East-West J. of Mathematics: Vol. 18, No 1 (2016) pp. 70-78

THE STABILITY OF SOLUTIONS OF GENERALIZED QUASI-EQUILIBRIUM PROBLEMS

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Abstract

In this paper, we study the generalized quasi-equilibrium problems and establish some results on the stability of solutions of generalized quasi-equilibrium problems and its applications.

1 Introduction

Throughout this paper X, Z and Y are supposed to be real topological locally convex Hausdorff spaces, $D \subset X, K \subset Z$ are nonempty subsets. Given multivaled mappings $P_1 : D \to 2^D, P_2 : D \to 2^D, Q : K \times D \to 2^K$ and $F : K \times D \times D \to 2^Y$, we are interested in the following problems:

Find $\bar{x} \in D$ such that

$$\bar{x} \in P_1(\bar{x})$$

and

$$0 \in F(y, \bar{x}, t), \text{for all } t \in P_2(\bar{x}) \text{ and } y \in Q(\bar{x}, t).$$

This problem is called the generalized quasi-equilibrium problem of type II.

In the problems, the multivalued mappings P_1, P_2 and Q are constraints mappings and F is an utility multivalued mapping that are often determined by equalities and inequalities, or by inclusions, not inclusions and intersections of other multivalued mappings, or by some relations in product spaces. This

Key words: Generalized quasi-equilibrium problems, upper and lower quasivariational inclusions, upper and lower semicontinuous multivalued mappings.
AMS Classification (2010): 41A65, 47H17, 47H20.

problem involves many problems in optimization theory as special cases, such as optimal control problems, Minty variational inequalities, Nash equilibrium problems... The optimal control problem is following:

Example 1.1. Let Ω be open bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and the boundary Γ of class C^1. We consider the problem of finding a control function $u \in L^p(\Omega)$ with $1 < p < +\infty$ and a coresponding state $y \in W^{1,r}(\Omega)$ which

$$\text{minimize} J(y,u) = \int\limits_{\Omega} L(x, y(x), u(x))dx \tag{1}$$

subject to

$$\begin{aligned} -\sum_{i,j=1}^{n} D_j\left((a_{\text{ij}}(x)).D_i y\right) + h(x,y) = u \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma, \end{aligned} \tag{2}$$

with one of following contrains:

1). Type 1: *Mixed constraints*

$$\begin{aligned} g_i\left(x, y(x), u(x)\right) \leq 0, \text{a.e.} x \in \Omega, \\ i = 1, ..., n; \end{aligned} \tag{3}$$

2). Type 2: *Homogeneous constraints*

$$\begin{aligned} &g(x, y(x)) \leq 0, \text{ vi mi } x \in \Omega, \\ &u(x) \in U, \text{ a.e., } x \in \Omega; \end{aligned} \tag{4}$$

3). Type 3: *Mixed and homogeneous constraints*

$$\begin{aligned} &g(x, y(x)) \leq 0, \text{ for all } x \in \Omega; \\ &f_i(x, y(x), u(x)) \leq 0, \text{ a.e. } x \in \Omega, \\ &i = 1, ..., n, \end{aligned} \tag{5}$$

where $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, h : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathodory function, $g_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is functions.

We defined the mappings $K(y,u) = Ay + h(.,y) - u$; $G_i(y,u) = g_i(.,y,u)$. When $g_i(.,y,u) \in C(\bar{\Omega})$, we defined the mapping

$$\phi_i(y,u) = \max_{x \in \Omega} g_i(x, y(x), u(x)).$$

The problem (1)-(3) becomes:

$$\begin{aligned} &\text{minimize} J(y,u), \\ \text{subject to} \quad &K(y,u) = 0, \text{ and } \phi_i(y,u) \leq 0, \\ &i = \overline{1,n}. \end{aligned}$$

We set

$$\begin{aligned} F(y,u,z,w) &= J(y,u) - J(z,w) + \mathbb{R}_+; \\ G(y,u,z,w) &= \left(K(y,u), \prod_{i=1}^{n} \Phi_i(y,u) - \mathbb{R}_+ \right). \end{aligned}$$

The above problems equivalent with the problems: Find $(\bar{y}, \bar{u}) \in W_0^{1,r}(\Omega) \times L^p(\Omega)$ such that

$$0 \in F(\bar{y}, \bar{u}, z, w) \times \left(K(y,u), \prod_{i=1}^{n} \Phi_i(y,u) - \mathbb{R}_+ \right),$$

which means that,

$$\begin{gathered} J(\bar{y}, \bar{u}) \leq J(z,w) \\ \text{for all} \quad (z,w) \in W_0^{1,r}(\Omega) \times L^p(\Omega); \\ K(\bar{y}, \bar{u}) = 0; \\ \Phi_i(\bar{y}, \bar{u}) \leq 0, i = 1, ..., m. \end{gathered}$$

This optimal are studied by Bui Trong Kien [3].

