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AN APPROXIMATION THEOREM FOR SOLUTIONS OF DEGENERATE NONLINEAR DIRICHLET PROBLEMS

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Abstract

The main result establishes that a weak solution of degenerate nonlinear linear elliptic equations can be approximated by a sequence of solutions for non-degenerate nonlinear linear elliptic equations.

1 Introduction

Let L be a degenerate elliptic operator in divergence form

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)\, D_i u(x)),\ \ D = \frac{\partial}{\partial x_j}, \tag{1.1}$$

where the coefficients a_{ij} are measurable, real-valued functions whose coefficient matrix $\mathcal{A} = (a_{ij})$ is symmetric and satisfies the degenerate ellipticity condition

$$\lambda|\xi|^2\omega(x) \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2\omega(x), \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and almost everywhere $x \in \Omega$, where Ω is a bounded open set in $\mathbb{R}^n$ and we assume that Ω has a Lipschitz boundary $\partial\Omega$ with outward unit normal $\vec{\eta}(x) = (\eta_1(x), ..., \eta_n(x))$, ω is a weight function, λ and Λ are positive constants.

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The main purpose of this paper (see Theorem 1.2) is to establish that a weak solution $u \in W_0^{1,2}(\Omega, \omega)$ for the nonlinear degenerate problem

$$(P)\begin{cases} Lu(x) + b(x)\,u(x) + \operatorname{div}(\Phi(u(x))) = g(x) - \displaystyle\sum_{j=1}^{n} D_j f_j(x) \ \text{ in } \ \Omega, \\ u(x) = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

can be approximated by a sequence of solutions of non-degenerate nonlinear elliptic equations, where $\Phi : \mathbb{R} \to \mathbb{R}^n$ and $b : \Omega \to \mathbb{R}$.

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various kinds of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1], [2], [3],[4] and [7]). Type of a weight depends on the equation type.

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [8]). These classes have found many useful applications in harmonic analysis (see [9]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [7]). There are, in fact, many interesting examples of weights (see [6] for p-admissible weights).

The following lemma can be proved in exactly the same way as Lemma 2.1 in [4] (see also, Lemma 3.1 and Lemma 4.13 in [1]). Our lemma provides a general approximation theorem for A_p weights $(1 \le p < \infty)$ by means of weights which are bounded away from 0 and infinity and whose A_p-constants depend only on the A_p-constant of ω. Lemma 1.1 is the key point for Theorem 1.2, and the crucial point consists of showing that a weak limit of a sequence of solutions of approximate problems is in fact a solution of the original problem.

Lemma 1.1. *Let $\alpha, \beta > 1$ be given and let $\omega \in A_p$ $(1 \le p < \infty)$, with A_p-constant $C(\omega, p)$ and let $a_{ij} = a_{ji}$ be measurable, real-valued functions satisfying*

$$\lambda\,\omega(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda\,\omega(x)\,|\xi|^2, \tag{1.3}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$. Then there exist weights $\omega_{\alpha\beta} \ge 0$ a.e. and measurable real-valued functions $a_{ij}^{\alpha\beta}$ such that the following conditions are met.

(i) $c_1(1/\beta) \le \omega_{\alpha\beta} \le c_2\,\alpha$ in Ω, where c_1 and c_2 depend only on ω and Ω.

(ii) There exist weights $\tilde{\omega}_1$ *and* $\tilde{\omega}_2$ *such that* $\tilde{\omega}_1 \leq \omega_{\alpha\beta} \leq \tilde{\omega}_2$, *where* $\tilde{\omega}_i \in A_p$ *and* $C(\tilde{\omega}_i, p)$ *depends only on* $C(\omega, p)$ $(i = 1, 2)$.

(iii) $\omega_{\alpha\beta} \in A_p$, *with constant* $C(\omega_{\alpha\beta}, p)$ *depending only on* $C(\omega, p)$ *uniformly on* α *and* β.

(iv) There exists a closed set $F_{\alpha\beta}$ *such that* $\omega_{\alpha\beta} \equiv \omega$ *in* $F_{\alpha\beta}$ *and* $\omega_{\alpha\beta} \sim \tilde{\omega}_1 \sim \tilde{\omega}_2$ *in* $F_{\alpha\beta}$ *with equivalence constants depending on* α *and* β *(i.e., there are positive constants* $c_{\alpha\beta}$ *and* $C_{\alpha\beta}$ *such that* $c_{\alpha\beta}\,\tilde{\omega}_i \leq \omega_{\alpha\beta} \leq C_{\alpha\beta}\,\tilde{\omega}_i$, $i = 1, 2$*). Moreover,* $F_{\alpha\beta} \subset F_{\alpha'\beta'}$ *if* $\alpha \leq \alpha'$, $\beta \leq \beta'$, *and the complement of* $\bigcup_{\alpha,\beta \geq 1} F_{\alpha\beta}$ *has zero measure.*

(v) $\omega_{\alpha\beta} \to \omega$ *a.e. in* $\mathbb{R}^n$ *as* $\alpha, \beta \to \infty$.

(vi) $\lambda\,\omega_{\alpha\beta}(x)\,|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x)\,\xi_i\xi_j \leq \Lambda\omega_{\alpha\beta}(x)\,|\xi|^2$, *for every* $\xi \in \mathbb{R}$ *and a.e.* $x \in \Omega$.

Proof. See [1], Lemma 3.1 or Lemma 4.13. □

The following theorem will be proved in section 3.

