East-West J. of Mathematics: Vol. 5, No 1 (2003) pp. 89-96

CCSR–MODULES AND WEAK LIFTING MODULES

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Abstract

In this note we introduce $CCSR$–modules and weak lifting modules as two generalizations of lifting modules. We obtain some properties, characterizations and decompositions of $CCSR$–modules and weak lifting modules.

1. Introduction

In what follows, all rings have identities and all modules are unital right modules. $N \leq M$ and $N \ll M$ mean N is a submodule and N is a small submodule of M, respectively. $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ will indicate the Jacobson radical of M and the socle of M, respectively.

Let R be a ring and M an R–module. Let K be a submodule of M. A *supplement* of K in M is a submodule N of M minimal with respect to the property $M = N+K$, equivalently, $M = N+K$ and $N \cap K \ll N$. A submodule N of M is called a *supplement* in M provided there exists a submodule K of M such that N is a supplement of K in M. It is easy to check that if N is a supplement in M, then $\mathrm{Rad}(N) = N \cap \mathrm{Rad}(M)$. The module M is called *amply supplemented* if for any two submodules A and B with $M = A + B$, B contains a supplement of A in M.

Let M be a module and N a submodule of M. Following [4], N is called *coclosed* in M if $N/K \ll M/K$ implies $N = K$ for all submodules K of M contained in N. The module M is called *lifting* if for every submodule N of M there is a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$.

Keywords: Lifting modules, CCSR–modules and weak lifting modules
(2000) Mathematics Subject Classification:

From [9, **?**], M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand of M.

In [8], P.F. Smith introduced weak CS–modules and $CESS$–modules to generalize CS–modules. In this note, as two generalizations of lifting modules we introduce weak lifting modules and $CCSR$–modules as follows, which are dual to weak CS–modules and $CESS$–modules, respectively. Any module M will be called a *weak lifting module* provided, for each semisimple submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. Any module M will be called a *$CCSR$–module* if, M is amply supplemented and every coclosed submodule N with $\mathrm{Rad}(N) \ll N$ is a direct summand of M. Clearly an amply supplemented module M with $\mathrm{Rad}(M) = M$ is a $CCSR$–module and a quasi-projective $CCSR$–module is lifting by [9, **?**]. Therefore by [9, **?**], amply supplemented divisible $\mathbb{Z}$–modules are $CCSR$–modules. It is easy to check that the ring of integers $\mathbb{Z}$ is weak lifting as a $\mathbb{Z}$–module, but is neither lifting nor $CCSR$. More generally, any module M with $\mathrm{Soc}(M) = 0$ is weak lifting. Therefore every torsion-free $\mathbb{Z}$–module is weak lifting by [9, **?**].

We start with the following fundamental lemma.

Lemma 1.1 Let N be a submodule of any module M. Consider the following statements:

- N is a supplement in M.
- N is coclosed in M.
- for all $K \leq N$, $K \ll M$ implies $K \ll N$.

Then (1)$\Longrightarrow$(2)$\Longrightarrow$(3) holds. If M is amply supplemented then (3)$\Longrightarrow$(1) holds.

Proof Clear by definitions. □

Lemma 1.2 *Any $CCSR$–module M with* $\mathrm{Rad}(M) \ll M$ *is lifting.*

Proof Let N be a coclosed (equivalently, supplement) submodule of M. Since $\mathrm{Rad}(N) \leq \mathrm{Rad}(M)$, then $\mathrm{Rad}(N) \ll M$. By Lemma 1.1, $\mathrm{Rad}(N) \ll N$. Since M is $CCSR$, then N is a direct summand of M. Hence M is lifting. □

Proposition 1.3 *Let M be a module.*

- *If M is a lifting module then M is a $CCSR$–module.*
- *If M is a $CCSR$–module then M is a weak lifting module.*

Proof

(1) Let M be a lifting module. Every coclosed submodule of M is a direct summand. Hence M is $CCSR$.

(2) Assume M is a $CCSR$–module. Let N be a semisimple submodule of M. Since M is amply supplemented, then there exists a coclosed submodule K of M such that $K \leq N$ and $N/K \ll M/K$ by [6, **?**]. Since $\mathrm{Rad}(K) \leq \mathrm{Rad}(N) = 0$, then $\mathrm{Rad}(K) = 0 \ll K$. By assumption, K is a direct summand of M. Thus M is a weak lifting module. □

Lemma 1.4 *Any direct summand of a $CCSR$–module is also a $CCSR$–module.*

Proof Let M be a $CCSR$–module and N a direct summand of M. Let X be a coclosed submodule of N with $\mathrm{Rad}(X) \ll X$. Then X is coclosed in M. By hypothesis, X is a direct summand of M, and so it is a direct summand of N. By [9, **?**], N is also amply supplemented. □

Lemma 1.5 Any direct summand of a weak lifting module is also a weak lifting module.

