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THE STABILITY OF SOLUTIONS OF GENERALIZED QUASI-EQUILIBRIUM PROBLEMS

Nguyen Thi Quynh Anh

The People's Police Univ. of Tech. and Logistics
Thuan Thanh, Bac Ninh, Vietnam
email: namlinhtn@gmail.com

Abstract

In this paper, we study the generalized quasi-equilibrium problems and establish some results on the stability of solutions of generalized quasi-equilibrium problems and its applications.

1 Introduction

Throughout this paper X, Z and Y are supposed to be real topological locally convex Hausdorff spaces, $D \subset X, K \subset Z$ are nonempty subsets. Given multivaled mappings $P_1 : D \to 2^D, P_2 : D \to 2^D, Q : K \times D \to 2^K$ and $F : K \times D \times D \to 2^Y$, we are interested in the following problems:

Find $\bar{x} \in D$ such that

$$\bar{x} \in P_1(\bar{x})$$

and

$$0 \in F(y, \bar{x}, t), \text{for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This problem is called the generalized quasi-equilibrium problem of type II.

In the problems, the multivalued mappings P_1, P_2 and Q are constraints mappings and F is an utility multivalued mapping that are often determined by equalities and inequalities, or by inclusions, not inclusions and intersections of other multivalued mappings, or by some relations in product spaces. This

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problem involves many problems in optimization theory as special cases, such as optimal control problems, Minty variational inequalities, Nash equilibrium problems... The optimal control problem is following:

Example 1.1. Let Ω be open bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and the boundary Γ of class C^1. We consider the problem of finding a control function $u \in L^p(\Omega)$ with $1 < p < +\infty$ and a coresponding state $y \in W^{1,r}(\Omega)$ which

$$\text{minimize} J(y,u) = \int\limits_{\Omega} L(x, y(x), u(x))dx \tag{1}$$

subject to

$$\begin{aligned} -\sum_{i,j=1}^{n} D_j\left(\left(a_{\mathrm{ij}}(x)\right).D_i y\right) + h(x,y) = u \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma, \end{aligned} \tag{2}$$

with one of following contrains:

1). Type 1: *Mixed constraints*

$$\begin{aligned} g_i\left(x, y(x), u(x)\right) \leq 0, \text{a.e.}\, x \in \Omega, \\ i = 1, ..., n; \end{aligned} \tag{3}$$

2). Type 2: *Homogeneous constraints*

$$\begin{aligned} &g(x, y(x)) \leq 0, \text{ vi mi } x \in \Omega, \\ &u(x) \in U, \text{ a.e., } x \in \Omega; \end{aligned} \tag{4}$$

3). Type 3: *Mixed and homogeneous constraints*

$$\begin{aligned} &g(x, y(x)) \leq 0, \text{ for all } x \in \Omega; \\ &f_i(x, y(x), u(x)) \leq 0, \text{ a.e. } \ x \in \Omega, \\ &i = 1, ..., n, \end{aligned} \tag{5}$$

where $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, h : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathodory function, $g_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is functions.

We defined the mappings $K(y,u) = Ay + h(.,y) - u;\ G_i(y,u) = g_i(.,y,u)$. When $g_i(.,y,u) \in C(\bar{\Omega})$, we defined the mapping

$$\phi_i(y,u) = \max_{x \in \Omega} g_i(x, y(x), u(x)).$$

The problem (1)-(3) becomes:

$$\text{subject to} \quad \begin{aligned} &\text{minimize} J(y,u), \\ &\ K(y,u) = 0, \text{ and } \phi_i(y,u) \leq 0, \\ &i = \overline{1,n}. \end{aligned}$$

We set

$$\begin{aligned} F(y,u,z,w) &= J(y,u) - J(z,w) + \mathbb{R}_+; \\ G(y,u,z,w) &= \left(K(y,u), \prod_{i=1}^{n} \Phi_i(y,u) - \mathbb{R}_+ \right). \end{aligned}$$

The above problems equivalent with the problems: Find $(\bar{y},\bar{u}) \in W_0^{1,r}(\Omega) \times L^p(\Omega)$ such that

$$0 \in F(\bar{y},\bar{u},z,w) \times \left(K(y,u), \prod_{i=1}^{n} \Phi_i(y,u) - \mathbb{R}_+ \right),$$

which means that,

$$\begin{gathered} J(\bar{y},\bar{u}) \le J(z,w) \\ \text{for all} \quad (z,w) \in W_0^{1,r}(\Omega) \times L^p(\Omega); \\ K(\bar{y},\bar{u}) = 0; \\ \Phi_i(\bar{y},\bar{u}) \le 0, i = 1, ..., m. \end{gathered}$$

This optimal are studied by Bui Trong Kien [3].

