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n-MULTIPLICATIVE GENERALIZED DERIVATIONS WHICH ARE ADDITIVE

João Carlos da Motta Ferreira*
and
Maria das Graças Bruno Marietto†

Center for Mathematics, Computation and Cognition
Federal University of ABC
Santa Adélia Street, 166, 09210-170, Santo André, Brazil
*e-mail: *joao.cmferreira@ufabc.edu.br; †graca.marietto@ufabc.edu.br*

Abstract

In this paper, we present a unified technique to discuss the additivity of n-multiplicative generalized derivations.

1 Introduction

Let R be an associative ring and n be a positive integer ≥ 2. A mapping $d : \mathrm{R} \rightarrow \mathrm{R}$ is called a *n-multiplicative derivation of* R if

$$d(a_1 \cdots a_n) = \sum_{i=1}^{n} a_1 \cdots d(a_i) \cdots a_n,$$

for arbitrary elements $a_1, \cdots, a_n \in \mathrm{R}$ [4]. If $d(a_1a_2) = d(a_1)a_2 + a_1d(a_2)$ for arbitrary elements $a_1, a_2 \in \mathrm{R}$, we just say that d is a *multiplicative derivation* of R [1].

A mapping $h : \mathrm{R} \rightarrow \mathrm{R}$ is called *additive* if $h(a_1 + a_2) = h(a_1) + h(a_2)$, for arbitrary elements $a_1, a_2 \in \mathrm{R}$.

The following definition is based on [2, pp. 32] and [4, pp. 2351].

*Corresponding author
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A mapping $g : \mathrm{R} \to \mathrm{R}$ is called *n-multiplicative generalized derivation* if there is an additive n-multiplicative derivation of R d such that

$$g(a_1 a_2 \cdots a_n) = g(a_1)a_2 \cdots a_n + \sum_{i=2}^{n} a_1 a_2 \cdots d(a_i) \cdots a_n,$$

for arbitrary elements $a_1, a_2, \cdots, a_n \in \mathrm{R}$. If $g(a_1 a_2) = g(a_1)a_2 + a_1 d(a_2)$ for arbitrary elements $a_1, a_2 \in \mathrm{R}$, we just say that g is a *multiplicative generalized derivation* of R.

The authors in [2] characterized the additivity of multiplicative generalized derivations on the class of associative rings R containing a non-trivial idempotent satisfying certain conditions, based on Martindale's conditions [3, pp. 695]. Their main result as follows:

Theorem 1.1. *[2, Theorem 2.1.] Let* R *be an associative ring containing an idempotent e which satisfies the following conditions,*

(i) $x\mathrm{R}e = 0$ *implies* $x = 0$ *(and hence* $x\mathrm{R} = 0$ *implies* $x = 0$*).*

(ii) $exe\mathrm{R}(1-e) = 0$ *implies* $exe = 0$.

(iii) $(1-e)xe\mathrm{R}(1-e) = 0$ *implies* $(1-e)xe = 0$.

If g is any multiplicative generalized derivation of R, *i.e.* $g(xy) = g(x)y + xd(y)$, *for arbitrary elements* $x, y \in \mathrm{R}$ *and some derivation d of* R, *then g is additive.*

In this paper we present a unified technique, based on the ideas of Wang [4], to discuss the additivity of n-multiplicative generalized derivations. As an application of the obtained results, we generalize the Theorem 1.1 for the class of n-multiplicative generalized derivations of an arbitrary associative ring containing a non-trivial idempotent satisfying the Daif and El-Sayiad's conditions (i)-(iii).

2 The main result

Our main result is as follows:

Theorem 2.1. *Let* R *be an associative ring containing a non-trivial idempotent e which satisfies the following conditions:*

(i) $x\mathrm{R}e = 0$ *implies* $x = 0$ *(and hence* $x\mathrm{R} = 0$ *implies* $x = 0$*);*

(ii) $exe\mathrm{R}(1-e) = 0$ *implies* $exe = 0$;

(iii) $(1-e)xe\mathrm{R}(1-e) = 0$ *implies* $(1-e)xe = 0$.

Suppose that $f : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ *is a mapping and k a positive integer satisfying:*

(iv) $f(x, 0) = f(0, y) = 0$;

(v) $f(\mathrm{R}e, \mathrm{R}e) \subseteq \mathrm{R}e$;

(vi) $f(u_1 \cdots u_k x, u_1 u_2 \cdots u_k y) = 0$;

(vii) $f(x, y)u_1 u_2 \cdots u_k = f(x u_1 u_2 \cdots u_k, y u_1 u_2 \cdots u_k)$;

for arbitrary elements $x, y, u_1, u_2, \cdots, u_k \in \mathrm{R}$.

