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GENERALIZED QUASIVARIATIONAL INEQUALITY PROBLEMS AND RELATED PROBLEMS

Bui The Hung

Pedagogical University of Thai Nguyen, Viet Nam
e-mail: hungbt.math@gmail.com

Abstract

In this paper, we study the existence theorems of solution for generalized quasivariational inequality problems. Some sufficient conditions on the existence of solutions of Pareto quasi-equilibrium problems, Pareto quasivariational inclusion problems and Pareto quasi-optimization problem with multivalued mappings are shown.

1 Introduction

Throuthout this paper, X, Z denote real Hausdorff locally convex topological vector spaces; Y be a real topological vector space and let $C \subseteq Y$ be a cone. We put $l(C) = C \cap (-C)$. If $l(C) = \{0\}, C$ is said to be pointed. Let Y^* be the topological dual space of Y. We denote by $\langle \xi, y \rangle$ the duality pair between $\xi \in Y^*$ and $y \in Y$. The topological dual cone C' and strict topological dual cone C'^{+} of C are defined as

$$C' := \{\xi \in Y' : \langle \xi, c \rangle \geq 0 \text{ for all } c \in C\},$$

$$C'^{+} := \{\xi \in Y' : \langle \xi, c \rangle > 0 \text{ for all } c \in C \backslash l(C)\}.$$

In this paper, we assume that C be a pointed cone with $C'^{+} \neq \emptyset$. Let $D \subseteq X, K \subseteq Z$ be nonempty subsets. Given multivalued mappings $P_1 : D \to$

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$2^D, P_2 : D \to 2^D, Q : D\times D \to 2^K$ with nonempty values and $F : K\times D\times D \to 2^Y$ with nonempty compact values. For any $\xi \in C'^+$, we consider the following problems:

(A_ξ) Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$\max_{z\in F(y,\bar{x},\bar{x})} \langle \xi, z\rangle \le \max_{z\in F(y,x,\bar{x})} \langle \xi, z\rangle \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x,\bar{x}).$$

(B_ξ) Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$\min_{z\in F(y,\bar{x},\bar{x})} \langle \xi, z\rangle \le \min_{z\in F(y,x,\bar{x})} \langle \xi, z\rangle \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x,\bar{x}).$$

There problems are called generalized quasivariational inequality problem in which the multivalued mappings P_1, P_2, Q are constraints, F is a utility multivalued mapping and $\xi \in C'^+$ is parameterized. In the cases F is a real function on $K \times D \times D$ and $C = \mathbb{R}_+$ then problems $(A_\xi), (B_\xi)$ becomes to find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$F(y,\bar{x},\bar{x}) \le F(y,x,\bar{x}) \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x,\bar{x}).$$

This is scalar quasivariational inequality problem studied in [7]. We know that scalar quasivariational inequality problem as generalizations of variational inequalities and optimization problems, including also many other related optimization problems such as fixed point problems, complementarity problems, Nash equilibria problems, minimax problems, etc. The purpose of this article is to establish sufficient conditions for the existence of solutions to problems $(A_\xi), (B_\xi)$. Moreover, we obtain some sufficient conditions for the existence of solutions of Pareto quasi-equilibrium problems, Pareto quasivariational inclusion problems and Pareto quasi-optimization problem with multivalued mappings.

2 Preliminaries

Given a subset $D \subseteq X$, we consider a multivalued mapping $F : D \to 2^Y$. The definition domain and the graph of F are denoted by

$$\operatorname{dom} F := \{x \in D : F(x) \neq \emptyset\},$$

$$\operatorname{gph} F := \{(x,y) \in D \times Y : y \in F(x)\},$$

respectively. We recall that F is said to be a closed (respectively, open) mapping if $\operatorname{gph} F$ is a closed (respectively, open) subset in the product space $X \times Y$. A multivalued mapping $F : D \to 2^Y$ is said to be upper (lower) semicontinuous in $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$ (respectively, $F(\bar{x}) \cap V \neq \emptyset$) there exists an open set U of $\bar{x}$ such that $F(x) \subseteq V$ (respectively, $F(x) \cap V \neq \emptyset$) for all $x \in U$.

Definition 2.1. Let Y be a linear space and C a nontrivial convex cone in Y. A nonempty convex subset B of C is called a base for C if each nonzero element $x \in C$ has a unique representation of the form $x = \lambda b$ with $\lambda > 0$ and $b \in B$.

Proposition 2.2 (See [3]). *Let Y be a Hausdorff locally convex space and C is a nontrivial cone in Y. Then C has a base B with $0 \notin \operatorname{cl} B$ if and only if $C'^{+} \neq \emptyset$.*

Remark 2.3. *If Y is locally convex Hausdorff space, C has a convex weakly* compact base, then $C'^{+} \neq \emptyset$.*

The following definitions will be used in the sequel.

Definition 2.4. Let $F : D \to 2^Y$ be a multivalued mapping. We say that F is a upper (lower) C-continuous in $\bar{x} \in \operatorname{dom} F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that:

$$F(x) \subseteq F(\bar{x}) + V + C$$

$$(F(\bar{x}) \subseteq F(x) + V - C, \text{ respectively})$$

holds for all $x \in U \cap \operatorname{dom} F$.

If F is upper C-continuous, lower C-continuous in any point of dom$F$, we say that it is upper C-continuous, lower C-continuous on D.

Definition 2.5. Let $F : D \times D \to 2^Y$ be a multivalued mapping. We say that:

(i) F is diagonally upper (lower) C-convex in the first variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0 (j = 1, 2, ..., n), \sum_{j=1}^{n} \alpha_j = 1$, one have

$$\sum_{j=1}^{n} \alpha_j F(x_j, x) \subseteq F(x, x) + C,$$

$$(\text{ respectively, } F(x, x) \subseteq \sum_{j=1}^{n} \alpha_j F(x_j, x) - C).$$

(ii) F is diagonally upper (lower) C-quasiconvex-like in the first variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0 (j = 1, 2, ..., n), \sum_{j=1}^{n} \alpha_j = 1$, there exists an index $i \in \{1, ..., n\}$ such that

$$F(x_i, x) \subseteq F(x, x) + C,$$

$$(\text{ respectively, } F(x, x) \subseteq F(x_i, x) - C).$$

Definition 2.6. Let $F : K \times D \times D \to 2^Y, Q : D \times D \to 2^K$ be multivalued mappings. We say that:

(i) F is diagonally upper (lower) (Q, C)-convex in the second variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0 (j = 1, 2, ..., n), \sum_{j=1}^{n} \alpha_j = 1$, there is an index $i \in \{1, 2, ..., n\}$ such that

$$\sum_{j=1}^{n} \alpha_j F(y, x_j, x) \subseteq F(y, x, x) + C \text{ for all } y \in Q(x_i, x),$$

$$(\text{ respectively, } F(y, x, x) \subseteq \sum_{j=1}^{n} \alpha_j F(y, x_j, x) - C \text{ for all } y \in Q(x_i, x)).$$

(ii) F is diagonally upper (lower) (Q, C)-quasiconvex-like in the second variable if for any finite set $\{x_1, ..., x_n\} \subseteq D, x = \sum_{j=1}^{n} \alpha_j x_j, \alpha_j \geq 0 (j = 1, 2, ..., n), \sum_{j=1}^{n} \alpha_j = 1$, there exists $i \in \{1, ..., n\}$ such that

$$F(y, x_i, x) \subseteq F(y, x, x) + C \text{ for all } y \in Q(x_i, x),$$

$$(\text{ respectively, } F(y, x, x) \subseteq F(y, x_i, x) - C \text{ for all } y \in Q(x_i, x)).$$

To prove the main results, we need the following theorems.

Theorem 2.7. (Fan- Browder, see [1]) *Let D be a nonempty convex compact subset of a topological vector space, $F : D \to 2^D$ be a multivalued map. Suppose that*

(i) *$F(x)$ is a nonempty convex subset of D for each $x \in D$;*

(ii) *$F^{-1}(x)$ is open in D for each $x \in D$.*

Then there exists $\bar{x} \in D$ such that $\bar{x} \in F(\bar{x})$.

Theorem 2.8. (See [2]) *Let X be a topological vector space and A and B are nonempty, and V is an open subsets of X. Assume that $G : A \to 2^B$ is a lower semicontinuous set-valued map. Then the set-valued map $\Phi(x) = (G(x)+V) \cap B$ is open in $A \times B$.*

Theorem 2.9. (See [11]) *Let X, Y be topological spaces and $F, G : X \to 2^Y$ be multivalued maps satisfy the following:*

(i) *F is open.*

(ii) *G is lower semicontinuous.*

Then the map $F \cap G$ is lower semicontinuous.

3 Existence of solutions for generalized quasivariational inequality problems

In this section we wish to establish an existence criterion for solutions of generalized quasivariational inequality problems. Let $M : K \times D \to 2^D$ be a multivalued mapping. First of all, we prove the following proposition.

Proposition 3.1. *Suppose that D is a nonempty convex compact subset and K is a nonempty subset and the multivalued mappings P_1, P_2, Q and M satisfy the following conditions:*

(i) *the set $W := \{x \in D : x \in P_1(x)\}$ is nonempty closed;*

(ii) *P_2 has nonempty values, $P_2^{-1}(x)$ is open and $\mathrm{co}(P_2(x)) \subseteq P_1(x)$ for all $x \in D$;*

(iii) *for each $t \in D$, the set*

$$B_t := \{x \in D : t \in M(y, x) \text{ for all } y \in Q(t, x)\}$$

is closed in D;

(iv) *for any finite subset $\{t_1, t_2, ..., t_n\}$ in D and $x \in \mathrm{co}\{t_1, t_2, ..., t_n\}$, there exists an index $j \in \{1, 2, ..., n\}$ such that $t_j \in M(y, x)$ for all $y \in Q(t_j, x)$.*
Then there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$x \in M(y, \bar{x}) \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

Proof. We define the multivalued mapping $G : D \to 2^D$ by

$$G(x) = \{t \in D : t \notin M(y, x) \text{ for some } y \in Q(t, x)\}.$$

By (iii) we have $G^{-1}(t) = D \backslash B_t$ is open in D, for all $t \in D$. We show that there exists $\bar{x} \in W$ such that $G(\bar{x}) \cap P_2(\bar{x}) = \emptyset$. On the contrary, suppose that $G(x) \cap P_2(x) \neq \emptyset$ for all $x \in W$. Now, we define the multivalued mapping $H : D \to 2^D$ by

$$H(x) = \begin{cases} \mathrm{co}\, G(x) \cap \mathrm{co}\, P_2(x), \text{ if } x \in W, \\ \mathrm{co}\, P_2(x), \text{ otherwise.} \end{cases}$$

Then $H(x)$ are nonempty convex for all $x \in D$ and

$$H^{-1}(x') = [(\mathrm{co}\, G)^{-1}(x') \cap (\mathrm{co}\, P_2)^{-1}(x')] \cup [(\mathrm{co}\, P_2)^{-1}(x') \cap D \backslash W] \text{ is open in } D.$$

By Fan- Browder fixed point theorem, there exists $x^* \in D$ such that $x^* \in H(x^*)$. Hence $x^* \in P_1(x^*)$ and $x^* \in \mathrm{co}\, G(x^*)$. This implies, there exists $\{t_1, t_2, ..., t_n\} \subseteq G(x^*)$ such that $x^* \in co\{t_1, t_2, ..., t_n\}$. By definition of G, for each $i \in \{1, 2, ..., n\}$, $t_i \notin M(y, x^*)$ for some $y \in Q(t_i, x^*)$. This contradicts with (iv). Hence there exists $\bar{x} \in W$ such that $G(\bar{x}) \cap P_2(\bar{x}) = \emptyset$. This implies $\bar{x} \in P_1(\bar{x})$ and

$$x \in M(y, \bar{x}) \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

The proof is complete. □

Now we are able to establish sufficient conditions for existence of solutions to (A_ξ).

Theorem 3.2. *Suppose that D is a nonempty convex compact subset, K is a nonempty subset and F with nonempty compact values. Assume that there exists $\xi \in C'^+$ such that the following conditions are fulfilled:*

(i) *the set $W := \{x \in D : x \in P_1(x)\}$ is nonempty closed;*

(ii) *P_2 has nonempty values, $P_2^{-1}(x)$ is open and $\mathrm{co}(P_2(x)) \subseteq P_1(x)$ for all $x \in D$;*

(iii) *for each $t \in D$, the set*

$$\Gamma_\xi^t := \{x \in D : \max_{z \in F(y,x,x)} \langle \xi, z \rangle \leq \max_{z \in F(y,t,x)} \langle \xi, z \rangle \text{ for all } y \in Q(t, x)\}$$

is closed in D;

(iv) *for any finite subset $\{t_1, t_2, ..., t_n\}$ in D and $x \in \mathrm{co}\{t_1, t_2, ..., t_n\}$, there exists an index $j \in \{1, 2, ..., n\}$ such that*

$$\max_{z \in F(y,x,x)} \langle \xi, z \rangle \leq \max_{z \in F(y,t_j,x)} \langle \xi, z \rangle \text{ for all } y \in Q(t_j, x).$$

Then (A_ξ) has a solution.