Theorem 1.2. *Let* Ω *be an open bounded set in* $\mathbb{R}^n$ *with a Lipschitz boundary* $\partial\Omega$. *Suppose that*

(H1) $f_j/\omega \in L^p(\Omega, \omega)$, $(j = 1, ..., n)$ *with* $p > nr \geq 4$;

(H2) $g/\omega \in L^q(\Omega, \omega)$, *with* $1/q = 1/p + 1/nr$;

(H3) $\omega \in A_r$, *with* $1 < r < p'$ *(where* $1/p + 1/p' = 1$*)*;

(H4) $b(x) \geq 0$ *for a.e.* $x \in \Omega$ *and* $b/\omega \in L^\infty(\Omega)$;

(H5) $\Phi : \mathbb{R} \to \mathbb{R}^n$ $(\Phi = (\Phi_1, ..., \Phi_n))$, *with* $|\Phi| \in L^\infty(\mathbb{R})$, $\Phi(0) = 0$ *and the functions* Φ_j *are continuous* $(j = 1, ..., n)$.

(H6) $|\Phi(u(x)) - \Phi(v(x))| \leq C_0\,\omega(x)\,|u(x) - v(x)|$ *for all* $u, v \in W_0^{1,2}(\Omega, \omega)$, *a.e.* $x \in \Omega$ *and* C_0 *is a positive constant.*

Then the problem (P) *has a unique solution and*

$$\|u\|_{W_0^{1,2}(\Omega,\omega)} \leq C\left(C_\Omega\,[\mu(\Omega)]^{1/2-1/q}\,\|g/\omega\|_{L^q(\Omega,\omega)} + [\mu(\Omega)]^{1/2-1/p}\sum_{j=1}^{n}\|f_j/\omega\|_{L^p(\Omega,\omega)}\right), \tag{1.4}$$

where $\mu(\Omega) = \int_\Omega \omega(x)\,dx$ *and* $C = \dfrac{(C_\Omega^2 + 1)^{1/2}}{M}$, $M = \lambda - C_0\,C_\Omega > 0$, C_Ω *the constant as in Theorem 2.1. Moreover,* u *is the weak limit in* $W_0^{1,2}(\Omega, \tilde{\omega}_1)$ *of a sequence of solutions* $u_m \in W_0^{1,2}(\Omega, \omega_m)$ *of the problems*

$$(P_m)\begin{cases} L_m u_m(x) + b_m(x)\,u_m(x) + \operatorname{div}[\Phi(u_m(x))] = g_m(x) - \displaystyle\sum_{j=1}^{n} D_j f_{jm}(x) \quad \text{in} \quad \Omega, \\ u_m(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

with $L_m u_m = -\sum_{i,j=1}^{n} D_j(a_{ij}^{mm}(x) D_i u_m(x))$, $g_m = g(\omega_m/\omega)^{1/q'}$, $f_{jm} = f_j(\omega_m/\omega)^{1/p'}$ *and* $b_m = b\,\omega/\omega_m$ *(where* ω_{mm}, a_{ij}^{mm} *and* $\tilde{\omega}_1$ *are as Lemma 1.1).*

2 Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C(p, \omega)$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [5],[6] or [10] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x; 2\,r)) \leq C\,\mu(B(x; r))$ for every ball $B = B(x; r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [6]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [9]).

If $\omega \in A_p$, then $\left(\frac{|E|}{|B|}\right)^p \leq C\frac{\mu(E)}{\mu(B)}$ whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [6]). Therefore, $\mu(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p \omega\,dx\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [10]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be open, k be a nonnegative integer and $\omega \in A_p$ $(1 < p < \infty)$. We define the weighted Sobolev space $W^{k,p}(\Omega, \omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$ for $1 \le |\alpha| \le k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p\,\omega\,dx + \sum_{1\le|\alpha|\le k} \int_\Omega |D^\alpha u|^p\,\omega\,dx \right)^{1/p}. \tag{2.1}$$

We also define $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1).

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Corollary 2.1.6 in [10]). The spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces.

It is evident that the weight function ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (c_1 and c_2 positive constants), gives nothing new (the space $\mathrm{W}_0^{k,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}_0^{k,p}(\Omega)$). Consequently, we shall be interested above in all such weight functions ω which either vanish in somewhere $\Omega \cup \partial\Omega$ or increase to infinity (or both).

The dual space of $W_0^{1,p}(\Omega,\omega)$ is the space

$$\begin{aligned}
&[W_0^{1,p}(\Omega,\omega)]^* = W^{-1,p'}(\Omega,\omega) \\
&= \{T = f_0 - \operatorname{div} F : F = (f_1, ..., f_n),\ \frac{f_j}{\omega} \in L^{p'}(\Omega,\omega)\}.
\end{aligned}$$

Definition 2.3. We say that an element $u \in W_0^{1,2}(\Omega,\omega)$ is weak solution of problem (P) if

$$\begin{aligned}
&\int_\Omega \langle \mathcal{A}\nabla u, \nabla\varphi\rangle\,dx + \int_\Omega b\,u\,\varphi\,dx - \int_\Omega \langle \Phi(u), \nabla\varphi\rangle\,dx \\
&= \int_\Omega g\,\varphi\,dx + \sum_{j=1}^n \int_\Omega f_j\,D_j\varphi\,dx,
\end{aligned}$$

for every $\varphi \in W_0^{1,2}(\Omega,\omega)$, where $\langle .,. \rangle$ denotes here the Euclidian scalar product in $\mathbb{R}^n$,

$$\langle \mathcal{A}\nabla u, \nabla\varphi\rangle = \sum_{i,j=1}^n a_{ij} D_i u D_j\varphi \quad\text{and}\quad \langle \Phi(u), \nabla\varphi\rangle = \sum_{j=1}^n \Phi_j(u) D_j\varphi.$$

