6^6$ are admissible in $\mathbf{P}_+^{13}(6)$

$$\begin{array}{lll}
c_{40} = x_1x_2x_3^2x_4^2x_5x_6^6 & c_{41} = x_1x_2^2x_3x_4^2x_5x_6^6 & c_{42} = x_1x_2^2x_3x_4^6x_5x_6^2 \\
c_{43} = x_1x_2x_3^2x_4^6x_5x_6^2 & c_{44} = x_1x_2x_3^2x_4x_5^6x_6^2 & c_{45} = x_1x_2x_3^2x_4x_5^2x_6^6 \\
c_{46} = x_1x_2x_3x_4^2x_5^2x_6^6 & c_{47} = x_1x_2^6x_3x_4x_5^2x_6^2 & c_{48} = x_1x_2^6x_3x_4^2x_5x_6^2 \\
c_{49} = x_1x_2^2x_3x_4x_5^2x_6^6 & c_{50} = x_1x_2^2x_3x_4x_5^6x_6^2 & c_{51} = x_1x_2x_3^6x_4x_5^2x_6^2 \\
c_{52} = x_1x_2x_3^6x_4^2x_5x_6^2 & c_{53} = x_1x_2x_3x_4^2x_5^6x_6^2 & c_{54} = x_1x_2x_3x_4^6x_5^2x_6^2 \\
c_{55} = x_1x_2x_3^2x_4^2x_5^6x_6 & c_{56} = x_1x_2x_3^2x_4^6x_5^2x_6 & c_{57} = x_1x_2x_3^6x_4^2x_5^2x_6 \\
c_{58} = x_1x_2^6x_3x_4^2x_5^2x_6 & c_{59} = x_1x_2^2x_3x_4^2x_5^6x_6 & c_{60} = x_1x_2^2x_3x_4^6x_5^2x_6
\end{array}$$

We see that if $x \neq c_t, \forall t, 40 \leq t \leq 60$, then x is of one of the following forms:

(i) $x_1^2x_2^{m_2}...x_6^{m_6}$,

(ii) $x_1^6x_2^{m_2}...x_6^{m_6}$,

(iii) $x_1x_2^2x_3^2x_4^{m_4}...x_6^{m_6}$,

(iv) $x_1x_2^6x_3^2x_4^{m_4}...x_6^{m_6}$,

(v) $x_1x_2^2x_3^6x_4^{m_4}...x_6^{m_6}$,

all of which are which are clearly inadmissible. For example

$$\begin{aligned}
x_1x_2^6x_3^2x_4^2x_5x_6 \equiv\ & x_1x_2^6x_3^2x_4x_5^2x_6 + x_1x_2^6x_3^2x_4x_5x_6^2 + x_1x_2^6x_3x_4^2x_5^2x_6 \\
& + x_1x_2^6x_3x_4^2x_5x_6^2 + x_1x_2^6x_3x_4x_5^2x_6^2 \mod \mathcal{A}^+\mathbf{P}(6)
\end{aligned}$$

That the monomials $c_t, 40 \leq t \leq 54$, are admissible follows from Theorem 2.7.

We claim that the following permutation representative of $x_1x_2x_3^2x_4^2x_5^2x_6^5$ are admissible in $\mathbf{P}_+^{13}(6)$

$$\begin{array}{lll}
c_{61} = x_1x_2x_3^2x_4^2x_5^5x_6^2 & c_{62} = x_1x_2x_3^2x_4^5x_5^2x_6^2 & c_{63} = x_1x_2^2x_3x_4^2x_5^5x_6^2 \\
c_{64} = x_1x_2^2x_3^5x_4x_5^2x_6^2 & c_{65} = x_1x_2^2x_3^5x_4^2x_5x_6^2 & c_{66} = x_1x_2^2x_3x_4^5x_5^2x_6^2 \\
c_{67} = x_1x_2x_3^2x_4^2x_5^2x_6^5 & c_{68} = x_1x_2^2x_3x_4^2x_5^2x_6^5 & c_{69} = x_1x_2^2x_3^5x_4^2x_5^2x_6
\end{array}$$