Proof. We define the multivalued mapping $M : K \times D \to 2^D$ by

$$M(y, x) = \{t \in D : \max_{z \in F(y,x,x)} \langle \xi, z \rangle \leq \max_{z \in F(y,t,x)} \langle \xi, z \rangle\}.$$

For any fixed $t \in D$, the set

$$\begin{aligned} B_t &= \{x \in D : t \in M(y, x) \text{ for all } y \in Q(t, x)\} \\ &= \{x \in D : \max_{z \in F(y,x,x)} \langle \xi, z \rangle \leq \max_{z \in F(y,t,x)} \langle \xi, z \rangle \text{ for all } y \in Q(t, x)\} \\ &= \Gamma_\xi^t \end{aligned}$$

is closed set in D. Now, let $\{t_1, t_2, ..., t_n\} \subseteq D$ and $x \in \mathrm{co}\{t_1, t_2, ..., t_n\}$. By (iv) we have there exists an index $j \in \{1, 2, ..., n\}$ such that

$$\max_{z \in F(y,x,x)} \langle \xi, z \rangle \leq \max_{z \in F(y,t_j,x)} \langle \xi, z \rangle \text{ for all } y \in Q(t_j, x).$$

This implies $t_j \in M(y,x)$ for all $y \in Q(t_j,x)$. Therefore, all the conditions of Proposition 3.1 are satisfied. Applying Proposition 3.1, there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$x \in M(y,\bar{x}) \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x,\bar{x}).$$

This implies $\bar{x} \in P_1(\bar{x})$ and

$$\max_{z\in F(y,\bar{x},\bar{x})} \langle \xi, z\rangle \le \max_{z\in F(y,x,\bar{x})} \langle \xi, z\rangle \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x,\bar{x}).$$

The proof is complete. □

Example 3.3. Consider problem (A_ξ) where $X = Y = Z = \mathbb{R}, C = \mathbb{R}_- := (-\infty,0], D = [0,1], K = (-1,2], P_1(x) = P_2(x) = Q(x,t) = [0,1]$ for all $x,t \in [0,1]$ and the multivalued mapping $F : K \times D \times D \to 2^{\mathbb{R}}$ by

$$F(y,x,t) = \begin{cases} [0,x], & \text{if } x \le t,\\ [x,1], & \text{otherwise.}\end{cases}$$

We easily check that $C'^{+} = (-\infty,0)$. Moreover, for each $\xi \in C'^{+}$ and $x,t \in [0,1]$, we have

$$\begin{aligned}\max_{z\in F(y,x,x)} \langle \xi, z\rangle &= \max_{z\in[0,x]} \langle \xi, z\rangle = 0,\\ \max_{z\in F(y,x,t)} \langle \xi, z\rangle &= \begin{cases} 0, & \text{if } x \le t,\\ \xi x, & \text{if } x > t.\end{cases}\end{aligned}$$

Then

$$\begin{aligned}\Gamma_\xi^t &:= \{x \in D : \max_{z\in F(y,x,x)} \langle \xi, z\rangle \le \max_{z\in F(y,t,x)} \langle \xi, z\rangle \text{ for all } y \in Q(t,x)\}\\ &= [0,t] \text{ is closed in } D.\end{aligned}$$

On the other hand, for any finite subset $\{t_1,t_2,...,t_n\}$ in D and $x = \sum_{i=1}^{n} \alpha_i t_i, \alpha_i \ge 0$
$(i = 1,2,...,n), \sum_{i=1}^{n} \alpha_i = 1$, there exists an index $j \in \{1,2,...,n\}$ such that $x \le t_j$.
This implies

$$\max_{z\in F(y,x,x)} \langle \xi, z\rangle = \max_{z\in F(y,t_j,x)} \langle \xi, z\rangle = 0 \text{ for all } y \in Q(t_j,x).$$

Then the assumptions in Theorem 3.2 are satisfied and $\bar{x} = 1$ is a unique solution of (A_ξ).

Theorem 3.4. *Suppose that D is a nonempty convex compact subset, K is a nonempty subset and F with nonempty compact vulues. Assume that there exists $\xi \in C'^{+}$ such that the satisfy conditions* (i), (iii), (iv) *of Theorem 3.2 and*

(ii') *P_2 is lower semicontinuous with nonempty values and $\mathrm{co}(P_2(x)) \subseteq P_1(x)$ for all $x \in D$.*
Then (A_ξ) has a solution.

Proof. Let $\mathcal{U}$ be a basis of convex neighborhood of the origin in the space X. For every $U \in \mathcal{U}$ we define the multivalued mappings $P_{1U}, P_{2U} : D \to 2^D$ by

$$P_{1U}(x) = (P_1(x) + \mathrm{cl}\, U) \cap D, P_{2U}(x) = (P_2(x) + U) \cap D,$$

where "cl" denotes the operation of taking the closure. It is easy to prove that $P_{2U}^{-1}(x)$ is open in D and $\mathrm{co}(P_{2U}(x)) \subseteq P_{1U}(x)$ for every $x \in D$. Therefore, all the conditions of Theorem 3.2 for P_{1U}, P_{2U}, Q and F are satisfied, there exists $\bar{x}_U \in D$ such that $\bar{x}_U \in P_{1U}(\bar{x}_U)$ and

$$\max_{z \in F(y, \bar{x}_U, \bar{x}_U)} \langle \xi, z \rangle \leq \max_{z \in F(y, x, \bar{x}_U)} \langle \xi, z \rangle \text{ for all } x \in P_{2U}(\bar{x}_U) \text{ and } y \in Q(x, \bar{x}_U).$$