In [5], Nguyen Xuan Tan and Nguyen Thi Quynh Anh showed the sufficient conditions for the existence of solutions of generalized quasi-equilibrium problems, one of them is the following:

Theorem 1.1. *The following conditions are sufficient for $(GEP)_{II}$ to have a solution:*

i) D is a nonempty convex compact subset;

ii) $P_1 : D \to 2^D$ is a multivalued mapping with a nonempty closed fixed point set $D_0 = \{x \in D |\ x \in P_1(x)\}$ in D;

iii) $P_2 : D \to 2^D$ is a multivalued mapping with $P_2^{-1}(x)$ open and the convex hull $coP_2(x)$ of $P_2(x)$ is contained in $P_1(x)$ for each $x \in D$;

iv) For any fixed $t \in D$, the set

$$B = \{x \in D |\ 0 \notin F(y,x,t), \text{ for some } \ y \in Q(x,t)\}$$

is open in D;

v) $F : K \times D \times D \to 2^Y$ is a $Q - KKM$ multivalued mapping.

Our aim is to finding sufficient conditions for solutions mapping to be stable.

2 Preliminaries and Definitions

Throughout this paper, as in the introduction, by X, Z, W and Y we denote real topological locally convex Hausdorff spaces. Given a subset $D \subseteq X$, we consider a multivalued mapping $F : D \to 2^Y$. The domain and the graph of F are denoted and defined by

$$\mathrm{dom}F = \{x \in D |\ F(x) \neq \emptyset\},$$

$$\mathrm{Gr}(F) = \{(x, y) \in D \times Y |\ y \in F(x)\},$$

respectively.

We recall that F is said to be a closed mapping if the graph $Gr(F)$ of F is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $clF(D)$ of its range $F(D)$ is a compact set in Y.

A multivalued mapping $F : D \to 2^Y$ is said to be upper (lower) semicontinuous, briefly: u.s.c (respectively, l.s.c) at $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$ (respectively, $F(\bar{x}) \cap U \neq \emptyset$), there exists an open set U of $\bar{x}$ that $F(x) \subseteq V$ (respectively, $F(x) \cap U \neq \emptyset$) for each $x \in U$ and F is said to be u.s.c (l.s.c) on D if it is u.s.c (respectively, l.s.c) at any point $x \in D$. These notions and definitions can be found in [2].

Let $D \subseteq X, K \subseteq Y$. The following proposition show the need and sufficient conditions to multivalued mappings be upper (lower) semicontinuous.

Proposition 2.1. ([4]) *Assume $F : D \to 2^Y$ is a multivalued mappings with compact valueds. Then, F is lower semicontinuous at $x \in D$ iff for all $y \in F(x)$ and for all net $\{x_\alpha\}$ in D $x_\alpha \to x$, exists a net $\{y_\alpha\}$ which satisfying $y_\alpha \in F(x_\alpha)$ for all α, $y_\alpha \to y$.*

Proposition 2.2. ([6]) *If the multivalued mappings F with $F^{-1}(x)$ open in D then F is lower semicontinuous.*

Proposition 2.3. ([1]) *If $F : D \to 2^K$ is an upper semicontinuous multivalued mapping with closed valueds, then it is closed. Ngc li, if F is a closed mapping and K is compact, then F is an upper semicontinuous mapping.*

3 The stability of solutions of generalized quasi-equilibrium problems

Let $X, Z, D, K, Y, \mathcal{C}$ as the above sections, Λ, Γ, Σ be real topological locally convex Hausdorff spaces, the mappings $P_i : D \times \Lambda \to 2^D$ with $i = 1, 2, Q : D \times D \times \Gamma \to 2^K$ and $F : K \times D \times D \times \Sigma \to 2^Y$. We are interested the generalized quasi-equilibrium problems dependent on parameter: Find $\bar{x} \in P_1(\bar{x}, \lambda)$ such that $0 \in F(y, \bar{x}, t, \mu)$ for all $t \in P_2(\bar{x}, \lambda), y \in Q(\bar{x}, t, \gamma)$.