We see that if $x \neq c_t, \forall t, 61 \leq t \leq 69$, then x is of one of the following forms:

(i) $x_1^5x_2^{m_2}...x_6^{m_6}$, (iii) $x_1x_2^2x_3^2x_4^{m_4}...x_6^{m_6}$, (v) $x_1x_2x_3^5x_4^{m_4}...x_6^{m_6}$,
(ii) $x_1^2x_2^{m_2}...x_6^{m_6}$, (iv) $x_1x_2^5x_3^{m_3}...x_6^{m_6}$,

all of which are clearly inadmissible. For example

$$\begin{aligned} x_1x_2^2x_3^2x_4x_5^2x_6^5 \equiv x_1x_2^2x_3x_4^2x_5^2x_6^5 + x_1x_2x_3^2x_4^2x_5^2x_6^5 + x_1x_2^2x_3x_4x_5^2x_6^6 \\ + x_1x_2x_3^2x_4x_5^2x_6^6 + x_1x_2x_3x_4^2x_5^2x_6^6 \mod \mathcal{A}^+\mathbf{P}(6) \end{aligned}$$

That the monomials $c_t, 61 \leq t \leq 66$, are admissible follows from Theorem 2.7.

Now, we prove that the set $\{c_t : 1 \leq t \leq 69\}$ is linearly independent. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1\leq t\leq 69} \gamma_t c_t \equiv 0$$

with $\gamma_t \in \mathbb{F}_2$, $1 \leq t \leq 69$. It is sufficient to prove that we must have $\gamma_t = 0$ for $55 \leq t \leq 60$ and $67 \leq t \leq 69$. By direct computation from the relations $p_{(i,j)}(\mathcal{S}) \equiv 0$, $1 \leq i \leq 5$, $1 < j \leq 6$, one gets $\gamma_{55} = \gamma_{56} = \gamma_{57} = \gamma_{58} = \gamma_{59} = \gamma_{60} = \gamma_{67} = \gamma_{68} = \gamma_{69} = 0$.

This shows that $\dim(\mathrm{QP}_+^{13}(6)) = 184 + 15 + 69 = 268$.

3.3 Dimension of $\mathbf{QP}_+^{13}(7)$

We now show that

$$\begin{aligned} \dim(\mathbf{QP}_+^n(n-6)) \geq\ & n - 6 + ((n-7)(n-8) - 2) + \binom{n-6}{2} + \binom{n-6}{3} \\ & + (n-6)\binom{n-8}{2} + \frac{(n-7)!}{4(n-11)!} + (n-8)\binom{n-9}{2} \\ & + \binom{n-9}{2} + \left(\binom{n-6}{3} - 1\right) + 2\binom{n-7}{3} \\ & - (n-9) + (n-6)\left(\binom{n-8}{4} - 1\right) + (n-8)\binom{n-9}{3} \end{aligned}$$

with equality when $n = 13$. In [4] it is shown that if $n \geq 11$ the monomials in $\mathbf{P}^n_+(n-6)$ that do not meet the hit criterion of Theorem 2.6 are :

$$\begin{aligned}
a_{n-6} &= x_1 ...\ x_{n-9}x^3_{n-8}x^3_{n-7}x^3_{n-6} \\
b_{n-6} &= x_1 ...\ x_{n-7}x^7_{n-6} \\
c_{n-6} &= x_1 ...\ x_{n-8}x^3_{n-7}x^5_{n-6} \\
d_{n-6} &= x_1 ...\ x_{n-10}x^2_{n-9}x^2_{n-8}x^3_{n-7}x^3_{n-6} \\
e_{n-6} &= x_1x_2 ...\ x_{n-9}x^2_{n-8}x^3_{n-7}x^4_{n-6} \\
f_{n-6} &= x_1x_2 ...\ x_{n-8}x^2_{n-7}x^6_{n-6} \\
g_{n-6} &= x_1x_2 ...\ x_{n-9}x^2_{n-8}x^2_{n-7}x^5_{n-6}
\end{aligned}$$