In [5], Nguyen Xuan Tan and Nguyen Thi Quynh Anh showed the sufficient conditions for the existence of solutions of generalized quasi-equilibrium problems, one of them is the following:

Theorem 1.1. *The following conditions are sufficient for $(GEP)_{II}$ to have a solution:*

i) D is a nonempty convex compact subset;

ii) $P_1 : D \to 2^D$ is a multivalued mapping with a nonempty closed fixed point set $D_0 = \{x \in D|\ x \in P_1(x)\}$ in D;

iii) $P_2 : D \to 2^D$ is a multivalued mapping with $P_2^{-1}(x)$ open and the convex hull $coP_2(x)$ of $P_2(x)$ is contained in $P_1(x)$ for each $x \in D$;

iv) For any fixed $t \in D$, the set

$$B = \{x \in D|\ 0 \notin F(y,x,t),\ \textit{for some}\ \ y \in Q(x,t)\}$$

is open in D;

v) $F : K \times D \times D \to 2^Y$ is a $\ Q - KKM$ multivalued mapping.

Our aim is to finding sufficient conditions for solutions mapping to be stable.

2 Preliminaries and Definitions

Throughout this paper, as in the introduction, by X, Z, W and Y we denote real topological locally convex Hausdorff spaces. Given a subset $D \subseteq X$, we consider a multivalued mapping $F : D \to 2^Y$. The domain and the graph of F are denoted and defined by

$$\mathrm{dom}F = \{x \in D|\ F(x) \neq \emptyset\},$$

$$\mathrm{Gr}(F) = \{(x, y) \in D \times Y|\ y \in F(x)\},$$

respectively.

We recall that F is said to be a closed mapping if the graph $Gr(F)$ of F is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $clF(D)$ of its range $F(D)$ is a compact set in Y.

A multivalued mapping $F : D \to 2^Y$ is said to be upper (lower) semicontinuous, briefly: u.s.c (respectively, l.s.c) at $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$ (respectively, $F(\bar{x}) \cap U \neq \emptyset$), there exists an open set U of $\bar{x}$ that $F(x) \subseteq V$ (respectively, $F(x) \cap U \neq \emptyset$) for each $x \in U$ and F is said to be u.s.c (l.s.c) on D if it is u.s.c (respectively, l.s.c) at any point $x \in D$. These notions and definitions can be found in [2].

Let $D \subseteq X, K \subseteq Y$. The following proposition show the need and sufficient conditions to multivalued mappings be upper (lower) semicontinuous.

Proposition 2.1. ([4]) *Assume $F : D \to 2^Y$ is a multivalued mappings with compact valueds. Then, F is lower semicontinuous at $x \in D$ iff for all $y \in F(x)$ and for all net $\{x_\alpha\}$ in D $x_\alpha \to x$, exists a net $\{y_\alpha\}$ which satisfying $y_\alpha \in F(x_\alpha)$ for all α, $y_\alpha \to y$.*

Proposition 2.2. ([6]) *If the multivalued mappings F with $F^{-1}(x)$ open in D then F is lower semicontinuous.*

Proposition 2.3. ([1]) *If $F : D \to 2^K$ is an upper semicontinuous multivalued mapping with closed valueds, then it is closed. Ngc li, if F is a closed mapping and K is compact, then F is an upper semicontinuous mapping.*

3 The stability of solutions of generalized quasi-equilibrium problems

Let $X, Z, D, K, Y, \mathcal{C}$ as the above sections, Λ, Γ, Σ be real topological locally convex Hausdorff spaces, the mappings $P_i : D \times \Lambda \to 2^D$ with $i = 1, 2, Q : D \times D \times \Gamma \to 2^K$ and $F : K \times D \times D \times \Sigma \to 2^Y$. We are interested the generalized quasi-equilibrium problems dependent on parameter: Find $\bar{x} \in P_1(\bar{x}, \lambda)$ such that $0 \in F(y, \bar{x}, t, \mu)$ for all $t \in P_2(\bar{x}, \lambda), y \in Q(\bar{x}, t, \gamma)$.