Proof Let M be a weak lifting module and N a direct summand of M. Let X be a semisimple submodule of N. Therefore there exists a direct summand K of M such that $K \leq X$ and $X/K \ll M/K$. Note that N/K is coclosed in M/K. Therefore $X/K \ll N/K$ by Lemma 1.1. Hence N is weak lifting. □

Proposition 1.6 Let M be a module. The following statements are equivalent.

- M is weak lifting.
- For every semisimple submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M_2$.
- Every semisimple submodule N of M can be written as $N = A \oplus S$ with A a direct summand of M and $S \ll M$.

Proof (1)$\Longrightarrow$(2) Let N be a semisimple submodule of M. Then there exists a direct summand M_1 of M such that $M_1 \leq N$ and $N/M_1 \ll M/M_1$. Write $M = M_1 \oplus M_2$ for some submodule M_2 of M. Consider the isomorphism $\alpha : M/M_1 \longrightarrow M_2$ with $\alpha(N/M_1) = N \cap M_2$. Therefore by [7, **?**], $N \cap M_2 \ll M_2$.

(2)$\Longrightarrow$(3) Let N be a semisimple submodule of M. Then there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M_2$. So, $N = M_1 \oplus (N \cap M_2)$, and the result follows with $A = M_1$ and $S = N \cap M_2$.

(3)$\Longrightarrow$(1) Let N be a semisimple submodule of M. Then $N = A \oplus S$ with A a direct summand of M and $S \ll M$. Clearly, with the natural epimorphism $\pi : M \longrightarrow M/A$, $N/A \ll M/A$. □

Proposition 1.7 *Let M be a weak lifting module. Then every semisimple supplement submodule of M is a direct summand.*

Proof Let N be a semisimple supplement submodule of M. Then there exists $K \leq M$ such that N is minimal with the property $M = N + K$. By Proposition

1.6, $N = A \oplus S$ with A a direct summand of M and $S \ll M$. Therefore $M = A + K$. By the minimality of N, $N = A$. □

Theorem 1.8 *Let M be an amply supplemented module. M is weak lifting if and only if every semisimple supplement submodule of M is a direct summand.*

Proof Necessity is clear from Proposition 1.7. Conversely, assume that every semisimple supplement submodule of M is a direct summand. Let N be a semisimple submodule of M. Then N has a supplement K and K has a supplement M_1 such that $M_1 \leq N$ and M_1 is a direct summand of M. Write $M = M_1 \oplus M_2$ for some submodule M_2 of M. Then $N = M_1 \oplus (N \cap M_2)$. Also, $M = M_1 + K$, and so $N = M_1 + (N \cap K)$. Let $\alpha : M_1 \oplus M_2 \longrightarrow M_2$ be the projection. Then $N \cap M_2 = \alpha(N) = \alpha(N \cap K)$. Since $N \cap K \ll M$, then $N \cap M_2 \ll M_2$ by [7, **?**]. Hence M is weak lifting by Proposition 1.6. □

Theorem 1.9 *Let M be an amply supplemented module. The following statements are equivalent.*

(1) M is a $CCSR$-module.

(2) For every coclosed submodule N of M with $\mathrm{Rad}(N) \ll N$, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M_2$.

(3) For every coclosed submodule N of M with $\mathrm{Rad}(N) \ll N$, there is a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$.

(4) Every coclosed submodule N of M with $\mathrm{Rad}(N) \ll N$ can be written as $N = A \oplus S$ with A a direct summand of M and $S \ll M$.

Proof (1)$\Longrightarrow$(2) Clear.

(2)$\Longrightarrow$(3) Let N be a coclosed submodule of M with $\mathrm{Rad}(N) \ll N$. Then there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M_2$. Consider the isomorphism $\alpha : M_2 \longrightarrow M/M_1$. Since $\alpha(N \cap M_2) = N/M_1$, then $N/M_1 \ll M/M_1$.

(3)$\Longrightarrow$(4) Let N be a coclosed submodule of M with $\mathrm{Rad}(N) \ll N$. Then there is a direct summand A of M such that $A \leq N$ and $N/A \ll M/A$. Write $M = A \oplus A'$. Clearly, $N = A \oplus (N \cap A')$. Hence we have the result with $S = N \cap A'$.

