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A NOTE ON ROUGH STATISCAL CONVERGENCE

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Abstract

In this paper, we introduce the notions of pointwise rough statistical convergence and rough statistically Cauchy sequences of real valued functions in the line of A. (Türkmenoglu) Gökhan and M. Güngör [6]. Furthermore we study their equivalence.

1 Introduction:

The idea of statistical convergence was first introduced by Fast [3] and also by Schoenberg[14] independently, by using the idea of natural density. Let K be a subset of the set of positive integers $\mathbb{N}$. Let K_n be a set defined by $K_n = \{k \in K : k \leq n\}$. Then the natural density of K is defined as $d(K) = \lim\limits_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n. Clearly finite set has natural density zero.

A real sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be pointwise statistically convergent ti ξ on a set S if for every $\varepsilon > 0$, $\lim\limits_{n\to\infty} |\{k \leq n : |x_n - \xi| \geq \varepsilon \text{ for } \ \xi \in S\}| = 0$.

The idea of rough convergence was introduced by Phu [12] in finite dimensional normed linear spaces. If $x = \{x_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers and r is a nonnegative real number, then x is said to be *rough convergent* to $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - \xi| < r + \varepsilon \text{ for all } n \geq N.$$

Key words: Pointwise rough statistical convergence, rough statistically Cauchy sequence.
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A lot of work has been done in this area by Phu [12],[13], Aytar [1], Pal, Chandra and Dutta [10] , Malik and Maity [8], [9].

Recently the idea of pointwise statistical convergence of a sequence of real valued functions is introduced by G*ö*khan and G*ü*ng*ö*r [6]. Some works in this direction can be found in [2], [7]. So it is quite natural to think whether the notion of rough convergence can be introduced for the pointwise statistical convergence of real valued functions. In this paper we do the same and introduce the notion of pointwise rough statistical convergence of sequence of real valued functions and study some basic properties of this type of convergence.

2 Rough Statistical Convergent Sequence of Real Valued Functions

Definition 2.1. A sequence of real valued functions $\{f_k\}_{k\in\mathbb{N}}$ on a set X is said to be *pointwise rough statistically convergent* to a function f on a set A if for every $\varepsilon > 0$ there exists a real number $r > 0$ such that

$$\lim_{n\to\infty}\frac{1}{n}|\{k \leq n : |f_k(x) - f(x)| \geq r + \varepsilon\}| = 0$$

for every $x \in X$. That is

$$|f_k(x) - f(x)| < r + \varepsilon \quad a.a.k. \tag{1}$$

for every $x \in X$

In this case we write $r\text{-}st\text{-}\lim f_k(x) = f(x)$ on A. This means that for every $\delta > 0$ there exists an integer N such that $\frac{1}{n}|\{k \leq n : |f_k(x)-f(x)| \geq r+\varepsilon\}| < \delta$ for every $x \in A$ and for all $n > N(= N(\varepsilon, \delta, x))$ and for every $\varepsilon > 0$.

We now define a sequence of real valued functions which is not pointwise statistically convergent but pointwise rough statistically convergent.

Example 2.1. *Define a sequence of real valued functions* $\{f_i\}_{i\in\mathbb{N}}$ *by*

$$\begin{aligned} f_i(x) &= \frac{1}{1+x^i} \quad \textit{if} \quad i \neq k^2, k = 1, 2, 3, \cdots \\ &= i, \quad \textit{otherwise} \end{aligned}$$

on $[0, 1]$.

The sequence $\{f_i\}_{i\in\mathbb{N}}$ is pointwise rough statistically convergent to zero with roughness degree 1. But this sequence is not pointwise statistically convergent.

Theorem 2.1. *Let* $\{f_k\}_{k\in\mathbb{N}}$ *and* $\{g_k\}_{k\in\mathbb{N}}$ *be two sequences of real valued functions defined on a set* X*. If* $r\text{-}st\text{-}\lim f_k(x) = f(x)$ *and* $r\text{-}st\text{-}\lim g_k(x) = g(x)$ *on* X*, then* $r\text{-}st\text{-}\lim(\alpha f_k(x) + \beta g_k(x)) = \alpha f_k(x) + \beta g_k(x)$ *for all* $\alpha, \beta \in \mathbb{R}$.