This implies

$$N(\bar{x}_U) \cap P_{2U}(\bar{x}_U) = \emptyset,$$

where $N(x) = \{x' \in D : \max_{z \in F(y,x,x)} \langle \xi, z \rangle > \max_{z \in F(y,x',x)} \langle \xi, z \rangle$ for some $y \in Q(x', x)\}$. Let $A_U := \{x \in P_{1U}(x) : N(x) \cap P_{2U}(x) = \emptyset\}$. Then $A_U \neq \emptyset$. On the other hand, for each $x' \in D$, $N^{-1}(x') = D \backslash \Gamma_\xi^{x'}$ is open in D by (iii) of Theorem 3.2. Therefore, N is lower semicontinuous, so $N \cap P_{2U}$. Since P_{1U} is closed, A_U is closed. Moreover, A_U is decreasing as U decreases, and therefore the family of nonempty compact sets $\{A_U\}_{U \in \mathcal{U}}$ has a common point, say $\bar{x}$. This implies $\bar{x} \in P_{1U}(\bar{x})$ for all $U \in \mathcal{U}$ and $N(\bar{x}) \cap P_2(\bar{x}) = \emptyset$. We claim that $\bar{x} \in P_1(\bar{x})$. Indeed, assume that $\bar{x} \notin P_1(\bar{x})$. Then there exists $U^* \in \mathcal{U}$ such that $\bar{x} \notin P_1(\bar{x}) + U^*$. Since $U^* \in \mathcal{U}$, there exists $U \in \mathcal{U}$ such that $\mathrm{cl}\, U \subseteq U^*$. Therefore, $\bar{x} \notin P_{1U}(\bar{x})$. This contradicts. Hence $\bar{x}$ is a solution of (A_ξ). The proof is complete. □

Example 3.5. Consider problem (A_ξ) where $X = Y = Z = \mathbb{R}, C = \mathbb{R}_- := (-\infty, 0], D = [0, 1], K = (-1, 2], P_1(x) = Q(x, t) = [0, 1], P_2(x) = [0, x]$ for all $x, t \in [0, 1]$ and the multivalued mapping $F : K \times D \times D \to 2^{\mathbb{R}}$ by

$$F(y, x, t) = \begin{cases} [0, x], & \text{if } x \leq t, \\ [x, 1], & \text{otherwise.} \end{cases}$$

It is clear that P_2 does not have open inverse values. Hence Theorem 3.2 does not work. But P_2 is lower semicontinuous and the hypotheses of Theorem 3.4 are satisfied. We easily check that $\bar{x} = 1$ is a unique solution of (A_ξ).

Remark 3.6. *For each* $\xi \in C'^{+}$, *the assumption* (iii) *in Theorem 3.2 is satisfied provided that:*

(a) *for each* $t \in D$, $Q(t,.)$ *is lower semicontinuous with nonempty compact values;*

(b) *for any* $t \in D$, $F(.,t,.)$ *is upper* $(-C)$*- continuous and the multivalued mapping* $G : K \times D \to 2^Y$ *defined by* $G(y,x) = F(y,x,x)$ *is lower* C*-continuous.*

Proof. For $\epsilon > 0$ be arbitrary, since the continuity of ξ, there exists a neighborhood V of the origin in Y such that $\xi(V) \subseteq (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. Let $\{x_\alpha\}$ be a net from Γ^t_ξ converging to x_0. Then, we have

$$\max_{z\in F(y,x_\alpha,x_\alpha)} \langle \xi, z\rangle \leq \max_{z\in F(y,t,x_\alpha)} \langle \xi, z\rangle \text{ for all } y \in Q(t,x_\alpha).$$

For each $y \in Q(t,x_0)$, by the lower semicontinuity of $Q(t,.)$, there exists $y_\alpha \in Q(t,x_\alpha)$ converging to y. We have

$$\max_{z\in F(y_\alpha,x_\alpha,x_\alpha)} \langle \xi, z\rangle \leq \max_{z\in F(y_\alpha,t,x_\alpha)} \langle \xi, z\rangle \text{ for all } \alpha.$$

On the other hand, since $F(.,t,.) : K \times D \to 2^Y$ is upper $(-C)$- continuous and the multivalued mapping $G : K \times D \to 2^Y$ defined by $G(y,x) = F(y,x,x)$ is lower C- continuous, there exists an index α_0 such that

$$\begin{aligned} F(y_\alpha,t,x_\alpha) &\subseteq F(y,t,x_0) - C + V, \\ F(y,x_0,x_0) &\subseteq F(y_\alpha,x_\alpha,x_\alpha) - C + V \text{ for all } \alpha \geq \alpha_0. \end{aligned}$$

It follows that

$$\begin{aligned} \max_{z\in F(y_\alpha,t,x_\alpha)} \langle \xi, z\rangle &< \max_{z\in F(y,t,x_0)} \langle \xi, z\rangle + \frac{\epsilon}{2}, \\ \max_{z\in F(y,x_0,x_0)} \langle \xi, z\rangle &< \max_{z\in F(y_\alpha,x_\alpha,x_\alpha)} \langle \xi, z\rangle + \frac{\epsilon}{2} \text{ for all } \alpha \geq \alpha_0. \end{aligned}$$

Hence

$$\max_{z\in F(y,x_0,x_0)} \langle \xi, z\rangle < \max_{z\in F(y,t,x_0)} \langle \xi, z\rangle + \epsilon.$$

Therefore,

$$\max_{z\in F(y,x_0,x_0)} \langle \xi, z\rangle \leq \max_{z\in F(y,t,x_0)} \langle \xi, z\rangle \text{ for all } y \in Q(t,x_0).$$

This shows $x_0 \in \Gamma^t_\xi$. Consequently, Γ^t_ξ is closed. □

Remark 3.7. *For $\xi \in C'^{+}$, the condition* (iv) *of Theorem 3.2 is satisfied if one of the following conditions is satisfied:*

1. *for each $x \in D$, the set*

$$\Omega_\xi^x := \{t \in D : \max_{z \in F(y,x,x)} \langle \xi, z \rangle > \max_{z \in F(y,t,x)} \langle \xi, z \rangle \text{ for some } y \in Q(t,x)\}$$

is convex.

2. *for each $y \in K$, $F(y,.,.) : D \times D \to 2^Y$ is diagonally lower C- convex in the first variable.*
3. *F is diagonally lower (Q, C)- quasiconvex-like in the second variable.*

Proof. Let $\{t_1, t_2, ..., t_n\} \subseteq D$ and $x \in \mathrm{co}\{t_1, t_2, ..., t_n\}$.