Then $f(x, y) = 0$, *for arbitrary elements* $x, y \in \mathrm{R}$.

Following the techniques presented by Daif and El-Sayiad [2] and Wang [4], we organize the proof of Theorem 2.1 in a series of Lemmas which have the same hypotheses. We begin with the following.

Lemma 2.2. $f(x, y)u = f(xu, yu)$ *for all elements* $x, y, u \in \mathrm{R}$.

Proof. For arbitrary elements $x, y, u, u_1, u_2, \cdots, u_k \in \mathrm{R}$ we have

$$\begin{aligned} f(x, y)uu_1 \cdots u_k = f(x, y)(uu_1) \cdots u_k &= f(x(uu_1) \cdots u_k, y(uu_1) \cdots u_k) \\ &= f((xu)u_1 \cdots u_k, (yu)u_1 \cdots u_k) = f(xu, yu)u_1 \cdots u_k. \end{aligned}$$

It follows that $(f(x, y)u - f(xu, yu))u_1 \cdots u_k = 0$. In view of condition (i) of the Theorem 2.1, we conclude that $f(x, y)u = f(xu, yu)$. □

Lemma 2.3. $f(x_{11} + x_{12}, y_{11} + y_{12}) = 0$, *for arbitrary elements* $x_{11}, y_{11} \in \mathrm{R}_{11}$ *and* $x_{12}, y_{12} \in \mathrm{R}_{12}$.

Proof. The result is a direct consequence of condition (vi) of the Theorem 2.1. □

Lemma 2.4. $f(x_{22}, y_{21}) = 0$, *for arbitrary elements* $x_{22} \in \mathrm{R}_{22}$ *and* $y_{21} \in \mathrm{R}_{21}$.

Proof. For an arbitrary element u_{1j} of R_{1j} $(j = 1, 2)$ we have

$$f(x_{22}, y_{21})u_{1j} = f(x_{22}u_{1j}, y_{21}u_{1j}) = f(0, y_{21}u_{1j}) = 0$$

which implies that $f(x_{22}, y_{21})\mathrm{R}_{1j} = 0$. Also, for an arbitrary element u_{2j} of R_{2j} $(j = 1, 2)$ we have

$$f(x_{22}, y_{21})u_{2j} = f(x_{22}u_{2j}, y_{21}u_{2j}) = f(x_{22}u_{2j}, 0) = 0$$

which results that $f(x_{22}, y_{21})\mathrm{R}_{2j} = 0$. It follows that $f(x_{22}, y_{21})\mathrm{R} = 0$ which implies that $f(x_{22}, y_{21}) = 0$, by condition (i) of the Theorem 2.1. □

Lemma 2.5. $f(x_{21}, y_{21}) = 0$, *for arbitrary elements* $x_{21}, y_{21} \in \mathrm{R}_{21}$.

Proof. For arbitrary elements z_{12} of R_{12} and u_{1j} of R_{1j} $(j = 1, 2)$ we have

$$f(x_{21}, x_{21})z_{12}u_{1j} = 0$$

which implies that $f(x_{21}, y_{21})z_{12}\mathrm{R}_{1j} = 0$. Also, for an arbitrary element u_{2j} of R_{2j} $(j = 1, 2)$ we have

$$\begin{aligned} f(x_{21}, y_{21})z_{12}u_{2j} &= f(x_{21}z_{12}u_{2j}, y_{21}z_{12}u_{2j}) \\ &= f(x_{21}z_{12}(u_{2j} + z_{12}u_{2j}), y_{21}(u_{2j} + z_{12}u_{2j})) \\ &= f(x_{21}z_{12}, y_{21})(u_{2j} + z_{12}u_{2j}) = 0, \end{aligned}$$

by Lemma 2.4, which results that $f(x_{21}, y_{21})z_{12}\mathrm{R}_{2j} = 0$. It follows that $f(x_{21}, y_{21})z_{12}\mathrm{R} = 0$ which implies that $f(x_{21}, y_{21})\mathrm{R}_{12} = 0$. From conditions (ii), (iii) and (v) of the Theorem 2.1, we conclude that $f(x_{21}, y_{21}) = 0$. □

Lemma 2.6. $f(x_{12} + x_{21}, y_{12} + y_{21}) = 0$, *for arbitrary elements* $x_{12}, y_{12} \in \mathrm{R}_{12}$ *and* $x_{21}, y_{21} \in \mathrm{R}_{21}$.