For any $\lambda \in \Lambda, \mu \in \Gamma, \gamma \in \Sigma$, we set $E(\lambda) = \{x \mid x \in P_1(x, \lambda)\}$; $M(\lambda, \gamma, \mu) = \{x \in D \mid x \in E(\lambda) \text{ and } 0 \in F(y, x, t, \mu) \text{ for all } t \in P_1(x, \lambda), y \in Q(x, t, \gamma)\}$.

Theorem 1.1 showed the sufficient conditions for solution mappings to have the valueds $M(\lambda, \gamma, \mu) \neq \emptyset$. Next, we state for the stability of solutions of the problem, involving: upper semicontinuous, lower semicontinuous in Berge's sense with respect to (λ, γ, μ).

Theorem 3.1. *Let $(\lambda_0, \gamma_0, \mu_0) \in \Lambda \times \Gamma \times \Sigma$. Assume that:*

1) *P_1 is an upper semicontinuous multivalued mappings with compact values; P_2 is a lower semicontinuous multivalued mappings;*

2) *Q is an lower semicontinuous multivalued mappings with compact valueds;*

3) *The set $A = \{(y, x, \lambda, \gamma, \mu) \mid x \in E(\lambda), 0 \in F(y, x, t, \gamma)$ for all $t \in P_2(x, \lambda), y \in Q(x, t, \mu)\}$ is closed.*

Then,
i) $M(\lambda_0, \gamma_0, \mu_0)$ is closed,
ii) M is upper semicontinuous multivalued mappings at $(\lambda_0, \gamma_0, \mu_0)$.

Proof Firstly, we proof that M is closed at $(\lambda_0, \gamma_0, \mu_0)$. We assume that M is not closed, it means existing the net $(x_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (x_0, \lambda_0, \gamma_0, \mu_0)$, with $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha), x_0 \notin M(\lambda_0, \gamma_0, \mu_0)$. From $x_\alpha \in E(\lambda_\alpha)$ and the closedness of E, it follows that $x_0 \in E(\lambda_0)$. Since $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)$, it follows that

$$0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha)$$
$$\text{for all } t_\alpha \in P_2(x_\alpha, \lambda_\alpha), y_\alpha \in Q(x_\alpha, t_\alpha).$$

Morever, $(y_\alpha, x_\alpha, t_\alpha) \in D$ is compact, without loss of generality, we may assume that $y_\alpha \to y_0, x_\alpha \to x_0, t_\alpha \to t_0$. We have $(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \in A$ and $(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0)$, it implies $(y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A$. Since that, we have $x_0 \in M(\lambda_0, \gamma_0, \mu_0)$. We have a contradition. So, M is closed at $(\lambda_0, \gamma_0, \mu_0)$.

Next, we proof that the mappings $M : \Lambda \times \Gamma \times \Sigma \to 2^D$ is upper semicontinuous at $(\lambda_0, \gamma_0, \mu_0)$. We assume that M is not upper semicontinuous at $(\lambda_0, \gamma_0, \mu_0)$. Then, exists the open set U contains $M(\lambda_0, \gamma_0, \mu_0)$ such that for any net $\{(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)\}$ converges to $(\lambda_0, \gamma_0, \mu_0)$, exists $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha), x_\alpha \notin U$. Since P_1 is a multivalued mappings with compact valueds, it implies P_1 (so, E) is a closed mappings. We may assume that $x_\alpha \to x_0$, so $x_0 \in E(\lambda_0)$. If $x_0 \notin M(\lambda_0, \gamma_0, \mu_0)$ then there exists $t_0 \in P_2(x_0, \lambda_0), y_0 \in Q(x_0, t_0, \gamma_0)$ such that

$$0 \notin F(y_0, x_0, t_0, \mu_0). \tag{6}$$

From $(x_\alpha, \lambda_\alpha) \to (x_0, \lambda_0)$, P_2 is lower semicontinuous at (x_0, λ_0), it implies that $t_\alpha \in P_2(x_\alpha, \lambda_\alpha), t_\alpha \to t_0$. Morever, Q is lower semicontinuous at (x_0, t_0, γ_0),

so $y_\alpha \in Q(x_\alpha, t_\alpha, \gamma_\alpha), y_\alpha \to y_0$. Since $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)$, we have $0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha)$. On the other hand, $(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0)$, $x_\alpha \in E(\lambda_\alpha), t_\alpha \in P_2(x_\alpha, \lambda_\alpha)$, $y_\alpha \in Q(x_\alpha, t_\alpha, \mu_\alpha)$, $0 \in F(y_\alpha, x_\alpha, t_\alpha, \mu_\alpha)$. In view of the closedness of A, we obtain

$$(y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A.$$

$$\begin{aligned} &x_0 \in E(\lambda_\alpha); \\ &0 \in F(y_0, x_0, t_0, \mu_0), \\ \text{for all} \quad &t_0 \in P_2(x_0, \lambda_0), y_0 \in Q(x_0, t_0, \mu_0). \end{aligned}$$