1. Assume that for each $j \in \{1, 2, ..., n\}$ we have

$$\max_{z \in F(y,x,x)} \langle \xi, z \rangle > \max_{z \in F(y,t_j,x)} \langle \xi, z \rangle \text{ for some } y \in Q(t_j, x).$$

This implies $t_j \in \Omega_\xi^x$ for $j = 1, 2, ..., n$. By Ω_ξ^x is convex set, $x \in \Omega_\xi^x$. This contradicts. Hence there exists an index $j \in \{1, 2, ..., n\}$ such that

$$\max_{z \in F(y,x,x)} \langle \xi, z \rangle \leq \max_{z \in F(y,t_j,x)} \langle \xi, z \rangle \text{ for all } y \in Q(t_j, x).$$

2. Since $F(y,.,.)$ is diagonally lower C- convex in the first variable, then

$$F(y, x, x) \subseteq \sum_{i=1}^{n} \alpha_i F(y, t_i, x) - C \text{ for all } y \in K,$$

where $x = \sum\limits_{i=1}^{n} \alpha_i t_i, \alpha_i \geq 0 (i = 1, 2, ..., n), \sum\limits_{i=1}^{n} \alpha_i = 1$. This implies

$$\begin{aligned}
\max_{z \in F(y,x,x)} \langle \xi, z \rangle &\leq \max_{z \in \sum_{i=1}^{n} \alpha_i F(y,t_i,x)} \langle \xi, z \rangle \\
&\leq \sum_{i=1}^{n} \alpha_i \max_{z \in F(y,t_i,x)} \langle \xi, z \rangle \\
&\leq \max_{1 \leq i \leq n} \max_{z \in F(y,t_i,x)} \langle \xi, z \rangle,
\end{aligned}$$

Thus, there exists an index $j \in \{1, 2, ..., n\}$ such that

$$\max_{z \in F(y,x,x)} \langle \xi, z \rangle \leq \max_{z \in F(y,t_j,x)} \langle \xi, z \rangle \text{ for all } y \in Q(t_j, x).$$

3. If F is diagonally lower (Q, C)- quasiconvex-like in the second variable, there exists an index $j \in \{1, 2, ..., n\}$,

$$F(y, x, x) \subseteq F(y, t_j, x) - C \text{ for all } y \in Q(t_j, x).$$

This yields

$$\max_{z\in F(y,x,x)}\langle \xi, z\rangle \leq \max_{z\in F(y,t_j,x)}\langle \xi, z\rangle \text{ for all } y\in Q(t_j,x).$$

□ Since Theorem 3.4, Remark 3.6 and Remark 3.7, we have following corollarys :

Corollary 3.8. *Suppose that D is a nonempty convex compact subset, K is a nonempty subset and F with nonempty compact values. Assume that there exists $\xi\in C'^{+}$ such that the following conditions are fulfilled:*
(i) *the set $W := \{x\in D : x\in P_1(x)\}$ is nonempty closed;*
(ii) *P_2 is lower semicontinuous with nonempty values and $\mathrm{co}(P_2(x))\subseteq P_1(x)$ for all $x\in D$;*
(iii) *for each $t\in D$, the set*

$$\Gamma^t_\xi := \{x\in D : \max_{z\in F(y,x,x)}\langle \xi, z\rangle \leq \max_{z\in F(y,t,x)}\langle \xi, z\rangle \text{ for all } y\in Q(t,x)\}$$

is closed in D;
(iv) *for each $x\in D$, the set*

$$\Omega^x_\xi := \{t\in D : \max_{z\in F(y,x,x)}\langle \xi, z\rangle > \max_{z\in F(y,t,x)}\langle \xi, z\rangle \text{ for some } y\in Q(t,x)\}$$

is convex.
Then (A_ξ) has a solution.

Corollary 3.9. *Suppose that D is a nonempty convex compact subset, K is a nonempty subset and the multivalued mappings P_1, P_2, Q and F satisfy the following conditions:*
(i) *P_1 is closed;*
(ii) *P_2 has nonempty values, $P_2^{-1}(x)$ is open and $\mathrm{co}(P_2(x))\subseteq P_1(x)$ for all $x\in D$;*
(iii) *for each $x\in D$, $Q(x,.)$ is lower semicontinuous with nonempty compact values;*
(iv) *F has nonempty compact values, for any $x'\in D$, $F(.,x',.)$ is upper $(-C)$- continuous and the multivalued mapping $G : K\times D\to 2^Y$ defined by $G(y,x) = F(y,x,x)$ is lower C- continuous;*
(v) *for each $y\in K$, $F(y,.,.) : D\times D\to 2^Y$ is diagonally lower C- convex in the first variable (or, F is diagonally lower (Q,C)- quasiconvex-like in the second variable).*
Then (A_ξ) has a solution, for all $\xi\in C'^{+}$.

Remark 3.10. *If $Y=\mathbb{R}, C=\mathbb{R}_+$ and $F : K\times D\times D\to\mathbb{R}$ is a single map, then Corollary 3.9 reduces to Corollary 2.4 in [7].*

Analogically, we obtain the following theorems.

Theorem 3.11. *Suppose that D is a nonempty convex compact subset, K is a nonempty subset and F with nonempty compact values. Assume that there exists $\xi \in C'^+$ such that the following conditions are fulfilled:*

(i) *the set $W := \{x \in D : x \in P_1(x)\}$ is nonempty closed;*

(ii) *P_2 has nonempty values, $P_2^{-1}(x)$ is open and $\operatorname{co}(P_2(x)) \subseteq P_1(x)$ for all $x \in D$;*

(iii) *for each $t \in D$, the set*

$$\gamma_\xi^t := \{x \in D : \min_{z\in F(y,x,x)} \langle \xi, z\rangle \leq \min_{z\in F(y,t,x)} \langle \xi, z\rangle \text{ for all } y \in Q(t,x)\}$$

is closed in D;

(iv) *for any finite subset $\{t_1, t_2, ..., t_n\}$ in D and $x \in \operatorname{co}\{t_1, t_2, ..., t_n\}$, there exists an index $j \in \{1, 2, ..., n\}$ such that*

$$\min_{z\in F(y,x,x)} \langle \xi, z\rangle \leq \min_{z\in F(y,t_j,x)} \langle \xi, z\rangle \text{ for all } y \in Q(t_j, x).$$

Then (B_ξ) has a solution.

Proof. We define the multivalued mapping $M : K \times D \to 2^D$ by

$$M(y, x) = \{t \in D : \min_{z\in F(y,x,x)} \langle \xi, z\rangle \leq \min_{z\in F(y,t,x)} \langle \xi, z\rangle\},$$

Following the same arguments as in the of Theorem 3.2 we show that there exists $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$x \in M(y, \bar{x}) \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

This implies $\bar{x} \in P_1(\bar{x})$ and

$$\min_{z\in F(y,\bar{x},\bar{x})} \langle \xi, z\rangle \leq \min_{z\in F(y,x,\bar{x})} \langle \xi, z\rangle \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

The proof is complete. □

Theorem 3.12. *Suppose that D is a nonempty convex compact subset, K is a nonempty subset and F with nonempty compact vulues. Assume that there exists $\xi \in C'^+$ such that the satisfy conditions* (i), (iii), (iv) *of Theorem 3.11 and*

(ii') *P_2 is lower semicontinuous with nonempty values and $\operatorname{co}(P_2(x)) \subseteq P_1(x)$ for all $x \in D$.*

Then (B_ξ) has a solution.

