East-West J. of Mathematics: Vol. 22, No 2 (2020) pp. 174-181
https://doi.org/10.36853/ewjm.2020.22.02/16

ON FULLY BOUNDED MODULES AND KRULL SYMMETRY

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Abstract

The concept of fully bounded modules introduced and as an special case of the Krull symmetry property, it is proved that if M_R is a fully bounded Noetherian projective module, which is a generator (NPG module for short) in the category of right R-modules (mod-R for short) and $S = End(M_R)$, then $k\text{-dim}\, M_R = k\text{-dim}\,{}_S M$.

1 Introduction

Throughout this paper, all rings are associative with $1 \neq 0$ and modules assumed to be unitary. An R-module M is called a generator if for every R-module N there is an epimorphism $M^{(I)} \to N$ for some indexed set I. Obviously, R is a generator and if an epimorphic image of M is a generator, then so is M. This implies that $R \oplus M$ is a generator for any R-module M, so every free R-module M is a generator. An R-module M is said to be self-generator if it generates all it's submodules. Note that an R-module M is self-generator if and only if for each submodule N of M, there exists $\Delta \subseteq S = End(M_R)$ such that $N = \sum_{f\in\Delta} f(M)$. Also, an R-module M is called quasi-projective (or self-projective) if it is M-projective. The notation $N \leq_e M$ means that

Key words: Keywords: bounded modules, fully bounded modules, Krull symmetry, Krull dimension, classical Krull dimension.
2010 AMS Mathematics Classification: Primary: 16P60, 16P20 ; Secondary: 16P40.

N is an essential (or large) submodule of M, that is $K \cap N \neq 0$ for every non-zero submodule K of M. Let R and S be two rings. the notion ${}_SM_R$ means that M is an $(S-R)$-bimodule, that is M is a left S-module and right R-module such that $(rm)s = r(ms)$ for all $r \in R, m \in M$ and $s \in S$. The bimodule ${}_SM_R$ will be called Noetherian if M is Noerherian both as a left $S-$module and as a right $R-$module. We shall say that a bimodule ${}_SM_R$ is Krull symmetric if M has Krull dimension both as a left S-module and as a right R-module and $k\text{-dim}\,{}_SM = k\text{-dim}\,M_R$, see also [1]. T. H. Lenagan proved the zero-dimensional case for Noetherian bimodules, that is $k\text{-dim}\,M_R = 0$ if and only if $k\text{-dim}\,{}_SM = 0$, for any Noetherian bimodule ${}_SM_R$, see [2, Theorem, 7.10]. Recall that a ring R is right (left) bounded if every essential right (left) ideal of R contains an (2-sided) ideal which is essential as a right (left) ideal. If both condition hold, R is called a bounded ring. Any commutative ring and any semisimple ring is bounded. Note that every nonzero ideal of a prime ring R is right and left essential, so a prime ring R is bounded if and only if every one sided ideal of R contains a nonzero (2-sided) ideal. A ring R is right (left) fully bounded if $\frac{R}{P}$ is right (left) bounded for any prime ideal P of R. If R is both left and right fully bounded, it called a fully bounded ring. A right (left) FBN ring is any right (left) fully bounded right (left) Noetherian ring. An FBN ring is any right and left FBN ring. Jategaonkar has shown that if S is a left FBN ring and R is a right FBN ring, then any Noetherian bimodule ${}_SM_R$ is Krull symmetric, see [2, Theorem 13.15]. In particular, $k\text{-dim}\,{}_RR = k\text{-dim}\,R_R$ for any FBN ring R. From these facts, naturally raised the Krull symmetry problems that is still open in general. The krull symmetry problems is as follows:

1. Does $k\text{-dim}\,R_R = k\text{-dim}_RR$, for any Noetherian ring R ?

2. Does $k\text{-dim}\,{}_SM = k\text{-dim}\,M_R$ for any Noetherian bimodule ${}_SM_R$?

After that, several authors tried to solve this problems and each one has investigated this issue in a special way, see for example [6]. The main result of this paper, Theorem 2.16, states another special case of Krull symmetry of modules. Now, let us give a brief outline of this paper. First of all, we introduce abreviated symbols in this paper as follows:

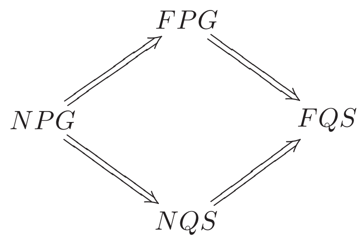
1. FQS module means a finitely generated quasi-projective module, which is self-generator.