Proof. For an arbitrary element u_{1j} of R_{1j} $(j = 1, 2)$ we have

$$\begin{aligned} f(x_{12} + x_{21}, y_{12} + y_{21})u_{1j} &= f((x_{12} + x_{21})u_{1j}, (y_{12} + y_{21})u_{1j}) \\ &= f(x_{21}u_{1j}, y_{21}u_{1j}) = f(x_{21}, y_{21})u_{1j} = 0, \end{aligned}$$

by Lemma 2.5, which implies that $f(x_{12} + x_{21}, y_{12} + y_{21})\mathrm{R}_{1j} = 0$. Also, for an arbitrary element u_{2j} of R_{2j} $(j = 1, 2)$ we have

$$\begin{aligned} f(x_{12} + x_{21}, y_{12} + y_{21})u_{2j} &= f((x_{12} + x_{21})u_{2j}, (y_{12} + y_{21})u_{2j}) \\ &= f(x_{12}u_{2j}, y_{12}u_{2j}) = f(x_{12}, y_{12})u_{2j} = 0, \end{aligned}$$

by Lemma 2.3, which results that $f(x_{12} + x_{21}, y_{12} + y_{21})\mathrm{R}_{2j} = 0$. It follows that $f(x_{12} + x_{21}, y_{12} + y_{21})\mathrm{R} = 0$ which allows us to conclude that $f(x_{12} + x_{21}, y_{12} + y_{21}) = 0$. □

Lemma 2.7. $f(x_{11} + x_{21}, y_{11} + y_{21}) = 0$, *for arbitrary elements* $x_{11}, y_{11} \in \mathrm{R}_{11}$ *and* $x_{21}, y_{21} \in \mathrm{R}_{21}$.

Proof. For arbitrary elements z_{12} of R_{12} and u_{1j} of R_{1j} $(j = 1, 2)$ we have

$$f(x_{11} + x_{21}, y_{11} + y_{21})z_{12}u_{1j} = 0$$

which implies that $f(x_{11} + x_{21}, y_{11} + y_{21})z_{12}\mathrm{R}_{1j} = 0$. Also, for an arbitrary element u_{2j} of R_{2j} $(j = 1, 2)$ we have

$$\begin{aligned} f(x_{11} + x_{21}, y_{11} + y_{21})z_{12}u_{2j} &= f((x_{11} + x_{21})z_{12}u_{2j}, (y_{11} + y_{21})z_{12}u_{2j}) \\ &= f((x_{11}z_{12} + x_{21})(u_{2j} + z_{12}u_{2j}), (y_{11}z_{12} + y_{21})(u_{2j} + z_{12}u_{2j})) \end{aligned}$$

$$= f(x_{11}z_{12} + x_{21}, y_{11}z_{12} + y_{21})(u_{2j} + z_{12}u_{2j}) = 0,$$

by Lemma 2.6, which results that $f(x_{11}+x_{21}, y_{11}+y_{21})z_{12}\mathrm{R}_{2j} = 0$. This implies that $f(x_{11}+x_{21}, y_{11}+y_{21})z_{12}\mathrm{R} = 0$ which yields that $f(x_{11}+x_{21}, y_{11}+y_{21})\mathrm{R}_{12} = 0$. From conditions (ii), (iii) and (v) of the Theorem 2.1, we conclude that $f(x_{11} + x_{21}, y_{11} + y_{21}) = 0$. □

Proof of Theorem 2.1. Let x, y and r be arbitrary elements of R. Then

$$f(x, y)re = f(xre, yre) = 0,$$

by Lemma 2.7. This results that $f(x, y)\mathrm{R}e = 0$ which allows us to conclude that $f(x, y) = 0$, by condition (i) of the Theorem 2.1. □

3 Some applications of the main result

In this section, we give some applications of our main result. We started by discussing the additivity of n-multiplicative generalized derivations.

Theorem 3.1. *Let* R *be a* $(n-1)$*-torsion free associative ring containing a non-trivial idempotent* e *which satisfies the following conditions:*

(i) $x\mathrm{R}e = 0$ *implies* $x = 0$ *(and hence* $x\mathrm{R} = 0$ *implies* $x = 0$*);*

(ii) $exe\mathrm{R}(1-e) = 0$ *implies* $exe = 0$;

(iii) $(1-e)xe\mathrm{R}(1-e) = 0$ *implies* $(1-e)xe = 0$.