Proof. The proof is similar to the one of Theorem 3.4. □

Remark 3.13. 1. *For each $\xi \in C'^{+}$, the assumption* (iii) *in Theorem 3.11 is satisfied provided that:*

(a) *for each $x \in D$, $Q(x,.)$ is lower semicontinuous with nonempty compact values;*

(b) *for any $x' \in D$, $F(.,x',.)$ is lower $(-C)$- continuous and the multivalued mapping $G : K \times D \to 2^Y$ defined by $G(y,x) = F(y,x,x)$ is upper C-continuous;*

2. *for $\xi \in C'^{+}$, if F is upper (Q,C)- quasiconvex-like in the second variable, then condition* (iv) *in Theorem 3.11 is satisfied.*

Proof. The proof is similar to the one of Remark 3.6 and Remark 3.7. □

Corollary 3.14. *Suppose that D is a nonempty convex compact subset, K is a nonempty subset and F with nonempty compact values. Assume that there exists $\xi \in C'^{+}$ such that the following conditions are fulfilled:*

(i) *the set $W := \{x \in D : x \in P_1(x)\}$ is nonempty closed;*

(ii) *P_2 has nonempty values, $P_2^{-1}(x)$ is open and* $\mathrm{co}(P_2(x)) \subseteq P_1(x)$ *for all $x \in D$;*

(iii) *for each $t \in D$, the set*

$$\gamma_\xi^t := \{x \in D : \min_{z \in F(y,x,x)} \langle \xi, z\rangle \le \min_{z \in F(y,t,x)} \langle \xi, z\rangle \text{ for all } y \in Q(t,x)\}$$

is closed in D;

(iv) *for each $x \in D$, the set*

$$\omega_\xi^x := \{t \in D : \min_{z \in F(y,x,x)} \langle \xi, z\rangle > \min_{z \in F(y,t,x)} \langle \xi, z\rangle \text{ for some } y \in Q(t,x)\}$$

is convex.

Then (B_ξ) has a solution.

Corollary 3.15. *Suppose that D is a nonempty convex compact subset, K is a nonempty subset and the multivalued mappings P_1, P_2, Q and F satisfy the following conditions:*

(i) *P_1 is upper semicontinuous with nonempty convex closed values;*

(ii) *P_2 has nonempty values, $P_2^{-1}(x)$ is open and* $\mathrm{co}(P_2(x)) \subseteq P_1(x)$ *for all $x \in D$;*

(iii) *for each $x \in D$, $Q(x,.)$ is lower semicontinuous with nonempty compact values;*

(iv) *F has nonempty compact values, for any $x' \in D$, $F(.,x',.)$ is lower $(-C)$- continuous and the multivalued mapping $G : K \times D \to 2^Y$ defined by $G(y,x) = F(y,x,x)$ is upper C- continuous;*

(v) *F is diagonally upper (Q,C)- quasiconvex-like in the second variable.*

Then (B_ξ) has a solution, for all $\xi \in C'^{+}$.

4 Pareto quasi-equilibrium problems

In this section, we apply the obtained results in Section 3 to Pareto quasi-equilibrium problems with multivalued mappings. Let D, K, P_1, P_2, Q and F be as Introduction. We consider the following Pareto quasi-equilibrium problems:

(UPQEP) Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$F(y, x, \bar{x}) \nsubseteq -C\backslash\{0\} \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

(LPQEP) Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$F(y, x, \bar{x}) \cap (-C\backslash\{0\}) = \emptyset \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

Corollary 4.1. *If $\bar{x}$ is a solution of (A_ξ) and $F(y, \bar{x}, \bar{x}) \cap C \neq \emptyset$ for all $y \in K$, then $\bar{x}$ is a solution of (UPQEP).*

Proof. Assume that $\bar{x}$ is a solution of (A_ξ), that is $\bar{x} \in P_1(\bar{x})$ and

$$\max_{z \in F(y, \bar{x}, \bar{x})} \langle \xi, z \rangle \leq \max_{z \in F(y, x, \bar{x})} \langle \xi, z \rangle \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

Since $F(y, \bar{x}, \bar{x}) \cap C \neq \emptyset$,

$$\max_{z \in F(y, \bar{x}, \bar{x})} \langle \xi, z \rangle \geq 0.$$

This implies

$$\max_{z \in F(y, x, \bar{x})} \langle \xi, z \rangle \geq 0 \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}). \tag{4.1}$$

We show that

$$F(y, x, \bar{x}) \nsubseteq -C\backslash\{0\} \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

Assume that there exists $x^* \in P_2(\bar{x})$ and $y^* \in Q(x^*, \bar{x})$ such that

$$F(y^*, x^*, \bar{x}) \subseteq -C\backslash\{0\}.$$

Thus,

$$\max_{z \in F(y^*, x^*, \bar{x})} \langle \xi, z \rangle < 0.$$

This contradicts (4.1).
Hence $\bar{x}$ is solution of (UPQEP). The proof of the corollary is complete. □

Corollary 4.2. *If $\bar{x}$ is a solution of (B_ξ) and $F(y, \bar{x}, \bar{x}) \subseteq C$ for all $y \in K$, then $\bar{x}$ is a solution of (LPQEP).*

Proof. Assume that $\bar{x}$ is a solution of (B_ξ), that is $\bar{x} \in P_1(\bar{x})$ and

$$\min_{z \in F(y,\bar{x},\bar{x})} \langle \xi, z \rangle \leq \min_{z \in F(y,x,\bar{x})} \langle \xi, z \rangle \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

By $F(y, \bar{x}, \bar{x}) \subseteq C$ we have

$$\min_{z \in F(y,\bar{x},\bar{x})} \langle \xi, z \rangle \geq 0.$$

This implies that

$$\min_{z \in F(y,x,\bar{x})} \langle \xi, z \rangle \geq 0 \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}). \tag{4.2}$$

We now claim that

$$F(y, x, \bar{x}) \cap (-C \backslash \{0\}) = \emptyset \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