Theorem 2.1. *(The weighted Sobolev inequality) Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ ($1 < p < \infty$). There exist positive constants C_Ω and δ such that for all $u \in W_0^{1,p}(\Omega,\omega)$ and all θ satisfying $1 \le \theta \le n/(n-1)+\delta$,*

$$\|u\|_{L^{\theta p}(\Omega,\omega)} \le C_\Omega \|\nabla u\|_{L^p(\Omega,\omega)}. \tag{2.2}$$

Proof. Its suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [3]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). □

3 Proof of Theorem 1.2

Step 1. The existence and uniqueness of solution $u \in W_0^{1,2}(\Omega,\omega)$ for the problem (P) has been demonstrated in [2], Theorem 1.1. In particular, for $\varphi = u$ in Definition 2.3. we have

$$\begin{aligned} &\int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle\, dx + \int_\Omega u^2\, b\, dx - \int_\Omega \langle \Phi(u), \nabla u\rangle\, dx \\ &= \int_\Omega g\, u\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j u\, dx. \end{aligned} \tag{3.1}$$

(i) By (1.2) we have

$$\sum_{i,j=1}^n \int_\Omega a_{ij}\, D_i u\, D_j u\, dx = \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle\, dx \geq \lambda \int_\Omega |\nabla u|^2\, \omega\, dx.$$

(ii) By (H4), $\displaystyle\int_\Omega u^2\, b\, dx \geq 0$.

(iii) By (H5)and (H6) we have $|\Phi(u)| \leq C_0\, |u|\, \omega$ a.e.. Using Theorem 2.1 (with $p = 2$ and $\theta = 1$) we obtain

$$\begin{aligned} \left| \int_\Omega \langle \Phi(u), \nabla u\rangle\, dx \right| &\leq \int_\Omega |\langle \Phi(u), \nabla u\rangle|\, dx \\ &\leq \int_\Omega |\Phi(u)|\, |\nabla u|\, dx \\ &\leq \int_\Omega C_0\, |u|\, |\nabla u|\, \omega\, dx \\ &\leq C_0 \left(\int_\Omega |u|^2\, \omega\, dx \right)^{1/2} \left(\int_\Omega |\nabla u|^2\, \omega\, dx \right)^{1/2} \\ &\leq C_0\, C_\Omega \int_\Omega |\nabla u|^2\, \omega\, dx. \end{aligned}$$

(iv) Using (H1) and (H2) (and since $q > 2$ and $\mu(\Omega) < \infty$), we have

$$\begin{aligned}\left|\int_\Omega g\,u\,dx\right| &\le \int_\Omega \frac{|g|}{\omega}\,|u|\,\omega\,dx \\ &\le \left(\int_\Omega \left(\frac{|g|}{\omega}\right)^2 \omega\,dx\right)^{1/2}\left(\int_\Omega |u|^2\omega\,dx\right)^{1/2} \\ &\le C_\Omega \|g/\omega\|_{L^2(\Omega,\omega)}\|\nabla u\|_{L^2(\Omega,\omega)} \\ &\le C_\Omega[\mu(\Omega)]^{1/2-1/q}\|g/\omega\|_{L^q(\Omega,\omega)}\|\nabla u\|_{L^2(\Omega,\omega)},\end{aligned}$$

and (since $p > 4$)

$$\begin{aligned}\left|\int_\Omega f_j\,D_j u\,dx\right| &\le \int_\Omega \frac{|f_j|}{\omega}\,|D_j|\,\omega\,dx \\ &\le \|f_j/\omega\|_{L^2(\Omega,\omega)}\,\|\nabla u\|_{L^2(\Omega,\omega)} \\ &\le [\mu(\Omega)]^{1/2-1/p}\|f_j/\omega\|_{L^p(\Omega,\omega)}\|\nabla u\|_{L^2(\Omega,\omega)}.\end{aligned}$$

Hence, in (3.1), we obtain

$$\begin{aligned}&\lambda\int_\Omega |\nabla u|^2\,\omega\,dx - C_0\,C_\Omega \int_\Omega |\nabla u|^2\,\omega\,dx \\ &\le \Bigg(C_\Omega[\mu(\Omega)]^{1/2-1/q}\|g/\omega\|_{L^q(\Omega,\omega)} \\ &\quad + [\mu(\Omega)]^{1/2-1/p}\sum_{j=1}^n \|f_j/\omega\|_{L^2(\Omega,\omega)}\Bigg)\,\|\nabla u\|_{L^2(\Omega,\omega)}.\end{aligned}$$