For any $\lambda \in \Lambda, \mu \in \Gamma, \gamma \in \Sigma$, we set $E(\lambda) = \{x \mid x \in P_1(x, \lambda)\}$; $M(\lambda, \gamma, \mu) = \{x \in D \mid x \in E(\lambda) \text{ and } 0 \in F(y, x, t, \mu) \text{ for all } t \in P_1(x, \lambda), y \in Q(x, t, \gamma)\}$.

Theorem 1.1 showed the sufficient conditions for solution mappings to have the valueds $M(\lambda, \gamma, \mu) \neq \emptyset$. Next, we state for the stability of solutions of the problem, involving: upper semicontinuous, lower semicontinuous in Berge's sense with respect to (λ, γ, μ).

Theorem 3.1. *Let $(\lambda_0, \gamma_0, \mu_0) \in \Lambda \times \Gamma \times \Sigma$. Assume that:*

1) P_1 is an upper semicontinuous multivalued mappings with compact values; P_2 is a lower semicontinuous multivalued mappings;

2) Q is an lower semicontinuous multivalued mappings with compact valueds;

3) The set $A = \{(y, x, \lambda, \gamma, \mu) \mid x \in E(\lambda), 0 \in F(y, x, t, \gamma) \text{ for all } t \in P_2(x, \lambda), y \in Q(x, t, \mu)\}$ is closed.

Then,

i) $M(\lambda_0, \gamma_0, \mu_0)$ is closed,

ii) M is upper semicontinuous multivalued mappings at $(\lambda_0, \gamma_0, \mu_0)$.

Proof Firstly, we proof that M is closed at $(\lambda_0, \gamma_0, \mu_0)$. We assume that M is not closed, it means existing the net $(x_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (x_0, \lambda_0, \gamma_0, \mu_0)$, with $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha), x_0 \notin M(\lambda_0, \gamma_0, \mu_0)$. From $x_\alpha \in E(\lambda_\alpha)$ and the closedness of E, it follows that $x_0 \in E(\lambda_0)$. Since $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)$, it follows that

$$\begin{gathered} 0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha) \\ \text{for all } t_\alpha \in P_2(x_\alpha, \lambda_\alpha), y_\alpha \in Q(x_\alpha, t_\alpha). \end{gathered}$$

Morever, $(y_\alpha, x_\alpha, t_\alpha) \in D$ is compact, without loss of generality, we may assume that $y_\alpha \to y_0, x_\alpha \to x_0, t_\alpha \to t_0$. We have $(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \in A$ and $(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0)$, it implies $(y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A$. Since that, we have $x_0 \in M(\lambda_0, \gamma_0, \mu_0)$. We have a contradition. So, M is closed at $(\lambda_0, \gamma_0, \mu_0)$.

Next, we proof that the mappings $M : \Lambda \times \Gamma \times \Sigma \to 2^D$ is upper semicontinuous at $(\lambda_0, \gamma_0, \mu_0)$. We assume that M is not upper semicontinuous at $(\lambda_0, \gamma_0, \mu_0)$. Then, exists the open set U contains $M(\lambda_0, \gamma_0, \mu_0)$ such that for any net $\{(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)\}$ converges to $(\lambda_0, \gamma_0, \mu_0)$, exists $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha), x_\alpha \notin U$. Since P_1 is a multivalued mappings with compact valueds, it implies P_1 (so, E) is a closed mappings. We may assume that $x_\alpha \to x_0$, so $x_0 \in E(\lambda_0)$. If $x_0 \notin M(\lambda_0, \gamma_0, \mu_0)$ then there exists $t_0 \in P_2(x_0, \lambda_0), y_0 \in Q(x_0, t_0, \gamma_0)$ such that

$$0 \notin F(y_0, x_0, t_0, \mu_0). \tag{6}$$

From $(x_\alpha, \lambda_\alpha) \to (x_0, \lambda_0)$, P_2 is lower semicontinuous at (x_0, λ_0), it implies that $t_\alpha \in P_2(x_\alpha, \lambda_\alpha), t_\alpha \to t_0$. Morever, Q is lower semicontinuous at (x_0, t_0, γ_0), so $y_\alpha \in Q(x_\alpha, t_\alpha, \gamma_\alpha), y_\alpha \to y_0$. Since $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)$, we have $0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha)$. On the other hand, $(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0)$, $x_\alpha \in E(\lambda_\alpha), t_\alpha \in P_2(x_\alpha, \lambda_\alpha)$, $y_\alpha \in Q(x_\alpha, t_\alpha, \mu_\alpha)$, $0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha)$. In view of the closedness of A, we obtain

$$(y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A.$$

$$\begin{aligned} & x_0 \in E(\lambda_\alpha); \\ & 0 \in F(y_0, x_0, t_0, \mu_0), \\ \text{for all} \quad & t_0 \in P_2(x_0, \lambda_0), y_0 \in Q(x_0, t_0, \mu_0). \end{aligned}$$