Proof The proof is immediate for $\alpha = 0$ and $\beta = 0$. Let $\alpha \neq 0$ and $\beta \neq 0$. Let $r > 0$ be a real number. Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}&\{k \leq n : |\alpha f_k(x) + \beta g_k(x) - (\alpha f(x) + \beta g(x))| \geq r + \varepsilon, \quad \text{for any} \quad x \in X\}\\ &\subseteq \{k \leq n : |\alpha f_k(x) - \alpha f(x)| \geq r + \varepsilon, \quad \text{for any} \quad x \in X\} \bigcup \\ &\{k \leq n : |\beta g_k(x) - \beta g(x))| \geq r + \varepsilon, \quad \text{for any} \quad x \in X\}.\end{aligned}$$

Since $|\alpha f_k(x) + \beta g_k(x) - \alpha f(x) + \beta g(x))| \geq \ |\alpha f_k(x) - (\alpha f(x)| + |\beta g_k(x) - \beta g(x)|$, hence we obtain r-st-$\lim(\alpha f_k(x) + \beta g_k(x)) = \alpha f(x) + \beta g(x)$ on X. $\square$

Now we introduce the rough statistical analog to the Cauchy convergence criterion.

Definition 2.2. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of real valued functions defined on a set X. Let $r > 0$ be a real number. The sequence $\{f_k\}_{k\in\mathbb{N}}$ is said to be *rough statistically Cauchy sequence* if for every $\varepsilon > 0$ there exists a natural number $N(= N(\varepsilon, x))$ such that $|f_k(x) - f_N(x)| < r + \varepsilon$ a.a.k. that is $\lim\limits_{n\to\infty} \frac{1}{n}|\{k \leq n : |f_k(x) - f_N(x)| \geq r + \varepsilon| = 0$ for every $x \in X$.

Theorem 2.2. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of real valued function defined on a set X. Then, the following statements are equivalent.*

(i) $\{f_k\}_{k\in\mathbb{N}}$ is a pointwise rough statistically convergent sequence on X.

(ii) $\{f_k\}_{k\in\mathbb{N}}$ is a rough statistically Cauchy sequence on X.

(iii) $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of real valued functions on X for which there exists a point wise rough statistically convergent sequence of real valued functions $\{g_k\}_{k\in\mathbb{N}}$ such that $f_k(x) = g_k(x)$ a.a.k. for every $x \in X$.

Proof $(i) \Rightarrow (ii)$. Let $r > 0$ be a real number. Let r-st-$\lim f_k(x) = f(x)$. Then for every $\varepsilon > 0$ $|f_k(x) - f(x)| < \frac{r+\varepsilon}{2}$ for every $x \in X$ a.a.k. Let N be a natural number so chosen that $|f_N(x) - f(x)| < \frac{r+\varepsilon}{2}$ for every $x \in X$. Then we have $|f_k(x) - f_N(x)| \leq |f_k(x) - f(x)| + |f_N(x) - f(x)| < \frac{r+\varepsilon}{2} + \frac{r+\varepsilon}{2} = r + \varepsilon$ a.a.k. for every $x \in X$. Hence $\{f_k\}_{k\in\mathbb{N}}$ is a rough statistically Cauchy sequence.

$(ii) \Rightarrow (iii)$. Assume that (ii) is true. Choose a natural number N such that the band $[f_N(x) - 1, f_N(x) + 1] = I$ contains $f_k(x)$ a.a.k. for every $x \in X$. By (ii) we get a natural number M such that $[f_M(x) - \frac{1}{2}, f_M(x) + \frac{1}{2}] = I'$ contains $f_k(x)$ a.a.k. for every $x \in X$. Hence $I \cap I' = I_1$ contains $f_k(x)$ a.a.k. for every $x \in X$. Now $\{k \leq n : f_k(x) \notin I \cap I', \quad \text{for every} \quad x \in X\} = \{k \leq n : f_k(x) \notin I, \quad \text{for every} \quad x \in X\} \bigcup \{k \leq n : f_k(x) \notin I', \quad \text{for every} \quad x \in X\}$.
Hence $\lim\limits_{n\to\infty} \frac{1}{n}|\{k \leq n : f_k(x) \notin I \cap I' \quad \text{for every} x \in X\}| \leq \lim\limits_{n\to\infty} \frac{1}{n}|\{k \leq n : f_k(x) \notin I \quad \text{for every} x \in X\}| + \lim\limits_{n\to\infty} \frac{1}{n}|\{k \leq n : f_k(x) \notin I' \quad \text{for every} x \in X\}| = 0$. Therefore I_1 is a closed band of height less than or equal to 1 which contains $f_k(x)$ a.a.k. for every $x \in X$.