Therefore

$$\begin{aligned}\|\nabla u\|_{L^2(\Omega,\omega)} \qquad &\le \frac{1}{M}\Bigg(C_\Omega[\mu(\Omega)]^{1/2-1/q}\|g/\omega\|_{L^q(\Omega,\omega)} \\ &\quad + [\mu(\Omega)]^{1/2-1/p}\sum_{j=1}^n \|f_j/\omega\|_{L^p(\Omega,\omega)}\Bigg),\end{aligned}$$

where $M = \lambda - C_0\,C_\Omega > 0$. Consequently, we obtain

$$\begin{aligned}&\|u\|^2_{W_0^{1,2}(\Omega,\omega)} = \int_\Omega |u|^2\,\omega\,dx + \int_\Omega |\nabla u|^2\,\omega\,dx \\ &\le (C_\Omega^2+1)\int_\Omega |\nabla u|^2\,\omega, dx \\ &\le \frac{(C_\Omega^2+1)}{M^2}\left(C_\Omega[\mu(\Omega)]^{1/2-1/q}\|g/\omega\|_{L^q(\Omega,\omega)} + [\mu(\Omega)]^{1/2-1/p}\sum_{j=1}^n \|f_j/\omega\|_{L^p(\Omega,\omega)}\right)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|u\|_{W_0^{1,2}(\Omega,\omega)} &\le \frac{(C_\Omega^2+1)^{1/2}}{M}\Big(C_\Omega[\mu(\Omega)]^{1/2-1/q}\|g/\omega\|_{L^q(\Omega,\omega)}\\ &+ [\mu(\Omega)]^{1/2-1/p}\sum_{j=1}^n \|f_j/\omega\|_{L^p(\Omega,\omega)}\Big)\\ &= C\Big(C_\Omega[\mu(\Omega)]^{1/2-1/q}\|g/\omega\|_{L^q(\Omega,\omega)}\\ &+ [\mu(\Omega)]^{1/2-1/p}\sum_{j=1}^n \|f_j/\omega\|_{L^p(\Omega,\omega)}\Big). \end{aligned} \tag{3.2}$$

Step 2. First, if $g_m = g(\omega_m/\omega)^{1/q'}$, $f_{jm} = f_j(\omega_m/\omega)^{1/p'}$ and $b_m = b\,\omega_m/\omega$, we note that

$$\|g_m/\omega_m\|_{L^q(\Omega,\omega_m)} = \|g/\omega\|_{L^q(\Omega,\omega)}, \quad \|f_{jm}/\omega_m\|_{L^p(\Omega,\omega_m)} = \|f_j/\omega\|_{L^p(\Omega,\omega)},$$

$$b_m \ge 0 \quad \text{and} \quad \|b_m/\omega_m\|_{L^\infty(\Omega)} = \|b/\omega\|_{L^\infty(\Omega)}.$$

By Lemma 1.1, $\omega_m \le \tilde\omega_2$. Then $\mu_m(\Omega) = \int_\Omega \omega_m\,dx \le \int_\Omega \tilde\omega_2\,dx = \tilde\mu_2(\Omega)$.
If $u_m \in W_0^{1,2}(\Omega,\omega_m)$ is a unique solution of problem (P_m), we have (by (3.2))

$$\begin{aligned}\|u_m\|_{W_0^{1,2}(\Omega,\omega_m)} &\le C\Big(C_\Omega\,[\mu_m(\Omega)]^{1/2-1/q}\|g_m/\omega_m\|_{L^q(\Omega,\omega_m)}\\ &\quad + [\mu_m(\Omega)]^{1/2-1/p}\sum_{j=1}^n \|f_{jm}/\omega_m\|_{L^p(\Omega,\omega_m)}\Big)\\ &\le C\Big(C_\Omega\,[\tilde\mu_2(\Omega)]^{1/2-1/q}\|g/\omega\|_{L^q(\Omega,\omega)}\\ &\quad + [\tilde\mu_2(\Omega)]^{1/2-1/p}\sum_{j=1}^n \|f_j/\omega\|_{L^p(\Omega,\omega)}\Big) = C_1.\end{aligned}$$

Using Lemma 1.1, $\tilde\omega_1 \le \omega_m$, we obtain

$$\|u_m\|_{W_0^{1,2}(\Omega,\tilde\omega_1)} \le \|u_m\|_{W_0^{1,2}(\Omega,\omega_m)} \le C_1. \tag{3.3}$$

Consequently, $\{u_m\}$ is a bounded sequence in $W_0^{1,2}(\Omega,\tilde\omega_1)$. Therefore, there is a subsequence, again denoted by $\{u_m\}$, and $\tilde u \in W_0^{1,2}(\Omega,\tilde\omega_1)$ such that

$$u_m \rightharpoonup \tilde u \text{ in } L^2(\Omega,\tilde\omega_1), \tag{3.4}$$

$$\nabla u_m \rightharpoonup \nabla\tilde u \text{ in } L^2(\Omega,\tilde\omega_1), \tag{3.5}$$

$$u_m \to \tilde u \text{ a.e. in } \Omega, \tag{3.6}$$

where the symbol "$\rightharpoonup$" denotes weak convergence (see Theorem 1.31 in [6]).