This conflict with (6), we have M is an upper semicontinuous multivalued mapping. □

Theorem 3.2. *Assume that:*

1) *E is a lower semicontinuous multivalued mappings at λ_0;*
2) *Q is an upper semicontinuous mapping with compact values;*
3) *P_2 is a closed multivalued mapping;*
4) *The set $A = \{(y, x, t, \lambda, \gamma, \mu) \in D \times D \times D \times \Lambda \times \Gamma \times \Sigma \mid x \in P_1(x, \lambda), 0 \notin F(y, x, t, \lambda, \gamma, \mu), t \in P_2(x, \lambda), y \in Q(x, t, \mu)\}$ is closed.*

Then, M is a lower semicontinuous multivalued mapping at $(\lambda_0, \gamma_0, \mu_0)$.

Proof We assume that M is not lower semicontinuous at $(\lambda_0, \gamma_0, \mu_0)$, i.e. there exists the net $(\lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (\lambda_0, \gamma_0, \mu_0)$, $x_0 \in M(\lambda_0, \gamma_0, \mu_0)$ such that $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha), x_\alpha \not\to x_0$. By the lower semicontinuty of E, $x_0 \in E(\lambda_0), \lambda_\alpha \to \lambda_0$, there exists $x'_\alpha \in E(\lambda_\alpha), x'_\alpha \to x_0, x'_\alpha \notin M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)$. Therefore, there exists $t_\alpha \in P_2(x_\alpha, \lambda_\alpha), y_\alpha \in Q(x_\alpha, t_\alpha, \mu_\alpha)$ such that $0 \notin F(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha)$. One the other hand, Q is an upper semicontinuous mapping with compact valueds, P_2 is a closed mapping, $\{t_\alpha\} \subseteq D, \{y_\alpha\} \subseteq K$ is l comparative compact. Hence, we may assume that $y_\alpha \to y_0, t_\alpha \to t_0$ and $y_0 \in Q(x_0, t_0, \mu_0), t_0 \in P_2(x_0, \lambda_0)$. We have

$$(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \in A, (y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0).$$

So $(y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A$, i.e.

$$\begin{aligned} 0 \notin F(y_0, x_0, t_0, \mu_0), x_0 \in P_1(x_0, \lambda_0), \\ t_0 \in P_2(x_0, \lambda_0), y_0 \in Q(x_0, t_0, \mu_0). \end{aligned}$$

This is a contradition with $x_0 \in M(\lambda_0, \gamma_0, \mu_0)$. The proof are complete. □

Example 3.1. Consider in optimal control problem on times (see in Introduction):

Let Ω be open bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and the boundary Γ of class C^1. We consider the problem of finding a control function $(u, \gamma, \mu) \in L^p(\Omega) \times \Gamma \times \Sigma$ with $1 < p < +\infty$ and a coresponding state $(y, \gamma, \mu) \in W^{1,r}(\Omega)$ which

$$\text{minimize} J(y, u, \mu) = \int_\Omega L(x, y(x), u(x), \mu) dx \tag{7}$$

subject to

$$\begin{aligned} -\sum_{i,j=1}^{n} D_j \left((a_{\mathrm{ij}}(x)) . D_i y \right) + h(x, y, \gamma) = u \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma, \end{aligned} \tag{8}$$

with one of following contrains:

1). Type 1: *Mixed constraints*

$$\begin{aligned} g_i \left(x, y(x), u(x), \gamma \right) \leq 0, \text{a.e. } x \in \Omega, \\ i = 1, ..., n; \end{aligned} \tag{9}$$

2). Type 2: *Homogeneous constraints*

$$\begin{aligned} & g(x, y(x), \gamma) \leq 0, \text{ vi mi } x \in \Omega, \\ & u(x) \in U, \text{ a.e., } x \in \Omega; \end{aligned} \tag{10}$$

3). Type 3: *Mixed and homogeneous constraints*

$$\begin{aligned} & g(x, y(x), \gamma) \leq 0, \text{ for all } x \in \Omega; \\ & f_i(x, y(x), u(x), \gamma) \leq 0, \text{ a.e. } x \in \Omega, \\ & i = 1, ..., n, \end{aligned} \tag{11}$$