Continuing by the same manner we can get N_2 such that $I^{''} = [f_{N_2}(x) - \frac{1}{4}, f_{N_2}(x) + \frac{1}{4}]$ contains $f_k(x)$ a.a.k. and by above argument $I \cap I^{''} = I_2$ contains $f_k(x)$ a.a.k for every $x \in X$ and the height of I_2 is less than or equal to $\frac{1}{2}$. By induction principle, we can construct a sequence $\{I_m\}_{m\in\mathbb{N}}$ of closed band such that for each $m, I_m \supseteq I_{m+1}$ and the height of I_m is not greater than 2^{1-m} and $f_k(x) \in I_m$ a.a.k. for every $x \in X$. Thus there exists a function $f(x)$ defined on X such that $\{f(x)\}$ is equal to $\bigcap\limits_{m=1}^{\infty} I_m$. Using the fact that $f_k(x) \in I_m$ a.a.k. for every $x \in X$, we can choose an increasing positive integer sequence $\{J_m\}_{m\in\mathbb{N}}$ such that

$$\frac{1}{n}|\{k \le n : f_k(x) \notin I_m \quad \text{for every} \quad x \in X\}| < \frac{1}{m} \ \text{ if } \ n > J_m \tag{2}$$

We now construct a subsequence $\{h_k(x)\}$ of $\{f_k(x)\}$ containing of all terms $f_k(x)$ such that $k > J_1$ and when $J_m < k \le J_{m+1}$ then $f_k(x) \notin I_m$ for every $x \in X$.

Define a sequence of real valued functions $\{g_k(x)\}$ by

$$\begin{aligned} g_k(x) &= f(x), \ \text{ if } \ f_k(x) \quad \text{is a term of} \quad \{h_k(x)\} \\ &= f_k(x), \ \ \text{otherwise} \end{aligned}$$

for every $x \in X$. Then $\lim\limits_{k\to\infty} g_k(x) = f(x)$ on X. If $\varepsilon > \frac{1}{m} > 0$ and $k > J_m$, then either $f_k(x)$ is a term of $\{h_k(x)\}$, which means

$$\begin{aligned} g_k(x) &= f(x) \ \text{ on } \ X \\ \text{or } \ g_k(x) &= f_k(x) \in I_m \ \text{ on } \ X. \end{aligned}$$

and $|g_k(x) - f(x)| \le$ height of $I_m \le 2^{1-m}$ for every $x \in X$.

Now if $J_m < n < J_{m+1}$, then $\{k \le n : g_k(x) \ne f_k(x) \ \text{ for every } \ x \in X\} \subseteq \{k \le n : f_k(x) \notin I_m \ \text{ for every } \ x \in X\}$ so by (10)

$$\begin{aligned} &\tfrac{1}{n}|\{k \le n : g_k(x) \ne f_k(x) \text{ for every } x \in X\}| \\ &\qquad\qquad \le \tfrac{1}{n}|\{k \le n : f_k(x) \notin I_m \text{ for every } x \in X\}| \\ &\qquad\qquad < \tfrac{1}{m}. \end{aligned}$$

By taking limit $n \to \infty$ we get the limit is 0 and consequently we say that $f_k(x) = g_k(x)$ a.a.k. for every $x \in X$. Therefore (ii) implies (iii).

(iii) $\Rightarrow$ *(i)*. Let us assume that (iii) holds. That is $f_k(x) = g_k(x)$ a.a.k. for every $x \in X$ and $\lim\limits_{k\to\infty} g_k(x) = f(x)$ on X. Let $r > 0$ be real number and $\varepsilon > 0$. Then for every n, $\{k \le n : |f_k(x) - f(x)| \ge r + \varepsilon \quad \text{for every} \quad x \in$

$X\} \subseteq \{k \leq n : f_k(x) \neq g_k(x) \quad \text{for every} \quad x \in X\} \bigcup \{k \leq n : |g_k(x) - f(x)| \geq r + \varepsilon \quad \text{for every} \quad x \in X\}$; since $\lim\limits_{k\to\infty} g_k(x) = f(x)$ on X, the later set contains a finite number of integers, say $p = p(\varepsilon, x)$. Since $f_k(x) = g_k(x)$ a.a.k for every $x \in X$, we get

$$\begin{aligned} &\lim_{n\to\infty} \tfrac{1}{n} |\{k \leq n : |f_k(x) - f(x)| \geq r + \varepsilon \quad \text{for every} \quad x \in X\}| \\ \leq &\lim_{n\to\infty} \tfrac{1}{n} |\{k \leq n : f_k(x) \neq g_k(x)| \quad \text{for every} \quad x \in X\}| + \lim_{n\to\infty} \tfrac{p}{n} = 0 \end{aligned}$$

Hence $|f_k(x) - f(x)| < r + \varepsilon$ a.a.k for every $x \in X$, so (i) holds. □

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