and their permutation representatives, while

$$\begin{aligned}
h_{n-6} &= x_1x_2 ...\ x_{n-11}x^2_{n-10}x^2_{n-9}x^2_{n-8}x^2_{n-7}x^3_{n-6} \\
k_{n-6} &= x_1x_2 ...\ x_{n-10}x^2_{n-9}x^2_{n-8}x^2_{n-7}x^4_{n-6}
\end{aligned}$$

and their permutation representatives have to be added to the list when $n \geq 13$ and

$$l_{n-6} = x_1x_2 ...\ x_{n-12}x^2_{n-11}x^2_{n-10}x^2_{n-9}x^2_{n-8}x^2_{n-7}x^2_{n-6}$$

and its permutation representatives has to be added to the list when $n \geq 14$. In [4] it is shown that the number of permutation representatives of a_{n-6}, b_{n-6}, c_{n-6}, d_{n-6}, e_{n-6}, f_{n-6}, g_{n-6} that are admissible is

$$\begin{aligned}
&n-6+((n-7)(n-8)-2)+\binom{n-6}{2}+\binom{n-6}{3} \\
&+(n-6)\binom{n-8}{2}+\frac{(n-7)!}{4(n-11)!}+(n-8)\binom{n-9}{2} \\
&+\binom{n-9}{2}+\left(\binom{n-6}{3}-1\right)+2\binom{n-7}{3}-(n-9)
\end{aligned}$$

When $n = 13$ this formula yields 357 admissible monomials in $\mathbf{P}^{13}_+(7)$.

We now compute the number of permutation representatives of h_{n-6} and k_{n-6} that are admissible when $n = 13$. By Theorem 2.7 the following permutation representatives of $x_1x_2x_3^2x_4^2x_5^2x_6^2x_7^3$ are admissible:

$$\begin{array}{lll}
d_1 = x_1^3x_2x_3x_4^2x_5^2x_6^2x_7^2 & d_2 = x_1x_2^3x_3x_4^2x_5^2x_6^2x_7^2 & d_3 = x_1x_2x_3^3x_4^2x_5^2x_6^2x_7^2 \\
d_4 = x_1x_2x_3^2x_4^3x_5^2x_6^2x_7^2 & d_5 = x_1x_2x_3^2x_4^2x_5^3x_6^2x_7^2 & d_6 = x_1x_2x_3^2x_4^2x_5^2x_6^3x_7^2 \\
d_7 = x_1x_2x_3^2x_4^2x_5^2x_6^2x_7^3 & d_8 = x_1^3x_2x_3^2x_4x_5^2x_6^2x_7^2 & d_9 = x_1x_2^3x_3^2x_4x_5^2x_6^2x_7^2 \\
d_{10} = x_1x_2^2x_3^3x_4x_5^2x_6^2x_7^2 & d_{11} = x_1x_2^2x_3x_4^3x_5^2x_6^2x_7^2 & d_{12} = x_1x_2^2x_3x_4^2x_5^3x_6^2x_7^2 \\
d_{13} = x_1x_2^2x_3x_4^2x_5^2x_6^3x_7^2 & d_{14} = x_1x_2^2x_3x_4^2x_5^2x_6^2x_7^3 & d_{15} = x_1^3x_2x_3^2x_4^2x_5x_6^2x_7^2 \\
d_{16} = x_1x_2^3x_3^2x_4^2x_5x_6^2x_7^2 & d_{17} = x_1x_2^2x_3^3x_4^2x_5x_6^2x_7^2 & d_{18} = x_1x_2^2x_3^2x_4^3x_5x_6^2x_7^2 \\
d_{19} = x_1x_2^2x_3^2x_4x_5^3x_6^2x_7^2 & d_{20} = x_1x_2^2x_3^2x_4x_5^2x_6^3x_7^2 & d_{21} = x_1x_2^2x_3^2x_4x_5^2x_6^2x_7^3 \\
d_{22} = x_1^3x_2x_3^2x_4^2x_5^2x_6x_7^2 & d_{23} = x_1x_2^3x_3^2x_4^2x_5^2x_6x_7^2 & d_{24} = x_1x_2^2x_3^3x_4^2x_5^2x_6x_7^2 \\
d_{25} = x_1x_2^2x_3^2x_4^3x_5^2x_6x_7^2 & d_{26} = x_1x_2^2x_3^2x_4^2x_5^3x_6x_7^2 & d_{27} = x_1x_2^2x_3^2x_4^2x_5x_6^3x_7^2 \\
d_{28} = x_1x_2^2x_3^2x_4^2x_5x_6^2x_7^3 & &
\end{array}$$