(4)$\Longrightarrow$(1) Let N be a supplement submodule of M with $\mathrm{Rad}(N) \ll N$. Then $N = A \oplus S$ with A a direct summand of M and $S \ll M$ by (4). Since N is a supplement, there exists a submodule K of M such that N is minimal with the property $M = N + K$. By the minimality of N, $N = A$. Thus M is $CCSR$. □

2. decompositions of ccsr–modules and weak lifting modules

Proposition 2.1 *Let M be a module such that $M = M_1 \oplus M_2$ for some weak lifting module M_1 and injective weak lifting module M_2. Then M is weak lifting.*

Proof Let S be any semisimple submodule of M. Then $S = (S \cap M_2) \oplus S'$ for some submodule S' of S. Hence $(S' + M_2)/S' \cong M_2$, an injective module. Thus $M/S' = (S' + M_2)/S' \oplus M'/S'$ for some submodule M' of M. Note that $M = M' \oplus M_2$. Thus $M' \cong M_1$. It follows that M' is a weak lifting module. There exists a direct summand K' of M' such that $K' \leq S'$ and $S'/K' \ll M'/K'$. Now $S \cap M_2$ is a semisimple submodule of the injective weak lifting module M_2. Therefore there exists a direct summand K of M_2 such that $K \leq S \cap M_2$ and $(S \cap M_2)/K \ll M_2/K$. Note that

$K \oplus K'$ is a direct summand of M and $S/(K \oplus K') \ll M/(K \oplus K')$.

This completes the proof. □

Note that every $\mathbb{Z}$–module M has a decomposition $M = C \oplus D$, with D an injective $\mathbb{Z}$–module and C a $\mathbb{Z}$–module not containing a non-zero injective submodule

[9, **?**]. Therefore we have

Theorem 2.2 A $\mathbb{Z}$–module M is weak lifting if and only if $M = M_1 \oplus M_2$ for some weak lifting module M_1 not containing a non-zero injective submodule and injective weak lifting module M_2.

Proof Sufficiency is clear by Theorem 2.1. Conversely, let M be a $\mathbb{Z}$–module and assume that M is a weak lifting module. Then by [9, **?**], $M = M_1 \oplus M_2$ for some $\mathbb{Z}$–module M_1 not containing a non-zero injective submodule and injective $\mathbb{Z}$–module M_2. By Lemma 1.5, M_1 and M_2 are weak lifting. Thus M has the decomposition, as required. □

Lemma 2.3 *Let M be a weak lifting module. Then $M = M_1 \oplus M_2$, where M_1 is a semisimple module and M_2 is a weak lifting module with* $\mathrm{Soc}(M_2) \ll M_2$

Proof Let M be a weak lifting module. There exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq \mathrm{Soc}(M)$ and $M_2 \cap \mathrm{Soc}(M) = \mathrm{Soc}(M_2) \ll M_2$. Therefore M_1 is semisimple and by Lemma 1.5, M_2 is weak lifting. □

Proposition 2.4 *Let M be a module such that $M = M_1 \oplus M_2$ for some weak lifting module M_1 and semisimple module M_2. Then M is weak lifting.*

Proof Let S be any semisimple submodule of M. Note that $S + M_1 = M_1 \oplus [(S + M_1) \cap M_2]$. Since M_2 is semisimple, $M_2 \cap (S + M_1)$ is a direct summand of M_2. Therefore $S + M_1$ is a direct summand of M. Since S is semisimple, $S \cap M_1$ is a direct summand of S, namely, there exists a submodule S' of S such that $S = (S \cap M_1) \oplus S'$. Then we have $S + M_1 = (S \cap M_1) + S' + M_1 = M_1 \oplus S'$.

Now since M_1 is weak lifting, there exists a direct summand K of M_1 such that $(S \cap M_1)/K \ll M_1/K$. Then $(S \cap M_1)/K \ll M/K$ by [7, **?**].

We claim that $S/(K \oplus S') = [(S \cap M_1) \oplus S']/(K \oplus S') \ll M/(K \oplus S')$. Let $M/(K \oplus S') = [(S \cap M_1) \oplus S']/(K \oplus S') + L/(K \oplus S')$ for some submodule L of M with $K \oplus S' \leq L$. Then $M = (S \cap M_1) + S' + L = (S \cap M_1) + L$ and so $M/K = (S \cap M_1)/K + L/K$.

Since $(S \cap M_1)/K \ll M/K$, $M/K = L/K$. Hence $M/(K \oplus S') = L/(K \oplus S')$. Clearly, $K \oplus S'$ is a direct summand of $M_1 \oplus S'$. Since $S + M_1 = M_1 \oplus S'$ and $S + M_1$ is a direct summand of M, $K \oplus S'$ is a direct summand of M. □

Theorem 2.5 *Let M be a module. M is weak lifting if and only if M has a decomposition $M = M_1 \oplus M_2$ for some semisimple module M_1 and weak lifting module M_2.*

Proof It is clear by Lemma 2.3 and Proposition 2.4. □

Theorem 2.6 *A weak lifting module M satisfies the ascending (respectively, descending) chain condition on small submodules if and only if $M = M_1 \oplus M_2$ for some semisimple module M_1 and module M_2 with* $\mathrm{Rad}(M_2)$ *Noetherian (respectively, Artinian).*

Proof Necessity: By [2, **?**], $\mathrm{Rad}(M)$ is Noetherian (Artinian). Since M is weak lifting, then $M = M_1 \oplus M_2$ with M_1 semisimple and M_2 a weak lifting module. Therefore $\mathrm{Rad}(M) = \mathrm{Rad}(M_2)$ is Noetherian (Artinian).