2. NQS module means a Noetherian quasi-projective module, which is self-generator.

3. FPG module means a finitly-generated projective module, which is generator and finally,

4. NPG module means a Noetherian projective module, which is a generator in mod-R.

Note that



In [3], we investigate FQS modules. We show that if M is an FQS module, then M is Goldie (critical, dual critical, etc.) if and only if the endomorphism ring $S = End(M_R)$ has all the latter properties for right hand, respectively. We also prove that an FQS module M has the classical Krull dimension, denoted by $cl.k\text{-}\dim M$ if and only if it satisfies ACC on prime submodules. The reader is referred to [3] for more details.
In this paper we will discuss some other properties of the FQS modules. First, we defined a bounded and a fully bounded module. Then we proved that if M is an FQS module and $S = End(M_R)$, then M is bounded (resp, fully bounded) if and only if S is right bounded (resp, fully bounded) ring. Also we show that if M is an NQS-module, which is FBN, then $k\text{-}\dim M_R = k\text{-}\dim S_S = cl.k\text{-}\dim S = cl.k\text{-}\dim M_R$. Consequently, if M be a faithful and balanced right R-module and $S = End(M_R)$, then $cl.k\text{-}\dim M_R = cl.k\text{-}\dim_S M$, if either side exists. Finally, as the main result we show that if R is a left FBN-ring and M_R is a fully bounded NPG module, then $k\text{-}\dim M_R = k\text{-}\dim {}_SM$. It is convenient, when we are dealing with the above dimensions, to begin our list of ordinals with -1.

2 main results

We recall that a fully invariant submodule of M is any sub-bimodule of ${}_SM_R$, where $S = End(M_R)$. Note that, X is a fully invariant submodule of M, if $f(X) \subseteq X$ for any $f \in S = End(M_R)$. According to [5, Definition 1.1], a prime submodule of an R-module M is a fully invariant proper submodule P such that $IX \subseteq P$ implies $X \subseteq P$ or $IM \subseteq P$, for any ideal I of $S = End(M_R)$ and any fully invariant submodule X of M. An R-module M is called prime if 0 is a prime submodule of M. Any maximal of the set of all fully invariant submodules of M is prime, see [5, Proposition 1.6]. The set of all prime submoules of M will be denoted by $spec(M)$. Next, we introduce the concept of bounded and fully bounded modules and investigate some related properties, similar to the definitions in ring theory. We note that by a fully invarint essential submodule of M_R, we mean a fully invariant submodule which is essential as an R-submodule.

Definition 2.1. An R-module M is bounded if every essential submodule of M, contains a fully invariant essential submodule.

Note that, if M is an FQS prime module, then every nonzero fully invariant submodule of M is an essential R-submodule of M, see [3, Lemma 3.11]. So a prime FQS module M is bounded if and only if every essential submodule of M contains a nonzero fully invariant submodule.

We recall that $I_X = \{f \in S : f(M) \subseteq X\}$ and $AM = \sum_{f\in A} f(M)$, for any submodule $X \subseteq M$ and $A \subseteq S = End(M_R)$.

Proposition 2.2. Let M be an FQS module and $S = End(M_R)$. M is bounded if and only if S is a right bounded ring.

Proof. Let M be bounded and A be an essential right ideal of S, then AM is an essential submodule of M, by [3, Lemma 3.4 and Theorem 3.5], so there exists a fully invariant essential submodule U such that $U \subseteq AM$. Then I_U is an essential ideal of S and $I_U \subseteq A$, by the above mentioned facts and therefore S is right boubded. Conversely, let S be right bounded and E be an essential submodule of M. Then I_E is an essential right ideal of S, so it contains an ideal X, which is right essential. Again, by the mentioned facts, XM is a fully invariant essential submodule of M and $XM \subseteq I_E M = E$. Therefore M is bounded. □

Definition 2.3. An R-module M is fully bounded if $\frac{M}{P}$ is bounded, for every $P \in spec(M)$.

Note that M is a fully bounded R-module if for any prime submodule P and any submodule A of M which $P \subsetneq A$, there exists a fully invariant essential submodule X of M such that $P \subsetneq X \subseteq A$.

Proposition 2.4. Let M be an FQS module and $S = End(M_R)$. M is fully bounded if and only if S if a right fully bounded ring.

Proof. First, let M be fully bounded. In order to show that S is a right fully bounded ring, let $P \in spec(S)$ and A be a right ideal of S such that $P \subsetneq A$. According to [3, Lemma 3.9], $PM \subsetneq AM$ are submodules of M and $PM \in spec(M)$. By definition, there exists a fully invariat essential submodule X of M such that $PM \subsetneq X \subseteq AM$. Then I_X is an ideal of S, which is essential as a right ideal and $P \subsetneq I_X \subseteq A$, by [3, Lemmas 3.4 and 3.5]. Therefore S is right fully bounded. Conversely, let S be a right fully bounded ring. In order to show that M is fully bounded, by the comment after Definition 2.3, let $P \in spec(M)$ and A be a submodule of M such that $P \subsetneq A$. Then I_P is a prime ideal of S properly contained in the right ideal I_A and so there exists an ideal X of S, which is essential as a right ideal and $I_P \subsetneq X \subseteq I_A$. In this case, XM is a fully invariant essential submodule of M and $P \subsetneq XM \subseteq A$. Therefore M is fully bounded and we are done. □

Let us recall two results to prove the next theorem.