We see that if $x \neq d_t, \forall t, 1 \leq t \leq 28$, then x is of the form $x_1^2x_2^{m_2}\dots\, x_{n-6}^{m_{n-6}}$ or $h_j^2(x_1x_2^2x_3^2x_4^2x_5^2x_6)$, $1 \leq j \leq 7$, or $x_1^3x_2^2x_3^{m_3}\dots\, x_{n-6}^{m_{n-6}}$ all of which are clearly inadmissible.

For all $n \geq 13$ the number of monomials in $\mathbf{QP}_+^n(n-6)$ that may be obtained from the monomials $d_t, 1 \leq t \leq 28$, by inductively applying Formula (2) is $(n-6)\left(\binom{n-8}{4} - 1\right)$.

We claim that the following permutation representative of $x_1x_2x_3x_4^2x_5^2x_6^2x_7^4$ are admissible in $\mathbf{P}_+^{13}(7)$

$$\begin{array}{lll}
d_{29} = x_1x_2^2x_3x_4x_5^2x_6^2x_7^4 & d_{30} = x_1x_2^2x_3x_4x_5^2x_6^4x_7^2 & d_{31} = x_1x_2^2x_3x_4^2x_5x_6^2x_7^4 \\
d_{32} = x_1x_2^2x_3x_4^2x_5x_6^4x_7^2 & d_{33} = x_1x_2^2x_3x_4^2x_5^4x_6x_7^2 & d_{34} = x_1x_2x_3^2x_4x_5^2x_6^2x_7^4 \\
d_{35} = x_1x_2x_3^2x_4x_5^2x_6^4x_7^2 & d_{36} = x_1x_2x_3^2x_4^2x_5x_6^2x_7^4 & d_{37} = x_1x_2x_3^2x_4^2x_5x_6^4x_7^2 \\
d_{38} = x_1x_2x_3^2x_4^2x_5^4x_6x_7^2 & d_{39} = x_1x_2^2x_3^4x_4x_5x_6^2x_7^2 & d_{40} = x_1x_2^2x_3^4x_4x_5^2x_6x_7^2 \\
d_{41} = x_1x_2x_3^2x_4^4x_5x_6^2x_7^2 & d_{42} = x_1x_2x_3^2x_4^4x_5^2x_6x_7^2 & d_{43} = x_1x_2x_3x_4^2x_5^2x_6^2x_7^4 \\
d_{44} = x_1x_2x_3x_4^2x_5^2x_6^4x_7^2 & d_{45} = x_1x_2^2x_3x_4^4x_5^2x_6x_7^2 & d_{46} = x_1x_2^2x_3x_4^4x_5x_6^2x_7^2 \\
d_{47} = x_1x_2x_3x_4^2x_5^4x_6^2x_7^2 & d_{48} = x_1x_2x_3^2x_4x_5^4x_6^2x_7^2 & d_{49} = x_1x_2^2x_3x_4x_5^4x_6^2x_7^2 \\
d_{50} = x_1x_2^2x_3x_4^2x_5^2x_6x_7^4 & d_{51} = x_1x_2^2x_3x_4^2x_5^2x_6^4x_7 & d_{52} = x_1x_2^2x_3x_4^2x_5^4x_6^2x_7 \\
d_{53} = x_1x_2^2x_3x_4^4x_5^2x_6^2x_7 & d_{54} = x_1x_2^2x_3^4x_4x_5^2x_6^2x_7 & d_{55} = x_1x_2x_3^2x_4^2x_5^2x_6x_7^4 \\
d_{56} = x_1x_2x_3^2x_4^2x_5^2x_6^4x_7 & d_{57} = x_1x_2x_3^2x_4^2x_5^4x_6^2x_7 & d_{58} = x_1x_2x_3^2x_4^4x_5^2x_6^2x_7
\end{array}$$