This conflict with (6), we have M is an upper semicontinuous multivalued mapping. □

Theorem 3.2. *Assume that:*

1) *E is a lower semicontinuous multivalued mappings at λ_0;*
2) *Q is an upper semicontinuous mapping with compact values;*
3) *P_2 is a closed multivalued mapping;*
4) *The set $A = \{(y, x, t, \lambda, \gamma, \mu) \in D \times D \times D \times \Lambda \times \Gamma \times \Sigma \mid x \in P_1(x, \lambda), 0 \notin F(y, x, t, \lambda, \gamma, \mu), t \in P_2(x, \lambda), y \in Q(x, t, \mu)\}$ is closed.*

Then, M is a lower semicontinuous multivalued mapping at $(\lambda_0, \gamma_0, \mu_0)$.

Proof We assume that M is not lower semicontinuous at $(\lambda_0, \gamma_0, \mu_0)$, i.e. there exists the net $(\lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (\lambda_0, \gamma_0, \mu_0)$, $x_0 \in M(\lambda_0, \gamma_0, \mu_0)$ such that $x_\alpha \in M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha), x_\alpha \not\to x_0$. By the lower semicontinuty of E, $x_0 \in E(\lambda_0), \lambda_\alpha \to \lambda_0$, there exists $x'_\alpha \in E(\lambda_\alpha), x'_\alpha \to x_0, x'_\alpha \notin M(\lambda_\alpha, \gamma_\alpha, \mu_\alpha)$. Therefore, there exists $t_\alpha \in P_2(x_\alpha, \lambda_\alpha), y_\alpha \in Q(x_\alpha, t_\alpha, \mu_\alpha)$ such that $0 \notin F(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha)$. One the other hand, Q is an upper semicontinuous mapping with compact valueds, P_2 is a closed mapping, $\{t_\alpha\} \subseteq D, \{y_\alpha\} \subseteq K$ is l comparative compact. Hence, we may assume that $y_\alpha \to y_0, t_\alpha \to t_0$ and $y_0 \in Q(x_0, t_0, \mu_0), t_0 \in P_2(x_0, \lambda_0)$. We have

$$(y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \in A, (y_\alpha, x_\alpha, t_\alpha, \lambda_\alpha, \gamma_\alpha, \mu_\alpha) \to (y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0).$$

So $(y_0, x_0, t_0, \lambda_0, \gamma_0, \mu_0) \in A$, i.e.

$$\begin{aligned} 0 \notin F(y_0, x_0, t_0, \mu_0), x_0 \in P_1(x_0, \lambda_0), \\ t_0 \in P_2(x_0, \lambda_0), y_0 \in Q(x_0, t_0, \mu_0). \end{aligned}$$

This is a contradition with $x_0 \in M(\lambda_0, \gamma_0, \mu_0)$. The proof are complete. □

Example 3.1. Consider in optimal control problem on times (see in Introduction):

Let Ω be open bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and the boundary Γ of class C^1. We consider the problem of finding a control function $(u, \gamma, \mu) \in L^p(\Omega) \times \Gamma \times \Sigma$ with $1 < p < +\infty$ and a coresponding state $(y, \gamma, \mu) \in W^{1,r}(\Omega)$ which

$$\text{minimize} J(y, u, \mu) = \int\limits_{\Omega} L(x, y(x), u(x), \mu) dx \tag{7}$$

subject to

$$\begin{aligned} -\sum_{i,j=1}^{n} D_j \left((a_{\text{ij}}(x)) . D_i y \right) + h(x, y, \gamma) = u \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma, \end{aligned} \tag{8}$$

with one of following contrains:

1). Type 1: *Mixed constraints*

$$\begin{aligned} g_i \left(x, y(x), u(x), \gamma \right) \leq 0, \text{a.e. } x \in \Omega, \\ i = 1, ..., n; \end{aligned} \tag{9}$$

2). Type 2: *Homogeneous constraints*

$$\begin{aligned} &g(x, y(x), \gamma) \leq 0, \text{ vi mi } x \in \Omega, \\ &u(x) \in U, \text{ a.e., } x \in \Omega; \end{aligned} \tag{10}$$