Then every n*-multiplicative generalized derivation of* R *is additive.*

The proof will be also organized in a series of lemmas. We begin with the following.

Let $g : \mathrm{R} \to \mathrm{R}$ be a n-multiplicative generalized derivation of R. Then there is an additive n-multiplicative derivation of R d such that

$$g(a_1a_2 \cdots a_n) = g(a_1)a_2 \cdots a_n + \sum_{i=2}^{n} a_1a_2 \cdots d(a_i) \cdots a_n,$$

for arbitrary elements $a_1, a_2, \cdots, a_n \in \mathrm{R}$. First, we note that

$$d(e) = d(\underbrace{e \cdots e}_{n \text{ terms}}) = \sum_{i=1}^{n} \underbrace{\overbrace{e \cdots d(e)}^{i \text{ terms}} \cdots e}_{n \text{ terms}} = d(e)e + (n-2)ed(e)e + ed(e)$$

which implies that $ed(e)e = 0$, since R is $(n-1)$-torsion free. Hence, if $d(e) = a_{11} + a_{12} + a_{21} + a_{22}$, where a_{ij} is an element of R_{ij} $(i, j = 1, 2)$, then $d(e) = a_{12} + a_{21}$. Also,

$$g(e) = g(\underbrace{e \cdots e}_{n \text{ terms}}) = \underbrace{g(e) \cdots e}_{n \text{ terms}} + \sum_{i=2}^{n} \underbrace{\overbrace{e \cdots d(e)}^{i \text{ terms}} \cdots e}_{n \text{ terms}} = g(e)e + ed(e).$$

Hence, if $g(e) = b_{11} + b_{12} + b_{21} + b_{22}$, where b_{ij} is an element of R_{ij} $(i, j = 1, 2)$, then $b_{11} + b_{12} + b_{21} + b_{22} = b_{11} + b_{21} + a_{12}$ which implies that $a_{12} = b_{12}$ and $b_{22} = 0$. This results that $g(e) = b_{11} + a_{12} + b_{21}$.

Let h be the inner derivation of R determined by the element $a_{12} - a_{21}$. Then $h(x) = [x, a_{12} - a_{21}]$ for an arbitrary element x of R. In particular, we have $h(e) = [e, a_{12} - a_{21}] = a_{12} + a_{21}$. Let H be the generalized inner derivation determined by the elements $b_{11} + b_{21}$ and $a_{12} - a_{21}$. Then $H(x) = (b_{11} + b_{21})x + x(a_{12} - a_{21})$ for an arbitrary element x of R. Similarly, we have $H(e) = b_{11} + a_{12} + b_{21}$.

Set the mappings $D, G : \mathrm{R} \to \mathrm{R}$ by $D = d - h$ and $G = g - H$. Then D is an additive n-multiplicative derivation of R and G is a n-multiplicative generalized derivation of R satisfying

$$G(a_1 a_2 \cdots a_n) = G(a_1) a_2 \cdots a_n + \sum_{i=2}^{n} a_1 a_2 \cdots D(a_i) \cdots a_n,$$

for arbitrary elements $a_1, a_2, \cdots, a_n \in \mathrm{R}$ and such that $D(e) = 0 = G(e)$. Moreover, the mapping g is additive if and only if G is additive.

From what we saw above, to prove the Theorem 3.1 we can, without loss of generality, replace the n-multiplicative derivation d by the n-multiplicative derivation D and the n-multiplicative generalized derivation g by the n-multiplicative generalized derivation G. Therefore, in the remaining part of this paper we will prove the additivity of the mapping G.

Lemma 3.2. $D(0) = 0$ *and* $G(0) = 0$.

Proof. We easily see that $D(0) = 0$. This results that

$$G(0) = G(\underbrace{0 \cdots 0}_{n \text{ terms}}) = \underbrace{G(0) \cdots 0}_{n \text{ terms}} + \sum_{i=2}^{n} \underbrace{0 \cdots D(0) \cdots 0}_{n \text{ terms}} = 0.$$

□

Lemma 3.3. $D(\mathrm{R}_{ij}) \subseteq \mathrm{R}_{ij}$ $(i, j = 1, 2)$.