Step 3. We have that $\tilde{u}\in W_0^{1,2}(\Omega,\omega)$. In fact, for F_k fixed, we have by (3.4) and (3.5), for all $\varphi\in W_0^{1,2}(\Omega,\tilde{\omega}_1)$,

$$\int_\Omega u_m\varphi\,\tilde{\omega}_1\,dx\to\int_\Omega \tilde{u}\,\varphi\,\tilde{\omega}_1\,dx,$$
$$\int_\Omega D_iu_mD_i\varphi\,\tilde{\omega}_1\,dx\to\int_\Omega D_i\tilde{u}\,D_i\varphi\,\tilde{\omega}_1\,dx.$$

If $\psi\in W_0^{1,2}(\Omega,\omega)$, then $\varphi=\psi\,\chi_{F_k}\in W_0^{1,2}(\Omega,\tilde{\omega}_1)$ (since $\omega\sim\tilde{\omega}_1$ in F_k, i.e., there is a constant $c>0$ such that $\tilde{\omega}_1\leq c\,\omega$ in F_k, and χ_E denotes the characteristic function of a measurable set $E\subset\mathbb{R}^n$) and

$$\int_\Omega\varphi^2\,\tilde{\omega}_1\,dx=\int_{F_k}\psi^2\,\tilde{\omega}_1\,dx\leq c\int_{F_k}\psi^2\omega\,dx\leq c\int_\Omega\psi^2\,\omega\,dx<\infty,$$
$$\int_\Omega(D_i\varphi)^2\,\tilde{\omega}_1\,dx\;=\int_{F_k}(D_i\psi)^2\tilde{\omega}_1\,dx\leq c\int_{F_k}(D_i\psi)^2\omega\,dx\leq c\int_\Omega(D_i\psi)^2\omega\,dx<\infty.$$

Consequently,

$$\int_\Omega u_m\psi\chi_{F_k}\,\tilde{\omega}_1\,dx\to\int_\Omega\tilde{u}\,\psi\,\chi_{F_k}\,\tilde{\omega}_1\,dx,$$
$$\int_\Omega D_iu_mD_i\psi\,\chi_{F_k}\,\tilde{\omega}_1\,dx\to\int_\Omega D_i\tilde{u}\,D_i\psi\,\chi_{F_k}\,\tilde{\omega}_1\,dx,$$

for all $\psi\in W_0^{1,2}(\Omega,\omega)$, that is, the sequence $\{u_m\,\chi_{F_k}\}$ is weakly convergent in $W_0^{1,2}(\Omega,\omega)$.
Therefore, we have

$$\|\nabla\tilde{u}\|^2_{L^2(F_k,\omega)}=\int_{F_k}|\nabla\tilde{u}|^2\omega\,dx\leq\limsup_{m\to\infty}\int_{F_k}|\nabla u_m|^2\,\omega\,dx,$$

and for $m\geq k$ we have $\omega=\omega_m$ in F_k. Hence, by (3.3), we obtain

$$\begin{aligned}\|\nabla\tilde{u}\|^2_{L^2(F_k,\omega)}&\leq\limsup_{m\to\infty}\int_{F_k}|\nabla u_m|^2\,\omega\,dx\\&=\limsup_{m\to\infty}\int_{F_k}|\nabla u_m|^2\,\omega_m\,dx\\&\leq\limsup_{m\to\infty}\int_\Omega|\nabla u_m|^2\omega_m\,dx\leq C_1^2.\end{aligned}$$

By the Monotone Convergence Theorem we obtain $\|\nabla\tilde{u}\|_{L^2(\Omega,\omega)}\leq C_1$. Therefore, we have $\tilde{u}\in W_0^{1,2}(\Omega,\omega)$.

Step 4. We need to show that $\tilde{u}$ is a solution of problem (P), i.e.,

$$\begin{aligned}&\int_\Omega \langle \mathcal{A}\nabla\tilde{u}, \nabla\varphi\rangle\, dx + \int_\Omega b\,\tilde{u}\,\varphi\, dx - \int_\Omega \langle \Phi(\tilde{u}), \nabla\varphi\rangle\, dx\\ &= \int_\Omega g\,\varphi\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi\, dx,\end{aligned}$$

for all $\varphi \in W_0^{1,2}(\Omega,\omega)$. Using that $u_m \in W_0^{1,2}(\Omega,\omega_m)$ is a solution of problem (P_m), we have

$$\begin{aligned}&\int_\Omega \langle \mathcal{A}^m\nabla u_m, \nabla\psi\rangle\, dx + \int_\Omega b_m\, u_m\,\psi\, dx - \int_\Omega \langle \Phi(u_m), \nabla\psi\rangle\, dx\\ &= \int_\Omega g_m\,\psi\, dx + \sum_{j=1}^n \int_\Omega f_{jm}\, D_j\psi\, dx,\end{aligned}$$

for all $\psi \in W_0^{1,2}(\Omega,\omega_m)$, where $\mathcal{A}^m = (a_{ij}^{mm})$. Moreover, over F_k (for $m \geq k$) we have the following properties:

(i) $\omega = \omega_m$; (ii) $g_m = g$; (iii) $f_{jm} = f_j$; (iv) $b_m = b$; (v) $a_{ij}^{mm}(x) = a_{ij}(x)$.

For $\varphi \in W_0^{1,2}(\Omega,\omega)$ and $k > 0$(fixed), we define $G_1, G_2 : W_0^{1,2}(\Omega,\tilde{\omega}_1) \to \mathbb{R}$ by

$$\begin{aligned}G_1(u) &= \int_\Omega \langle \mathcal{A}\nabla u, \nabla\varphi\rangle\, \chi_{F_k}\, dx + \int_\Omega b\, u\,\varphi\,\chi_{F_k}\, dx,\\ G_2(u) &= \int_\Omega \langle \Phi(u), \nabla\varphi\rangle\, \chi_{F_k}\, dx,\end{aligned}$$

where χ_E denotes the characteristic function of a set $E \subset \mathbb{R}^n$.