On the contrary, suppose that there exists $x^* \in P_2(\bar{x})$ and $y^* \in Q(x^*, \bar{x})$ such that

$$F(y^*, x^*, \bar{x}) \cap (-C \backslash \{0\}) \neq \emptyset.$$

There is $\bar{a}$ such that $\bar{a} \in F(y^*, x^*, \bar{x}) \cap (-C \backslash \{0\})$. Then, we have

$$\min_{z \in F(y^*,x^*,\bar{x})} \langle \xi, z \rangle \leq \langle \xi, \bar{a} \rangle < 0.$$

This contradicts (4.2).
Hence $\bar{x}$ is a solution of (LPQEP). The proof of the corollary is complete. $\square$

Example 4.3. Consider problem $(UPQEP)$ where $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = (-\infty, 0] \times (-\infty, 0], D = [0,1], K = (-1, 2], P_1(x) = P_2(x) = Q(x,t) = [0,1]$ for all $x, t \in [0,1]$ and the multivalued mapping $F : K \times D \times D \to 2^{\mathbb{R}^2}$ by

$$F(y, x, t) = [0, t] \times [0, xy].$$

We easily check that the assumptions in Theorem 3.2 are satisfied. Moreover, for each $\xi \in C'^{+} = (-\infty, 0) \times (-\infty, 0)$ and $\bar{x} \in [0,1]$, we have

$$\max_{z \in F(y,\bar{x},\bar{x})} \langle \xi, z \rangle = \max_{z \in F(y,x,\bar{x})} \langle \xi, z \rangle = 0.$$

Then $\bar{x} = 1$ is a unique solution of (A_ξ). On the other hand, we have $F(y, \bar{x}, \bar{x}) \cap C \neq \emptyset$ for all $y \in K$ and $\bar{x} = 1$ is a solution of $(UPQEP)$.

5 Pareto quasivariational inclusion problems

In recent years, there are many papers on quasivariational inclusion problems (see [4], [5], [6], [7], [8], [9], [10], ...). However, most of these articles deal with ideal quasivariational inclusion problems and there are only few articles for the study of Pareto quasivariational inclusion problems. In this section, we prove the existence of solutions to the following Pareto quasivariational inclusion problems:

(UPQVIP) Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$F(y, x, \bar{x}) \not\subseteq F(y, \bar{x}, \bar{x}) - C\backslash\{0\} \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

(LPQVIP) Find $\bar{x} \in D$ such that $\bar{x} \in P_1(\bar{x})$ and

$$F(y, \bar{x}, \bar{x}) \not\subseteq F(y, x, \bar{x}) + C\backslash\{0\} \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

Corollary 5.1. *If $\bar{x}$ is a solution of (A_ξ) then $\bar{x}$ is solution of (UPQVIP).*

Proof. Assume that $\bar{x}$ is a solution of (A_ξ), that is $\bar{x} \in P_1(\bar{x})$ and

$$\max_{z \in F(y,\bar{x},\bar{x})} \langle \xi, z \rangle \leq \max_{z \in F(y,x,\bar{x})} \langle \xi, z \rangle \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}). \tag{5.1}$$

We claim that

$$F(y, x, \bar{x}) \not\subseteq F(y, \bar{x}, \bar{x}) - C\backslash\{0\} \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

Assume that there exists $\hat{x} \in P_2(\bar{x})$ and $\hat{y} \in Q(\hat{x}, \bar{x})$ such that

$$F(\hat{y}, \hat{x}, \bar{x}) \subseteq F(\hat{y}, \bar{x}, \bar{x}) - C\backslash\{0\}.$$

It follows that

$$\max_{z \in F(\hat{y},\hat{x},\bar{x})} \langle \xi, z \rangle < \max_{z \in F(\hat{y},\bar{x},\bar{x})} \langle \xi, z \rangle.$$

This contradicts (5.1).
Thus, $\bar{x}$ is solution of (UPQVIP). The proof is complete. □

Corollary 5.2. *If $\bar{x}$ is a solution of (B_ξ) then $\bar{x}$ is solution of (LPQVIP).*

Proof. Assume that $\bar{x}$ is a solution of (B_ξ), that is $\bar{x} \in P_1(\bar{x})$ and

$$\min_{z \in F(y,\bar{x},\bar{x})} \langle \xi, z \rangle \leq \min_{z \in F(y,x,\bar{x})} \langle \xi, z \rangle \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}). \tag{5.2}$$

We show that

$$F(y, \bar{x}, \bar{x}) \not\subseteq F(y, x, \bar{x}) + C\backslash\{0\} \text{ for all } x \in P_2(\bar{x}) \text{ and } y \in Q(x, \bar{x}).$$

Assume that there exists $\hat{x} \in P_2(\bar{x})$ and $\hat{y} \in Q(\hat{x}, \bar{x})$ such that

$$F(\hat{y}, \bar{x}, \bar{x}) \subseteq F(\hat{y}, \hat{x}, \bar{x}) + C\backslash\{0\}.$$

We conclude

$$\min_{z \in F(\hat{y}, \hat{x}, \bar{x})} \langle \xi, z \rangle < \min_{z \in F(\hat{y}, \bar{x}, \bar{x})} \langle \xi, z \rangle.$$

This contradicts (5.2).
Hence $\bar{x}$ is solution of (LPQVIP). The proof is complete. □

Example 5.3. Consider problem $(UPQVIP)$ where $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = (-\infty, 0] \times (-\infty, 0], D = [0,1], K = (-1,2], P_1(x) = P_2(x) = [0,1], Q(x,t) = [0,t]$ for all $x, t \in [0,1]$ and the multivalued mapping $F : K \times D \times D \to 2^{\mathbb{R}^2}$ by

$$F(y, x, t) = [yt, 1] \times [x, 1].$$

We easily check that the assumptions in Theorem 3.2 are satisfied. Moreover, for each $\xi := (\xi_1, \xi_2) \in C'^{+} = (-\infty, 0) \times (-\infty, 0)$ and $\bar{x} \in [0,1]$, we have

$$\begin{aligned} \max_{z \in F(y, \bar{x}, \bar{x})} \langle \xi, z \rangle &= \max_{z \in [y\bar{x}, 1] \times [\bar{x}, 1]} \langle \xi, z \rangle = \xi_1 y \bar{x} + \xi_2 \bar{x}, \\ \max_{z \in F(y, x, \bar{x})} \langle \xi, z \rangle &= \max_{z \in [y\bar{x}, 1] \times [x, 1]} \langle \xi, z \rangle = \xi_1 y \bar{x} + \xi_2 x. \end{aligned}$$

Then $\bar{x}$ is a solution of (A_ξ) if and only if

$$\xi_1 y \bar{x} + \xi_2 \bar{x} \leq \xi_1 y \bar{x} + \xi_2 x \text{ for all } x \in [0,1] \text{ and } y \in [0, \bar{x}].$$

This inequality holds if and only if $\bar{x} = 1$. Thus, $\bar{x} = 1$ is a unique solution of (A_ξ) and so $(UPQVIP)$.