Theorem 2.5. [3, Theorem 4.3] Let M be an FQS module, $S = End_R(M)$ and X be a submodule of M. Then we have:

1. $G - dimM_R = G - dimS_S$.
2. $k\text{-dim}\, M_R = k\text{-dim}\, S_S$, if either side exists.
3. $k\text{-dim}\, X = k\text{-dim}\, I_X$.
4. $n\text{-dim}\, M_R = n\text{-dim}\, S_S$, if either side exists.
5. $n\text{-dim}\, X = n\text{-dim}\, I_X$.
6. $k\text{-dim}\, \frac{M}{X} = k\text{-dim}\, \frac{S}{I_X}$, if X is fully invariant.
7. $n\text{-dim}\, \frac{M}{X} = n\text{-dim}\, \frac{S}{I_X}$, if X is fully invariant.

Theorem 2.6. [3, Theorem 4.21] Let M be an FQS module and $S = End_R(M)$. Then M has classical Krull dimension if and only if S has classical Krull dimension and in this case $cl.k\text{-dim}\, M = cl.k\text{-dim}\, S$.

We recall that, in [4] Krause showed that if R is a right fully bounded ring with right Krull dimension then $cl.k\text{-dim}\, R \leq k\text{-dim}\, R$ and equality holds, where R is a right FBN ring. The next result is an analogue of this fact for modules.

Theorem 2.7. Let M be a fully bounded NQS-module. Then:

$$k\text{-dim}\, M_R = k\text{-dim}\, S_S = cl.k\text{-dim}\, S = cl.k\text{-dim}\, M_R.$$

Proof. Note that M is an FQS module and so $k\text{-dim}\, M_R = k\text{-dim}\, S_S$ and $cl.k\text{-dim}\, M_R = cl.k\text{-dim}\, S$, by Theorems 2.5, 2.6. Since M is fully bounded, S is a right fully bounded ring, by Proposition 2.4. Also, M is Noetherian and so S is right Noetherian, by Theorem 2.5. Thus S is a right FBN ring and so $k\text{-dim}\, S_S = cl.k\text{-dim}\, S$, by the above comment and therefore

$$k\text{-dim}\, M_R = k\text{-dim}\, S_S = cl.k\text{-dim}\, S = cl.k\text{-dim}\, M_R.$$

□

Let M be a right R-module, $S = End(M_R)$ and $T = End({}_SM)$ be the endomorphism ring of ${}_SM$. In the literature, T is called the biendomorphism ring of M_R, abbreviated $Biend(M_R)$. Recall that the canon $\psi : R \to T$, by $\psi(r) = \bar{r}$, where $\bar{r}(m) = mr$, is a ring homomorphism. The module M_R is called faithful, when the canonical map ψ is injective or equivalently, $ann_R(M) = \{r \in R : Mr = 0\} = 0$. In other case that ψ is surjective, M is called to be balanced. We cite the following fact.

Theorem 2.8. [7, 18.8] Let M be a right R-module and $S = End(M_R)$, then M_R is a generator if and only if

1. ${}_SM$ is FP(i.e., finitely generated and projective), and
2. $R \cong T = Biend(M_R)$

The following result is now immediate.

Corollary 2.9. Let M_R be a generator, then M_R is both faithful and balanced.

Now, we have the following.

Theorem 2.10. Let M_R be a module.

1. If M_R is FPG, then so is ${}_SM$.
2. If M_R is faithful and balanced, then the converse of (1) holds, i.e., if ${}_SM$ is an FPG-module, so is M_R.

Proof. (1) Since M_R is a generator, thus ${}_SM$ is FP and $R \cong T$, by Theorem 2.8. In this case, M_T is FP and $S = End(M_R) \cong End(M_T) = Biend({}_SM)$. Again, we may invoke the left-hand version of Theorem 2.8 to show that ${}_SM$ is a generator. Therefore ${}_SM$ is an FPG-module.
(2) If ${}_SM$ is FPG, then so is M_T, by item (1). If in addition, M_R is faithful and balanced, then $R \cong T$. Therefore M_R is an FPG-module and we are done. □

According to [5, Theorem 2.4], a fully invariant proper submodule P of an R-module M is prime if $\phi X \subseteq P$ implies that $X \subseteq P$ or $\phi M \subseteq P$ for any $\phi \in S = End(M_R)$ and any fully invariant submodule X of M. Similarly, a prime submodule of ${}_SM$ is a fully invariant submodule P of ${}_SM$ such that $X\psi \subseteq P$ implies that $X \subseteq P$ or $M\psi \subseteq P$, for any $\psi \in T = End({}_SM) = Biend(M_R)$ and fully invariant submodule X of ${}_SM$.