(a) We have that G_1 is linear and continuous functional. In fact, since the matrix $\mathcal{A} = (a_{ij})$ is symmetric, we have $|\langle \mathcal{A}\nabla u, \nabla\varphi\rangle| \leq \langle \mathcal{A}\nabla u, \nabla u\rangle^{1/2} \langle \mathcal{A}\nabla\varphi, \nabla\varphi\rangle^{1/2}$, where $\langle .,.\rangle$ denotes here the Euclidian scalar product in $\mathbb{R}^n$. We also have $\omega \sim \tilde{\omega}_1$

in F_k $(\omega \le c\,\tilde{\omega}_1)$. By (1.2) and (H4) we obtain

$$\begin{aligned}
|G_1(u)| &\le \int_{F_k} |\langle \mathcal{A}\nabla u, \nabla\varphi\rangle|\,dx + \int_{F_k} b\,|u|\,|\varphi|\,dx \\
&\le \int_{F_k} \langle \mathcal{A}\nabla u, \nabla u\rangle^{1/2}\,\langle \mathcal{A}\nabla\varphi, \nabla\varphi\rangle^{1/2}\,dx + \int_{F_k} \frac{b}{\omega}\,|u|\,|\varphi|\,\omega\,dx \\
&\le \left(\int_{F_k} \langle \mathcal{A}\nabla u, \nabla u\rangle\,dx\right)^{1/2}\left(\int_{F_k} \langle \mathcal{A}\nabla\varphi, \nabla\varphi\rangle^{1/2}\,dx\right)^{1/2} \\
&+ \|b/\omega\|_{L^\infty(\Omega)}\left(\int_{F_k} |u|^2\omega\,dx\right)^{1/2}\left(\int_{F_k} |\varphi|^2\,\omega\right)^{1/2} \\
&\le \Lambda\left(\int_{F_k} |\nabla u|^2\,\omega\,dx\right)^{1/2}\left(\int_{\Omega} |\nabla\varphi|^2\,\omega\,dx\right)^{1/2} \\
&+ \|b/\omega\|_{L^\infty(\Omega)}\left(\int_{F_k} |u|^2\,\omega\,dx\right)^{1/2}\left(\int_{\Omega} |\varphi|^2\omega\,dx\right)^{1/2} \\
&\le \Lambda\left(\int_{F_k} c\,|\nabla u|^2\tilde{\omega}_1\,dx\right)^{1/2}\left(\int_{\Omega} |\nabla\varphi|^2\,\omega\,dx\right)^{1/2} \\
&+ \|b/\omega\|_{L^\infty(\Omega)}\left(\int_{F_k} c\,|u|^2\tilde{\omega}_1\,dx\right)^{1/2}\left(\int_{\Omega} |\varphi|^2\omega\,dx\right)^{1/2} \\
&\le (\Lambda\,c^{1/2} + \|b/\omega\|_{L^\infty(\Omega)}\,c^{1/2})\|\varphi\|_{W_0^{1,2}(\Omega,\omega)}\,\|u\|_{W_0^{1,2}(\Omega,\tilde{\omega}_1)}.
\end{aligned}$$

(b) We have that G_2 is continuous functional. In fact, if $u_1, u_2 \in W_0^{1,2}(\Omega,\tilde{\omega}_1)$, we obtain (by (H6))

$$\begin{aligned}
|G_2(u_2) - G_2(u_1)| &\le \int_{F_k} |\langle \Phi(u_2) - \Phi(u_1), \nabla\varphi\rangle|\,dx \\
&\le \int_{F_k} |\Phi(u_2) - \Phi(u_1)|\,|\nabla\varphi|\,dx \\
&\le \int_{F_k} C_0\,|u_2 - u_1|\,|\nabla\varphi|\,\omega\,dx \\
&\le C_0\left(\int_{F_k} c\,|u_2 - u_1|^2\tilde{\omega}_1\,dx\right)^{1/2}\left(\int_{\Omega} |\nabla\varphi|^2\,\omega\,dx\right)^{1/2} \\
&\le C_0\,c^{1/2}\,\|\varphi\|_{W_0^{1,2}(\Omega,\omega)}\,\|u_2 - u_1\|_{W_0^{1,2}(\Omega,\tilde{\omega}_1)}.
\end{aligned}$$

If $\varphi \in W_0^{1,2}(\Omega,\omega)$, then $\psi = \varphi\,\chi_{F_k} \in W_0^{1,2}(\Omega,\tilde{\omega}_1)$ (for $m \ge k$). Using (a), (b) and properties (i), (ii), (iii), (iv), (v) and that $u_m \in W_0^{1,2}(\Omega,\omega_m)$ is solution of problem (P_m) we obtain