Assume that

$$\frac{1}{n} > \frac{1}{r} \geq \frac{1}{p} - \frac{1}{n}. \tag{12}$$

$(u, \gamma) \in W^{1,r}(\Omega) \times \Gamma, (y, \gamma) \in W_0^{1,r}(\Omega) \times \Gamma$ is a solution of (8) iff

$$\begin{aligned} \int_\Omega \left(\sum_{i,j=1}^{n} a_{\mathrm{ij}} D_i y D_j \varphi \right) dx + \int_\Omega h(x, y, \gamma) \varphi dx \\ = \langle u, \varphi \rangle \ \ \forall \varphi \in W_0^{1,r}(\Omega). \end{aligned}$$

From (12)and Sobolev and Rellich Theorem, we have $L^p(\Omega) \hookrightarrow W^{1,r}(\Omega)$. Hence, $(u, \gamma) \in L^p(\Omega) \times \Gamma$, the equation (8) c duy nht nghim $(y, \gamma), y \in W_0^{1,r}(\Omega) \hookrightarrow C(\bar{\Omega})$.

We define the mappings:

$$K(y, u, \gamma) = Ay + h(., y, \gamma) - u, G_i(y, u, \gamma) = g_i(., y, u, \gamma).$$

If $g_i(., y, u, \gamma) \in \mathcal{C}(\bar{\Omega})$, the mapping

$$\phi_i(y, u, \gamma) = \max_{x \in \Omega} g_i(x, y(x), u(x), \gamma)$$

is well defined. The above problem becomes:

$$\begin{array}{l} \text{minimize}\, J(y, u, \mu), \\ \text{s.t.} K(y, u, \gamma) = 0, \ \text{ v } \phi(y, u, \gamma) \leq 0. \end{array}$$

Setting

$$F(y, u, z, w, \mu) = J(y, u, \mu) - J(z, w, \mu) + \mathbb{R}_+,$$

$$G(y, u, z, w, \gamma) = (K(y, u, \gamma), \prod_{i=1}^{n} \Phi_i(y, u, \gamma) - \mathbb{R}_+).$$

The above problem equivalent with following problem: Find $(\bar{y}, \bar{u}, \bar{\gamma}, \bar{\mu}) \in W_0^{1,r}(\Omega) \times L^p(\Omega) \times \Gamma \times \Sigma$ such that

$$0 \in F(\bar{y}, \bar{u}, z, w, \bar{\mu}) \times \left(K(\bar{y}, \bar{u}, \bar{\gamma}), \prod_{i=1}^{n} \Phi_i(\bar{y}, \bar{u}, \bar{\gamma}) - \mathbb{R}_+ \right),$$

that is

$$J(\bar{y}, \bar{u}, \bar{\mu}) \leq J(z, w, \bar{\mu})$$
$$\text{for all } (z, w) \in W_0^{1,r}(\Omega) \times L^p(\Omega);$$

$$K(\bar{y}, \bar{u}, \bar{\gamma}) = 0, \Phi_i(\bar{y}, \bar{u}, \bar{\gamma}) \leq 0, i = 1, 2, ..., m.$$

Assume that P_1, Q are as in Section 2.5. We call $M : \Lambda \times \Gamma \times \Sigma \to W_0^{1,r}(\Omega) \times L^p(\Omega)$ is a solutions mapping of the optimal control problem: Find $(\bar{y}, \bar{u}, \bar{\gamma}, \bar{\mu}) \in W_0^{1,r}(\Omega) \times L^p(\Omega) \times \Gamma \times \Sigma$ such that

$$\begin{array}{rc} & J(\bar{y}, \bar{u}, \bar{\mu}) \leq J(z, w, \bar{\mu}) \\ \text{for all} & z, w \in W_0^{1,r}(\Omega) \times L^p(\Omega), \\ & K(\bar{y}, \bar{u}, \bar{\gamma}) = 0, \\ & \Phi_i(\bar{y}, \bar{u}, \bar{\gamma}) \leq 0, i = 1, 2, ..., m, \end{array}$$

for all $w \in P_1(\bar{x}, \bar{\lambda}), z \in Q(\bar{x}, w)$.

To show that M be an upper (lower) semicontinuous multivalued mapping at $(\bar{\gamma}, \bar{\mu})$, we find conditions to ensure that A (in 3.1, 3.2) is closed. For example, if J is a continuous function and P_1, Q as in 3.1, then A is closed.

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