3). Type 3: *Mixed and homogeneous constraints*

$$\begin{aligned} &g(x, y(x), \gamma) \leq 0, \text{ for all } x \in \Omega; \\ &f_i(x, y(x), u(x), \gamma) \leq 0, \text{ a.e. } x \in \Omega, \\ &i = 1, ..., n, \end{aligned} \tag{11}$$

Assume that

$$\frac{1}{n} > \frac{1}{r} \geq \frac{1}{p} - \frac{1}{n}. \tag{12}$$

$(u, \gamma) \in W^{1,r}(\Omega) \times \Gamma, (y, \gamma) \in W_0^{1,r}(\Omega) \times \Gamma$ is a solution of (8) iff

$$\begin{aligned} \int\limits_{\Omega} \left(\sum_{i,j=1}^{n} a_{\text{ij}} D_i y D_j \varphi \right) dx + \int\limits_{\Omega} h(x, y, \gamma) \varphi dx \\ = \langle u, \varphi \rangle \ \ \forall \varphi \in W_0^{1,r}(\Omega). \end{aligned}$$

From (12)and Sobolev and Rellich Theorem, we have $L^p(\Omega) \hookrightarrow W^{1,r}(\Omega)$. Hence, $(u, \gamma) \in L^p(\Omega) \times \Gamma$, the equation (8) c duy nht nghim $(y, \gamma), y \in W_0^{1,r}(\Omega) \hookrightarrow C(\bar{\Omega})$.

We define the mappings:

$$K(y,u,\gamma) = Ay + h(.,y,\gamma) - u, G_i(y,u,\gamma) = g_i(.,y,u,\gamma).$$

If $g_i(.,y,u,\gamma) \in \mathcal{C}(\bar{\Omega})$, the mapping

$$\phi_i(y,u,\gamma) = \max_{x\in\Omega} g_i(x,y(x),u(x),\gamma)$$

is well defined. The above problem becomes:

$$\begin{array}{l} \text{minimize } J(y,u,\mu), \\ \text{s.t.} K(y,u,\gamma) = 0, \ \text{ v } \phi(y,u,\gamma) \leq 0. \end{array}$$

Setting

$$F(y,u,z,w,\mu) = J(y,u,\mu) - J(z,w,\mu) + \mathbb{R}_+,$$

$$G(y,u,z,w,\gamma) = (K(y,u,\gamma), \prod_{i=1}^{n} \Phi_i(y,u,\gamma) - \mathbb{R}_+).$$

The above problem equivalent with following problem: Find $(\bar{y},\bar{u},\bar{\gamma},\bar{\mu}) \in W_0^{1,r}(\Omega) \times L^p(\Omega) \times \Gamma \times \Sigma$ such that

$$0 \in F(\bar{y},\bar{u},z,w,\bar{\mu}) \times \left(K(\bar{y},\bar{u},\bar{\gamma}), \prod_{i=1}^{n} \Phi_i(\bar{y},\bar{u},\bar{\gamma}) - \mathbb{R}_+ \right),$$

that is

$$\begin{array}{c} J(\bar{y},\bar{u},\bar{\mu}) \leq J(z,w,\bar{\mu}) \\ \text{for all } (z,w) \in W_0^{1,r}(\Omega) \times L^p(\Omega); \end{array}$$

$$K(\bar{y},\bar{u},\bar{\gamma}) = 0, \Phi_i(\bar{y},\bar{u},\bar{\gamma}) \leq 0, i = 1,2,...,m.$$

Assume that P_1, Q are as in Section 2.5. We call $M : \Lambda \times \Gamma \times \Sigma \to W_0^{1,r}(\Omega) \times L^p(\Omega)$ is a solutions mapping of the optimal control problem: Find $(\bar{y},\bar{u},\bar{\gamma},\bar{\mu}) \in W_0^{1,r}(\Omega) \times L^p(\Omega) \times \Gamma \times \Sigma$ such that

$$\begin{array}{cc} & J(\bar{y},\bar{u},\bar{\mu}) \leq J(z,w,\bar{\mu}) \\ \text{for all} & z,w \in W_0^{1,r}(\Omega) \times L^p(\Omega), \\ & K(\bar{y},\bar{u},\bar{\gamma}) = 0, \\ & \Phi_i(\bar{y},\bar{u},\bar{\gamma}) \leq 0, i = 1,2,...,m, \end{array}$$

for all $w \in P_1(\bar{x},\bar{\lambda}), z \in Q(\bar{x},w)$.

To show that M be an upper (lower) semicontinuous multivalued mapping at $(\bar{\gamma},\bar{\mu})$, we find conditions to ensure that A (in 3.1, 3.2) is closed. For example, if J is a continuous function and P_1, Q as in 3.1, then A is closed.

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