$$\begin{aligned}
&\int_{F_k} \langle \mathcal{A}\nabla \tilde{u}, \nabla\varphi\rangle\,dx + \int_{F_k} b\,\tilde{u}\,\varphi\,dx - \int_{F_k} \langle \Phi(\tilde{u}), \nabla\varphi\rangle\,dx \\
&= \lim_{m\to\infty} [\,G_1(u_m) + G_2(u_m)\,] \\
&= \lim_{m\to\infty} \biggl(\int_{F_k} \langle \mathcal{A}^m \nabla u_m, \nabla\varphi\rangle\,dx + \int_{F_k} b_m\,u_m\,\varphi\,dx \\
&\quad - \int_{F_k} \langle \Phi(u_m), \nabla\varphi\rangle\,dx \biggr) \\
&= \lim_{m\to\infty} \biggl(\int_{\Omega} \langle \mathcal{A}^m \nabla u_m, \nabla\varphi\rangle\,dx + \int_{\Omega} b_m\,u_m\,\varphi\,dx \\
&\quad - \int_{\Omega} \langle \Phi(u_m), \nabla\varphi\rangle\,dx \\
&\quad - \int_{\Omega\cap F_k^c} \langle \mathcal{A}^m \nabla u_m, \nabla\varphi\rangle\,dx - \int_{\Omega\cap F_k^c} b_m\,u_m\,\varphi\,dx \\
&\quad + \int_{\Omega\cap F_k^c} \langle \Phi(u_m), \nabla\varphi\rangle\,dx \biggr) \\
&= \lim_{m\to\infty} \biggl(\int_{\Omega} g_m\,\varphi\,dx + \sum_{j=1}^{n} \int_{\Omega} f_{jm}\,D_j\varphi\,dx \\
&\quad - \int_{\Omega\cap F_k^c} \langle \mathcal{A}^m \nabla u_m, \nabla\varphi\rangle\,dx - \int_{\Omega\cap F_k^c} b_m\,u_m\,\varphi\,dx \\
&\quad + \int_{\Omega\cap F_k^c} \langle \Phi(u_m), \nabla\varphi\rangle\,dx \biggr),
\end{aligned} \tag{3.7}$$

where E^c denotes the complement of a set $E\subset\mathbb{R}^n$.

(I) Suppose $\varphi\in W_0^{1,2}(\Omega,\omega)$. By a density argument (see Corollary 2.1.6 in [10]) we can suppose $\varphi\in C_0^\infty(\Omega)$. We have (by $\omega_m\to w$ a.e.)

$$g_m\,\varphi = g(\omega_m/\omega)^{1/q'} \to g\,\varphi \text{ a.e.}$$

and

$$\begin{aligned}
|g_m\,\varphi| &= |g|(\omega_m/\omega)^{1/q'}\,|\varphi| \\
&= \frac{|g|}{\omega^{1/q'}}\,\omega_m^{1/q'}\,\varphi \\
&\le C_\varphi \frac{|g|}{\omega^{1/q'}}\,\tilde{\omega}_2^{1/q'} \in L^1(\Omega),
\end{aligned}$$

since $\dfrac{|g|}{\omega^{1/q'}} \in L^q(\Omega)$ and $\tilde{\omega}_2^{1/q'} \in L^{q'}(\Omega)$. By the Lebesgue Dominated Convergence Theorem and $\tilde{\omega}_2 \in A_2$ ($\tilde{\omega}_2 \in A_r$ and $r < 2$) we obtain (as $m \to \infty$)

$$\int_\Omega g_m \, \varphi \, dx \to \int_\Omega g \, \varphi \; dx. \tag{3.8}$$

Analogously, we have

$$\int_\Omega f_{jm} \, D_j \varphi \, dx \to \int_\Omega f_j \, D_j \varphi \, dx. \tag{3.9}$$

(II) Since the matrix $\mathcal{A}^m = (a_{ij}^{mm})$ is symmetric, we have $|\langle \mathcal{A}^m \nabla u_m, \nabla \varphi \rangle| \leq \langle \mathcal{A}^m \nabla u_m, \nabla u_m \rangle^{1/2} \langle \mathcal{A}^m \nabla \varphi, \nabla \varphi \rangle^{1/2}$. Then, by (1.2) and (3.3), we obtain

$$\begin{aligned}
&\left| \int_{\Omega \cap F_k^c} \langle \mathcal{A}^m \nabla u_m, \nabla \varphi \rangle \, dx \right| \leq \int_{\Omega \cap F_k^c} |\langle \mathcal{A}^m \nabla u_m, \nabla \varphi \rangle| \, dx \\
&\leq \Lambda \left(\int_{\Omega \cap F_k^C} |\nabla u_m|^2 \omega_m \, dx \right)^{1/2} \left(\int_{\Omega \cap F_k^c} |\nabla \varphi|^2 \omega_m \, dx \right)^{1/2} \\
&\leq \Lambda \|u_m\|_{W_0^{1,2}(\Omega, \omega_m)} \left(\int_{\Omega \cap F_k^c} |\nabla \varphi|^2 \, w_m \, dx \right)^{1/2} \\
&\leq \Lambda \, C_1 \, C_\varphi \left(\int_{\Omega \cap F_k^c} w_m \, dx \right)^{1/2}.
\end{aligned} \tag{3.10}$$

(III) By (H4) and (3.3) we obtain

$$\begin{aligned}
&\left| \int_{\Omega \cap F_k^c} b_m \, u_m \, \varphi \, dx \right| \leq \int_{\Omega \cap F_k^c} \frac{b_m}{\omega_m} \, |u_m| \, |\varphi| \, \omega_m \, dx \\
&\leq \|b_m/\omega_m\|_{L^\infty(\Omega)} \left(\int_{\Omega \cap F_k^c} |u_m|^2 \, \omega_m \, dx \right)^{1/2} \left(\int_{\Omega \cap F_k^c} |\varphi|^2 \, \omega_m \, dx \right)^{1/2} \\
&\leq \|b/\omega\|_{L^\infty(\Omega)} \|u_m\|_{W_0^{1,2}(\Omega, \omega_m)} \left(\int_{\Omega \cap F_k^c} |\varphi|^2 \, \omega_m \, dx \right)^{1/2} \\
&\leq C_1 \, C_\varphi \, \|b/\omega\|_{L^\infty(\Omega)} \left(\int_{\Omega \cap F_k^c} \omega_m \, dx \right)^{1/2}.
\end{aligned} \tag{3.11}$$