Remark 2.11. Let M_R be a faithful and balanced module, then $R \cong T = Biend(M_R)$ and it is easy to see that X is a fully invariant submodule of M_R if and only if X is a fully invariant submodule of ${}_SM$ for any $X \subseteq M$. In other words, for both modules M_R and ${}_SM$, the sets of fully invariant submodules are coincide. Thus, according to the above comment, P is a prime submodule of ${}_SM$, if and only if, P is a fully invariant submodule of M_R such that $Xr \subseteq P$ implies that $X \subseteq P$ or $Mr \subseteq P$, for every $r \in R$ and fully invariant submodule X of M_R.

Now, with respect to the previous remark, we will have the following result. We note that, by $Spec(M_R)$ and $Spec({}_SM)$ we means the set of all prime submodules of M_R and ${}_SM$.

Theorem 2.12. Let M_R be an FPG module, $S = End(M_R)$ and $P \subseteq M$. Then $P \in Spec(M_R)$ if and only if $P \in Spec({}_SM)$.

Proof. First, let $P \in Spec(M_R)$. Since M_R is a generator, it is faithful and balanced and so P is a fully invariant submodule of ${}_SM$ too, by the previous remark. In order to show that $P \in Spec({}_SM)$, let X be a fully invariant submodule of M_R and $r \in R$ such that $Xr \subseteq P$. Define $\phi : M \to M$, by $\phi(m) = mr$, then clearly $\phi \in S$ and $Xr = \phi X \subseteq P$. Since $P \in Spec(M_R)$, we see that $X \subseteq P$ or $\phi M = Mr \subseteq P$. Now we use the previous remark to conclude that $P \in Spec({}_SM)$. Conversely, let $P \in Spec({}_SM)$. Note that ${}_SM$ is an FPG module, by Theorem 2.10. The previous reasoning implies that $P \in Spec(M_T)$, where $T = Biend(M_R)$. But M_R is faithful and balanced and so there is the ring isomorphism $R \cong T$. Consequently, $P \in Spec(M_R)$ and this copmlete the proof. □

We cite the following definition from [3, Definition 4.18].

Definition 2.13. Let M be an R-module. We denote the set of all prime submodules of M, by $Spec(M)$. Let $X(M) = Spec(M)$ and $X_0(M)$ denote the set of all maximal fully invariant submodules of M. For an ordinal number $\alpha > 0$, $X_\alpha(M)$ denote the set of all prime submodules P of M such that each prime submodule Q properly containing P, belongs to X_β for some $\beta < \alpha$. Hence we have $X_0(M) \subseteq X_1(M) \subseteq X_2(M) \subseteq \dots$. The smallest ordinal α for which $X_\alpha(M) = X(M)$ is called the classical Krull dimension of M and is denoted by $cl.k\text{-}\dim M$.

For left hand modules, provided a similar definition. Now we have the next important result.

Theorem 2.14. Let M_R be an FPG module and $S = End(M_R)$. Then $cl.k\text{-}\dim M_R = cl.k\text{-}\dim_S M$, if either side exists.

Proof. By Theorem 2.12, immediately it results that for every ordinal α, $X_\alpha(M_R) = X_\alpha({}_SM)$ and so we are done. □

Now we return to our main purpose to find a sort of modules M_R for which $k\text{-}\dim M_R = k\text{-}\dim {}_SM$. We cite the next known fact.

Lemma 2.15. [2, Theorem 13.15] Let S be a left FBN ring, R a right FBN ring and ${}_SM_R$ a bimodule, which is finitely generated on both sides. Then $k\text{-}\dim M_R = k\text{-}\dim {}_SM$.

We will now state our main result:

Theorem 2.16. Let R be a left FBN ring, M_R a fully bounded NPG module and $S = End(M_R)$. Then $k\text{-}\dim M_R = k\text{-}\dim {}_SM$.

Proof. Obviously M is both NQS and FPG, so by Theorems 2.7 and 2.14, $k\text{-}\dim M_R = cl.k\text{-}\dim M_R = cl.k\text{-}\dim_S M$. Also ${}_SM$ is FPG, by Theorem 2.10, so it is FQS. Since M is a generator, $R \cong T = End({}_SM)$ and so T is a left FBN ring. By Theorem 2.5 and left-hand version of Proposition 2.4, it followes that ${}_SM$ is fuly bounded and Noetherian, so it is NQS. Now we use the left-hand version of Theorem 2.7 to see that $k\text{-}\dim{}_S M = cl.k\text{-}\dim{}_S M$. Consequently, $k\text{-}\dim M_R = k\text{-}\dim{}_S M$. □

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