Proof. For an arbitrary element x_{11} of R_{11} we have $D(x_{11}) = D(\underbrace{e x_{11} e \cdots e}_{n \text{ terms}}) = eD(x_{11})e$ which is an element of R_{11}. Also, for an arbitrary element x_{12} of

R_{12}, then $D(x_{12}) = D(\underbrace{e\cdots ex_{12}}_{n\text{ terms}}) = eD(x_{12})$ and $0 = D(0) = D(\underbrace{x_{12}e\cdots e}_{n\text{ terms}}) = D(x_{12})e$. It follows that $D(x_{12})$ belongs to R_{12}. Similarly, we prove that for an arbitrary element x_{21} of R_{21}, $D(x_{21})$ belongs to R_{21}. Finally, for an arbitrary element x_{22} of R_{22}, then $0 = D(0) = D(\underbrace{e\cdots ex_{22}}_{n\text{ terms}}) = eD(x_{22})$ and $0 = D(0) = D(\underbrace{x_{22}e\cdots e}_{n\text{ terms}}) = D(x_{22})e$. Therefore $D(x_{22})$ is an element of R_{22}. This proves the Lemma. □

Lemma 3.4. *The following hold: (i)* $G(R_{1j}) \subseteq R_{1j}\,(j = 1, 2)$, *(ii)* $G(R_{11} + R_{21}) \subseteq R_{11} + R_{21}$ *and (iii)* $G(R_{22}) \subseteq R_{12} + R_{22}$. *Moreover* $G(x_{11} + x_{12}) = G(x_{11}) + G(x_{12})$, *for arbitrary elements* x_{11} *of* R_{11} *and* x_{12} *of* R_{12}.

Proof. Let x_{1j} be an arbitrary element of $R_{1j}\,(j = 1, 2)$. Then $G(x_{1j}) = G(\underbrace{e\cdots ex_{1j}}_{n\text{ terms}}) = \underbrace{G(e)e\cdots x_{1j}}_{n\text{ terms}} + \sum_{i=2}^{n} \underbrace{e\cdots D(e)\cdots x_{1j}}_{n\text{ terms}} = eD(x_{1j}) = D(x_{1j})$ which is an element of R_{1j}, by Lemma 3.3. Thus, for an arbitrary element $x_{11} + x_{12}$ of eR we have $G(x_{11} + x_{12}) = G(\underbrace{e\cdots e(x_{11} + x_{12})}_{n\text{ terms}}) = \underbrace{G(e)e\cdots(x_{11} + x_{12})}_{n\text{ terms}} + \sum_{i=2}^{n} \underbrace{e\cdots D(e)\cdots(x_{11} + x_{12})}_{n\text{ terms}} = eD(x_{11} + x_{12}) = D(x_{11}) + D(x_{12}) = G(x_{11}) + G(x_{12})$, by the preceding case. This allows us to conclude that $G(R_{1j}) \subseteq R_{1j}\,(j = 1, 2)$ and that $G(x_{11}+x_{12}) = G(x_{11})+G(x_{12})$. Also, for arbitrary elements x_{11} of R_{11} and x_{21} of R_{21}, we have $G(x_{11} + x_{21}) = G(\underbrace{(x_{11} + x_{21})e\cdots e}_{n\text{ terms}}) = G\underbrace{(x_{11} + x_{21})e\cdots e}_{n\text{ terms}} + \sum_{i=2}^{n} \underbrace{(x_{11} + x_{21})\cdots D(e)\cdots e}_{n\text{ terms}} = G(x_{11} + x_{21})e$. This results that $G(R_{11} + R_{21}) \subseteq R_{11}+R_{21}$. Yet, for an arbitrary element x_{22} of R_{22} write $G(x_{22}) = d_{11}+ d_{12} + d_{21} + d_{22}$. Then $0 = G(0) = \underbrace{G(x_{22}e\cdots e)}_{n\text{ terms}} = G\underbrace{(x_{22})e\cdots e}_{n\text{ terms}} + \sum_{i=2}^{n} \underbrace{x_{22}\cdots D(e)\cdots e}_{n\text{ terms}} = G(x_{22})e = d_{11} + d_{21}$. This shows that $G(x_{22}) = d_{12} + d_{22}$.