We see that if $x \neq d_t, \forall t, 29 \leq t \leq 58$, then x is of one of the following forms:

(i) $x_1^2x_2^{m_2}...x_7^{m_7}$, (iv) $x_1x_2^2x_3^4x_4^2...x_7^{m_7}$, (vii) $x_1x_2^4x_3^{m_3}...x_7^{m_7}$,

(ii) $x_1^4x_2^{m_2}...x_7^{m_7}$, (v) $x_1x_2x_3^4x_4^{m_4}...x_7^{m_7}$,

(iii) $x_1x_2^2x_3^2x_4^{m_4}...x_7^{m_7}$, (vi) $x_1x_2x_3x_4^4x_5^{m_5}...x_7^{m_7}$,

all of which are clearly inadmissible. For example

$$\begin{aligned}
x_1x_2^2x_3^2x_4^2x_5x_6^4x_7 \equiv\ & x_1x_2^2x_3^2x_4x_5^2x_6^4x_7 + x_1x_2^2x_3^2x_4x_5x_6^4x_7^2 + x_1x_2^2x_3x_4^2x_5^2x_6^4x_7 \\
&+ x_1x_2^2x_3x_4^2x_5x_6^4x_7^2 + x_1x_2^2x_3x_4x_5^2x_6^4x_7^2 + x_1x_2x_3^2x_4^2x_5^2x_6^4x_7 \\
&+ x_1x_2x_3^2x_4^2x_5x_6^4x_7^2 + x_1x_2x_3^2x_4x_5^2x_6^4x_7^2 \\
&+ x_1x_2x_3x_4^2x_5^2x_6^4x_7^2 \mod \mathcal{A}^+\mathbf{P}(7)
\end{aligned}$$

That the monomials $d_t,\ 29 \leq t \leq 49$, are admissible follows from Theorem 2.7.

Now, we prove that the set $\{d_t : 29 \leq t \leq 58\}$ is linearly independent in $\mathbf{QP}_+^{13}(7)$. Suppose that there is a linear relation

$$\mathcal{S} = \sum_{29 \leq t \leq 58} \gamma_t d_t \equiv 0$$

with $\gamma_t \in \mathbb{F}_2$, $36 \leq t \leq 65$. It is sufficient to prove that we must have $\gamma_t = 0$ for $50 \leq t \leq 58$. By direct computation from the relations $p_{(i,j)}(\mathcal{S}) \equiv 0$, $1 \leq i \leq 3$,$1 < j \leq 6$, one gets $\gamma_{50} = \gamma_{51} = \gamma_{52} = \gamma_{53} = \gamma_{54} = \gamma_{55} = \gamma_{56} = \gamma_{57} = \gamma_{58} = 0$.

For all $n \geq 13$ the number of monomials in $\mathbf{QP}_+^n(n-6)$ that may be obtained from the monomials $d_t,\ 29 \leq t \leq 58$, by inductively applying Formula (2) is $(n-8)\binom{n-9}{3}$. This establishes Inequality 7 of Lemma 3.1.

Thus $\dim(\mathbf{QP}_+^{13}(7)) = 357 + 58 = 415$.