6 Pareto quasi-optimization problem

Let $A \subseteq Y$ be a nonempty set and let $x_0 \in A$. We say that x_0 is an Pareto efficient point of A with respect to C if there is no $x \in A$ such that $x_0 - x \in C\backslash\{0\}$. The set of all Pareto efficient point of A with respect to C is denoted by $\mathrm{PMin}(A|C)$. Let $P : D \to 2^D, Q : D \to 2^K$ and $F : K \times D \times D \to 2^Y$ be multivalued mappings. In this section, we shall apply Corollary 3.15 to the following Pareto quasi-optimization problem: Find $\bar{x} \in D$ such that $\bar{x} \in P(\bar{x})$ and

$$F(y, \bar{x}, \bar{x}) \cap \mathrm{PMin}(F(y, \bar{x}, P(\bar{x}))|C) \neq \emptyset \text{ for all } y \in Q(\bar{x}).$$

Corollary 6.1. *If $\bar{x}$ is a solution of (B_ξ) for $P_1 = P_2 = P, Q$ and F, then $\bar{x}$ is a solution of Pareto quasi-optimization problem.*

Proof. Assume that $\bar{x}$ is a solution of (B_ξ), that is $\bar{x} \in P(\bar{x})$ and

$$\min_{z\in F(y,\bar{x},\bar{x})}\langle \xi, z\rangle \leq \min_{z\in F(y,x,\bar{x})}\langle \xi, z\rangle \text{ for all } x \in P(\bar{x}) \text{ and } y \in Q(\bar{x}). \tag{6.1}$$

Assume that there exists $\bar{y} \in Q(\bar{x})$ such that

$$F(\bar{y},\bar{x},\bar{x}) \cap \mathrm{PMin}(F(\bar{y},\bar{x},P(\bar{x}))|C) = \emptyset.$$

Let $\bar{v} \in F(\bar{y},\bar{x},\bar{x})$ such that

$$\langle \xi, \bar{v}\rangle = \min_{z\in F(\bar{y},\bar{x},\bar{x})}\langle \xi, z\rangle.$$

Since $\bar{v} \notin \mathrm{PMin}(F(\bar{y},\bar{x},P(\bar{x}))|C)$, there exists $x^* \in P(\bar{x})$ and $v^* \in F(\bar{y},\bar{x},x^*)$ such that

$$\bar{v} - v^* \in C\backslash\{0\}.$$

This implies

$$\min_{z\in F(\bar{y},\bar{x},\bar{x})}\langle \xi, z\rangle = \langle \xi, \bar{v}\rangle > \langle \xi, v^*\rangle \geq \min_{z\in F(\bar{y},\bar{x},x^*)}\langle \xi, z\rangle.$$

This contradicts (6.1).
Hence

$$F(y,\bar{x},\bar{x}) \cap \mathrm{PMin}(F(y,\bar{x},P(\bar{x}))|C) \neq \emptyset \text{ for all } y \in Q(\bar{x}).$$

The proof of the corollary is complete. □

Example 6.2. Consider Pareto quasi-optimization problem, where $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+ := [0,+\infty) \times [0,+\infty), D = [0,1], K = (-1,2], P(x) = Q(x) = [0,1]$ for all $x \in [0,1]$ and the multivalued mapping $F : K\times D\times D \to \mathbb{R}^2$ by

$$F(y,x,t) = (x - x^2, 1 - yt) \text{ for all } (y,x,t) \in K \times D \times D.$$

Now we check that the condition (v) in Corollary 3.15 satisfied. Indeed, let $\{x_1, x_2, ..., x_n\} \subseteq D$ and $x = \sum\limits_{i=1}^{n}\alpha_i x_i, \alpha_i \geq 0 (i = 1, 2, ..., n), \sum\limits_{i=1}^{n}\alpha_i = 1$. We have

$$\sum_{i=1}^{n}\alpha_i x_i - (\sum_{i=1}^{n}\alpha_i x_i)^2 = \sum_{i=1}^{n}\alpha_i x_i(1 - \sum_{i=1}^{n}\alpha_i x_i) \leq \max_{1\leq i\leq n} x_i(1 - \max_{1\leq i\leq n} x_i) = x_j - x_j^2,$$

where $x_j = \max\limits_{1\leq i\leq n} x_i$. This implies

$$\begin{aligned} F(y,x,x) &= (x - x^2; 1 - yx) = (\sum_{i=1}^{n}\alpha_i x_i - (\sum_{i=1}^{n}\alpha_i x_i)^2; 1 - yx) \\ &\in (x_j - x_j^2; 1 - yx) + \mathbb{R}^2_+ = F(y, x_j, x) + \mathbb{R}^2_+ \text{ for all } y \in [0,1]. \end{aligned}$$

Hence F is diagonally upper $(Q, \mathbb{R}^2_+)-$ quasiconvex-like in the second variable. Therefore, all conditions of Corollary 3.15 are fulfilled.
Moreover, for each $\xi := (\xi_1, \xi_2) \in C'^+ = (0, +\infty) \times (0, +\infty)$ and $\bar{x} \in [0, 1]$, we have

$$\begin{aligned} \min_{z \in F(y,\bar{x},\bar{x})} \langle \xi, z \rangle &= \xi_1(\bar{x} - \bar{x}^2) + \xi_2(1 - y\bar{x}), \\ \min_{z \in F(y,x,\bar{x})} \langle \xi, z \rangle &= \xi_1(x - x^2) + \xi_2(1 - y\bar{x}). \end{aligned}$$

Thus, $\bar{x}$ is a solution of (B_ξ) if and only if

$$\bar{x} - \bar{x}^2 \le x - x^2 \text{ for all } x \in [0, 1].$$

This inequality holds if and only if $\bar{x} \in \{0, 1\}$. Thus, $\{0, 1\}$ is solution set of (B_ξ).

A direct calculation shows that

$$F(y, 0, 0) \in \mathrm{PMin}(F(y, 0, P(0))|\mathbb{R}^2_+) \text{ for all } y \in Q(0).$$

$$F(y, 1, 1) \in \mathrm{PMin}(F(y, 1, P(1))|\mathbb{R}^2_+) \text{ for all } y \in Q(1).$$

Hence $\bar{x} = 0, \bar{x} = 1$ is solutions of Pareto quasi-optimization problem.

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