This proves the Lemma. □

Proof of Theorem 3.1. From the hypotheses, let g a n-multiplicative generalized derivation of R and d an additive n-multiplicative derivation of R such that

$$g(a_1\cdots a_n) = g(a_1)\cdots a_n + \sum_{i=2}^{n} a_1\cdots d(a_i)\cdots a_n,$$

for arbitrary elements $a_1, \cdots, a_n \in \mathrm{R}$. Set $f : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ by $f(x,y) = G(x+y) - G(x) - G(y)$, for arbitrary elements $x, y \in \mathrm{R}$. Then $f(x,0) = f(0,y) = 0$, for arbitrary elements $x, y \in \mathrm{R}$. Also, for arbitrary elements x_{11}, y_{11} of R_{11} and x_{21}, y_{21} of R_{21} we have $f(x_{11} + x_{21}, y_{11} + y_{21}) = G((x_{11} + x_{21}) + (y_{11} + y_{21})) - G(x_{11} + x_{21}) - G(y_{11} + y_{21}) = G((x_{11} + y_{11}) + (x_{21} + y_{21})) - G(x_{11} + x_{21}) - G(y_{11} + y_{21})$ which is an element of $\mathrm{R}_{11} + \mathrm{R}_{21}$, by Lemma 3.4(ii). This shows that $f(\mathrm{R}e, \mathrm{R}e) \subseteq \mathrm{R}e$. Yet, for arbitrary elements $x, y, u_1, \cdots, u_{n-1} \in \mathrm{R}$ we have

$$
\begin{aligned}
f(u_1 \cdots u_{n-1}x, u_1 \cdots u_{n-1}y) &= G(u_1 \cdots u_{n-1}x + u_1 \cdots u_{n-1}y) \\
&- G(u_1 \cdots u_{n-1}x) - G(u_1 \cdots u_{n-1}y) = G(u_1 \cdots u_{n-1}(x+y)) \\
&- G(u_1 \cdots u_{n-1}x) - G(u_1 \cdots u_{n-1}y) = G(u_1) \cdots u_{n-1}(x+y) \\
&+ \sum_{i=2}^{n} u_1 \cdots D(u_i) \cdots u_{n-1}(x+y) - G(u_1) \cdots u_{n-1}x \\
-\sum_{i=2}^{n} u_1 \cdots D(u_i) \cdots u_{n-1}x - G(u_1) \cdots u_{n-1}y &- \sum_{i=2}^{n} u_1 \cdots D(u_i) \cdots u_{n-1}y = 0
\end{aligned}
$$

and

$$
\begin{aligned}
f(x,y)u_1 \cdots u_{n-1} &= \big(G(x+y) - G(x) - G(y)\big)u_1 \cdots u_{n-1} \\
&= G(x+y)u_1 \cdots u_{n-1} - G(x)u_1 \cdots u_{n-1} - G(y)u_1 \cdots u_{n-1} \\
&= G(x+y)u_1 \cdots u_{n-1} + \sum_{i=2}^{n} (x+y)u_1 \cdots D(u_i) \cdots u_{n-1} \\
&\quad - G(x)u_1 \cdots u_{n-1} - \sum_{i=2}^{n} xu_1 \cdots D(u_i) \cdots u_{n-1} \\
&\quad - G(y)u_1 \cdots u_{n-1} - \sum_{i=2}^{n} yu_1 \cdots D(u_i) \cdots u_{n-1} \\
&= G((x+y)u_1 \cdots u_{n-1}) - G(xu_1 \cdots u_{n-1}) - G(yu_1 \cdots u_{n-1}) \\
&= f(xu_1 \cdots u_{n-1}, yu_1 \cdots u_{n-1}).
\end{aligned}
$$

□

Corollary 3.5. *Let* R *be a* $(n-1)$*-torsion free prime associative ring containing a non-trivial idempotent* e*. Then every* n*-multiplicative generalized derivation of* R *is additive.*

The ideas that follow below are similar those presented by Wang [4].

Let X be a Banach space. Denote by $\mathcal{B}(\mathrm{X})$ the algebra of all bounded linear operators on X. A subalgebra of $\mathcal{B}(\mathrm{X})$ is called a *standard operator algebra* if it contains all finite rank operators. It is well known that every standard operator algebra is prime. Moreover, if $\dim \mathrm{X} \geq 2$, then there exists a non-trivial idempotent operator of rank one in $\mathcal{B}(\mathrm{X})$. Therefore, it follows from Corollary 3.5 that:

Corollary 3.6. *Let* X *be a Banach space with* $\dim \mathrm{X} \geq 2$, A *be a standard operator algebra on* X. *Then every* n*-multiplicative generalized derivation of* A *is additive.*

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