3.4 Dimension of $\mathbf{QP}_+^{13}(8)$

We now show that

$$\begin{aligned}
\dim(\mathbf{QP}_+^n(n-5)) = n - 6 &+ \frac{(n-5)!}{2(n-7)(n-9)!} + \binom{n-5}{2} \\
&+ \binom{n-6}{2} + (n-5)\binom{n-7}{3} + 2\left(\sum_{i=2}^{n-8}\binom{i}{2}\right) \\
&+ \binom{n-8}{2} + \left(\binom{n-7}{2} - 1\right) + \left(\binom{n-6}{5} - 3\right)
\end{aligned}$$

In [4] it is shown that if $n \geq 9$, then the monomials in $\mathbf{P}_+^n(n-5)$ that do not meet the hit criterion of Theorem 2.6 are:

$$
\begin{aligned}
a_{n-5} &= x_1 \dots x_{n-8} x_{n-7}^2 x_{n-6}^3 x_{n-5}^3 \\
b_{n-5} &= x_1 x_2 \dots x_{n-7} x_{n-6}^3 x_{n-5}^4 \\
c_{n-5} &= x_1 x_2 \dots x_{n-7} x_{n-6}^2 x_{n-5}^5 \\
d_{n-5} &= x_1 x_2 \dots x_{n-6} x_{n-5}^6
\end{aligned}
$$

and their permutation representatives, while

$$
\begin{aligned}
e_{n-5} &= x_1 x_2 \dots x_{n-9} x_{n-8}^2 x_{n-7}^2 x_{n-6}^2 x_{n-5}^3 \\
f_{n-5} &= x_1 x_2 \dots x_{n-8} x_{n-7}^2 x_{n-6}^2 x_{n-5}^4
\end{aligned}
$$

and their permutation representatives have to be added to the list when $n \geq 10$ and

$$g_{n-5} = x_1 x_2 \dots x_{n-11} x_{n-10}^2 x_{n-9}^2 x_{n-8}^2 x_{n-7}^2 x_{n-6}^2 x_{n-5}^2$$

and its permutation representatives has to be added to the list when $n \geq 13$.

In [4] it is shown that the number of permutation representatives of a_{n-5}, b_{n-5}, c_{n-5}, d_{n-5}, e_{n-5}, f_{n-5} that are admissible is

$$
\begin{aligned}
& n-6+\frac{(n-5)!}{2(n-7)(n-9)!}+\binom{n-5}{2} \\
& +\binom{n-6}{2}+(n-5)\binom{n-7}{3}+2\left(\sum_{i=2}^{n-8}\binom{i}{2}\right) \\
& +\binom{n-8}{2}+\left(\binom{n-7}{2}-1\right)
\end{aligned}
$$

When $n=13$ this formula yields 420 admissible monomials in $\mathbf{P}_+^{13}(8)$.

We now compute the number of permutation representatives of g_{n-5} that are admissible when $n = 13$. We claim that the following permutation representative of $x_1x_2x_3x_4^2x_5^2x_6^2x_7^2x_7^2$ are admissible in $\mathbf{P}_+^{13}(8)$

$$
\begin{array}{lll}
e_1 = x_1x_2^2x_3x_4x_5^2x_6^2x_7^2x_8^2 & e_2 = x_1x_2^2x_3x_4^2x_5x_6^2x_7^2x_8^2 & e_3 = x_1x_2^2x_3x_4^2x_5^2x_6x_7^2x_8^2 \\
e_4 = x_1x_2^2x_3x_4^2x_5^2x_6^2x_7x_8^2 & e_5 = x_1x_2^2x_3^2x_4x_5x_6^2x_7^2x_8^2 & e_6 = x_1x_2^2x_3^2x_4x_5^2x_6x_7^2x_8^2 \\
e_7 = x_1x_2^2x_3^2x_4x_5^2x_6^2x_7x_8^2 & e_8 = x_1x_2^2x_3^2x_4^2x_5x_6x_7^2x_8^2 & e_9 = x_1x_2^2x_3^2x_4^2x_5x_6^2x_7x_8^2 \\
e_{10} = x_1x_2x_3^2x_4x_5^2x_6^2x_7^2x_8^2 & e_{11} = x_1x_2x_3^2x_4^2x_5x_6^2x_7^2x_8^2 & e_{12} = x_1x_2x_3^2x_4^2x_5^2x_6x_7^2x_8^2 \\
e_{13} = x_1x_2x_3^2x_4^2x_5^2x_6^2x_7x_8^2 & e_{14} = x_1x_2x_3x_4^2x_5^2x_6^2x_7^2x_8^2 & e_{15} = x_1x_2^2x_3x_4^2x_5^2x_6^2x_7^2x_8 \\
e_{16} = x_1x_2^2x_3^2x_4x_5^2x_6^2x_7^2x_8 & e_{17} = x_1x_2^2x_3^2x_4^2x_5x_6^2x_7^2x_8 & e_{18} = x_1x_2x_3^2x_4^2x_5^2x_6^2x_7^2x_8
\end{array}
$$