(IV) Since $\omega \in A_2$ ($A_r \subset A_2$) and by (H3) , (H5),(H6) we have

$$\begin{aligned}
\left| \int_{\Omega\cap F_k^c} \langle \Phi(u_m), \nabla\varphi\rangle\, dx \right| &\leq \int_{\Omega\cap F_k^c} |\Phi(u_m)|\, |\nabla\varphi|\, dx \\
&\leq \|\Phi\|_{L^\infty(\mathbb{R})} \int_{\Omega\cap F_k^c} |\nabla\varphi|\, dx \\
&= \|\Phi\|_{L^\infty(\mathbb{R})} \int_{\Omega\cap F_k^c} |\nabla\varphi|\, \omega^{1/2}\omega^{-1/2}\, dx \\
&\leq \|\Phi\|_{L^\infty(\mathbb{R})} \left(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2\, \omega\, dx \right)^{1/2} \left(\int_{\Omega\cap F_k^c} \frac{1}{\omega}\, dx \right)^{1/2} \\
&\leq C_\varphi \|\Phi\|_{L^\infty(\Omega)} \left(\int_{\Omega\cap F_k^c} \frac{1}{\omega}\, dx \right)^{1/2}. \qquad (3.12)
\end{aligned}$$

By Theorem 2.9, Chapter IV of [5](or Lemma 15.8 of [6]), there exist constants $\delta > 0$ and $C > 0$ such that, if $\bar{\Omega} \subset Q_0$ (Q_0 is a fixed cube), then

$$\omega_m(\Omega\cap F_k^c) \leq \tilde{\omega}_2(\Omega \,\cap\, F_k^c) \leq C\, \tilde{\omega}_2(Q_0) \left(\frac{|F_k^c|}{|Q_0|} \right)^{\delta},$$

which is independent of m. Using Lemma 1.1, we know that $|F_k^c| \to 0$ when $k \to \infty$. Then, we obtain in (3.10), (3.11) and (3.12)

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} \langle \mathcal{A}^m \nabla u_m, \nabla\varphi\rangle\, dx = 0, \qquad (3.13)$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} b_m\, u_m \varphi\, dx = 0, \qquad (3.14)$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} \langle \Phi(u_m), \nabla\varphi\rangle\, dx = 0. \qquad (3.15)$$

Therefore, by (3.7), (3.13),(3.14) and (3.15) we conclude that (when $k \to \infty$)

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) D_i\tilde{u}(x) D_j\varphi(x)\, dx + \int_\Omega b(x)\, \tilde{u}(x)\, \varphi(x)\, dx - \sum_{j=1}^{n} \int_\Omega \Phi_j(\tilde{u}(x))\, D_j\varphi(x)\, dx$$
$$= \int_\Omega g(x)\varphi(x)\, dx + \sum_{j=1}^{n} \int_\Omega f_j(x) D_j\varphi(x)\, dx,$$

for all $\varphi \in W_0^{1,2}(\Omega, \omega)$, that is, $\tilde{u}$ is a solution of problem (P).

Therefore, $u = \tilde{u}$ (by the uniqueness) and u is the weak limit in $W_0^{1,2}(\Omega, \tilde{\omega}_1)$ of a sequence of solutions $u_m \in W_0^{1,2}(\Omega, \omega_m)$ of the problems (P_m).

Example. Let us have $\Omega = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2<1\}$. Consider the weight $\omega(x,y,z) = (x^2+y^2+z^2)^{-1}\in A_{3/5}$ $(r = 3/5)$, $p = 5$, $q = 45/34$, $0<a_1<a_2<a_3$ and the functions

$$\begin{aligned}
&\Phi:\mathbb{R}\to\mathbb{R}^3,\quad \Phi(t) = (\sin(t), 1-\cos(t), \sin(t)),\\
&g(x,y,z) = \arctan(1/(x^2+y^2+z^2))\ ,\ b(x,y,z) = \mathrm{e}^{-(x^2+y^2+z^2)},\\
&f_1(x,y,z) = \frac{\cos(1/(x^2+y^2+z^2))}{(x^2+y^2+z^2)^{1/3}},\ f_2(x,y,z) = \frac{\sin(1/(x^2+y^2+z^2))}{(x^2+y^2+z^2)^{1/3}},\\
&f_3(x,y,z) = 0.
\end{aligned}$$

Let us consider the partial differential operator

$$\begin{aligned}
Lu(x,y,z) = \quad & -\frac{\partial}{\partial x}\left(a_1(x^2+y^2+z^2)^{-1}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_2(x^2+y^2+z^2)^{-1}\frac{\partial u}{\partial y}\right)\\
& -\frac{\partial}{\partial z}\left(a_3(x^2+y^2+z^2)^{-1}\frac{\partial u}{\partial z}\right).
\end{aligned}$$

By Theorem 1.2, the problem

$$(P)\begin{cases}
Lu(x,y,z) + b(x,y,z)\,u(x,y,z) + \mathrm{div}(\Phi(u(x,y,z)))\\
= g(x,y,z) - \displaystyle\sum_{j=1}^{n} D_j f_j(x,y,z) \ \text{ in } \ \Omega,\\
u(x,y,z) = 0 \ \text{ on } \ \partial\Omega,
\end{cases}$$

has a unique solution $u\in W_0^{1,2}(\Omega,\omega)$ and u can be approximated by a sequence of solutions of nonlinear non-degenerate elliptic equations.

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