Sufficiency: Let $M = M_1 \oplus M_2$ with M_1 semisimple and $\mathrm{Rad}(M_2)$ Noetherian (Artinian). Then $\mathrm{Rad}(M) = \mathrm{Rad}(M_2)$, and hence $\mathrm{Rad}(M)$ is Noetherian (Artinian). Now the result follows from [2, **?**]. □

Example 2.7 Let p be any prime and M the $\mathbb{Z}$–module $\mathbb{Z}/\imath\mathbb{Z} \oplus \mathbb{Z}/\imath^{\nvDash}\mathbb{Z}$. Then M is weak lifting which is not a $CCSR$–module.

Proof By Proposition 2.4, M is weak lifting. We know that M is not lifting (see, for example [5]), and so M is not $CCSR$–module by Lemma 1.2. □

Theorem 2.8 *Let M be a $CCSR$–module. Then $M = M_1 \oplus M_2$ for some lifting module M_1 with* $\mathrm{Rad}(M_1) \ll M_1$ *and module M_2 with* $\mathrm{Rad}(M_2) = M_2$.

Proof Since M is amply supplemented, there exists a submodule M_1 of M such that $M = M_1 + \mathrm{Rad}(M)$ and $\mathrm{Rad}(M_1) = M_1 \cap \mathrm{Rad}(M) \ll M_1$. Therefore M_1 is a direct summand of M. We can write $M = M_1 \oplus M_2$ for some submodule M_2 of M. Now, $\mathrm{Rad}(M) = \mathrm{Rad}(M_1) \oplus \mathrm{Rad}(M_2)$, and so $M = M_1 \oplus \mathrm{Rad}(M_2)$. It follows that $\mathrm{Rad}(M_2) = M_2$. By Lemma 1.2 and Lemma 1.4, M_1 is lifting.
□

The converse of Theorem 2.8 is not true as we see in the following example.

Example 2.9 Let $M_{\mathbb{Z}} = \mathbb{Z}/\imath\mathbb{Z} \oplus \mathbb{Q}$ where p is any prime integer. Clearly, M is not amply supplemented, and so is not $CCSR$. Also, $\mathrm{Rad}(\mathbb{Z}/p\mathbb{Z}) = 0 \ll \mathbb{Z}/\imath\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$ is lifting and $\mathrm{Rad}(\mathbb{Q}_{\mathbb{Z}}) = \mathbb{Q}_{\mathbb{Z}}$. □

Corollary 2.10 *Let R be a right perfect ring. Let M be a right R–module. Then the following statements are equivalent.*

(1) M is lifting.

(2) M is $CCSR$.

(3) $M = M_1 \oplus M_2$ for some lifting module M_1 with $\mathrm{Rad}(M_1) \ll M_1$ and module M_2 with $\mathrm{Rad}(M_2) = M_2$.

Proof (1)$\Longrightarrow$(2) follows by definitions. (2)$\Longrightarrow$(3) is Theorem 2.8.

(3)$\Longrightarrow$(1) Let $M = M_1 \oplus M_2$ with M_1 lifting, $\mathrm{Rad}(M_1) \ll M_1$ and $\mathrm{Rad}(M_2) = M_2$. By [1, **?**], $M_2 = 0$. Hence $M = M_1$ is lifting. □

Note that if R is a right perfect ring then a weak lifting module needs not be lifting. To see this, we give the following example.

Example 2.11 ([3]).Let R be a local Artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $dim({}_Q W) = 2$ and $dim(W_Q) = 1$. Then the indecomposable injective right R–module $U = [(R \oplus R)/D]_R$ with $D = \{\, (u\varrho, -v\varrho) \mid \varrho \in R \,\}$ in [3, Proposition 2(r)] is weak lifting, but is not lifting.

Proof By [7, Corollary 4.9], U is not lifting (the only small submodules of U are 0 and $\mathrm{Soc}(U) = \mathrm{Rad}(U)$, which is simple). Let N be a non-zero semisimple submodule of U. Then $N = \mathrm{Soc}(U) \ll U$. This implies that U is weak lifting. □

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