We see that if $x \neq e_t, \forall t, 1 \leq t \leq 18$, then x is of the form $x_1^2 x_2^{m_2} ... \; x_8^{m_8}$ or of the form $x_1 x_2^2 x_3^2 x_4^2 x_5^2 x_6^{m_6} ... \; x_8^{m_8}$, all of which are clearly inadmissible. For example

$$\begin{aligned} x_1x_2^2x_3^2x_4^2x_5^2x_6x_7^2x_8 \equiv\, & x_1x_2^2x_3^2x_4^2x_5x_6^2x_7^2x_8 + x_1x_2^2x_3^2x_4^2x_5x_6x_7^2x_8^2 \\ & + x_1x_2^2x_3^2x_4x_5^2x_6^2x_7^2x_8 + x_1x_2^2x_3^2x_4x_5^2x_6x_7^2x_8^2 \\ & + x_1x_2^2x_3x_4^2x_5^2x_6^2x_7^2x_8 + x_1x_2^2x_3x_4^2x_5^2x_6x_7^2x_8^2 \\ & + x_1x_2^2x_3x_4x_5^2x_6^2x_7^2x_8^2 + x_1x_2x_3^2x_4^2x_5^2x_6^2x_7^2x_8 \\ & + x_1x_2x_3^2x_4^2x_5^2x_6x_7^2x_8^2 + x_1x_2x_3^2x_4^2x_5x_6^2x_7^2x_8^2 \\ & + x_1x_2x_3x_4^2x_5^2x_6^2x_7^2x_8^2 \mod \mathcal{A}^+\mathbf{P}(8) \end{aligned}$$

That the monomials $e_t, \ 1 \leq t \leq 14$, are admissible follows from Theorem 2.7.

Now we prove that the set $\{e_t : 1 \leq t \leq 18\}$ is linearly independent in $\mathbf{QP}_+^{13}(8)$. Suppose that there is a linear relation

$$\mathcal{S} = \sum_{1 \leq t \leq 18} \gamma_t e_t \equiv 0$$

with $\gamma_t \in \mathbb{F}_2$, $1 \leq t \leq 18$. It is sufficient to prove that we must have $\gamma_t = 0$ for $15 \leq t \leq 18$. By direct computation from the relations $p_{(i,j)}(\mathcal{S}) \equiv 0$, $1 \leq i \leq 2$,$1 < j \leq 5$, one gets $\gamma_{15} = \gamma_{16} = \gamma_{17} = \gamma_{18} = 0$.

For all $n \geq 13$ the number of monomials in $\mathbf{QP}_+^n(n-5)$ that may be obtained from the monomials $e_t, 1 \leq t \leq 18$, by inductively applying Formula (2) is $\binom{n-6}{5} - 3$. It is easy to show that any other monomial not obtained from the monomials $e_t, 1 \leq t \leq 18$, in this way is inadmissible. This establishes Formula 6 of Lemma 3.1.

Thus $\dim(\mathbf{QP}_+^{13}(8) = 420 + 18 = 438$.

3.5 Dimension of $\mathbf{QP}_+^{13}(n)$, $9 \leq n \leq 13$

That

$$\dim(\mathbf{QP}_+^{13}(n)) = \begin{cases} 1 \text{ if } n = 13 \\ 11 \text{ if } n = 12 \\ 55 \text{ if } n = 11 \\ 164 \text{ if } n = 10 \\ 322 \text{ if } n = 9 \end{cases}$$

follows from the cases of Lemma 3.1 that are proved in [4].

This completes the proof of Lemma 3.1 hence that of Theorem 1.2.

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