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A REMARK ON A QUASI-VARIATIONAL INEQUALITY FOR THE MAXWELL TYPE EQUATION

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Abstract

In this paper, we remark that a class of quasi-variational inequality for the Maxwell type equation in a multiply-connected domain with holes has a solution . Our class contains, so called, p-curlcurl operator. The existence of solution heavily depends on the geometry of the domain and the boundary conditions. We consider the quasi-variational inequality with a tangent free boundary condition.

1 Introduction

Generalized Maxwell's equations in electromagnetic field in equilibrium written by

$$\begin{cases} \boldsymbol{j} = \operatorname{curl} \boldsymbol{h}, \\ \operatorname{curl} \boldsymbol{e} = \boldsymbol{f}, \\ \varepsilon \operatorname{div} \boldsymbol{e} = q, \\ \operatorname{div} \boldsymbol{h} = 0 \end{cases} \tag{1.1}$$

in Ω, where Ω is a bounded domain in $\mathbb{R}^3$ with a boundary Γ, $\boldsymbol{e}$ and $\boldsymbol{h}$ denote the electric and the magnetic fields, respectively, ε is the permittivity of the electric field, σ is the electric conductivity of the material, $\boldsymbol{j}$ is the total current

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density and q is the density of electric charge. We use the nonlinear extension of Ohm's law $|\boldsymbol{j}|^{p-2}\boldsymbol{j} = \sigma\boldsymbol{e}$. Then $\boldsymbol{h}$ satisfies the following equations

$$\begin{cases} \operatorname{curl}\,[\frac{1}{\sigma}|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}] = \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{h} = 0 \end{cases} \tag{1.2}$$

in Ω. Mathematically the left-hand side of (1.2) is, so called, p-curlcurl operator. We impose the natural boundary condition

$$\boldsymbol{h} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \tag{1.3}$$

where $\boldsymbol{n}$ denotes the outer normal unit vector field to Γ. Putting $\nu = 1/\sigma$, we must consider the following system.

$$\begin{cases} \operatorname{curl}\,[\nu|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}] = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{h} = 0 & \text{in } \Omega, \\ \boldsymbol{h} \times \boldsymbol{n} = 0, & \text{on } \Gamma. \end{cases} \tag{1.4}$$

In the case where Ω is a bounded simply-connected domain without holes, and $p > 2$, Yin et al. [11] obtained the existence theorem of a weak solution of (1.2) with (1.3). Miranda et al [9] considered the case the boundary condition $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ, in the simply-connected domain without holes.

The above generalization of the Ohm law arises in type-II superconductors and is known as an extension of the Bean critical-state model, in which $|\operatorname{curl}\boldsymbol{h}|$ cannot exceed some given critical value $j > 0$. In the present paper, we consider the case where this threshold j varies with the absolute value $|\boldsymbol{h}|$ of the magnetic field $\boldsymbol{h}$. That is to say,

$$\boldsymbol{e} = \begin{cases} \nu|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h} & \text{if } |\operatorname{curl}\boldsymbol{h}| < j(|\boldsymbol{h}|), \\ \nu(j^{p-2} + \lambda)\operatorname{curl}\boldsymbol{h} & \text{if } |\operatorname{curl}\boldsymbol{h}| = j(|\boldsymbol{h}|), \end{cases}$$

where $\lambda = \lambda(x) \geq 0$ is an unknown Lagrange multiplier and has support in the superconductivity region

$$S = \{x \in \Omega; |\operatorname{curl}\boldsymbol{h}(x)| = j(|\boldsymbol{h}(x)|\}.$$

This fact leads to a following quasi-variational inequality.

$$\int_\Omega \nu|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h} \cdot \operatorname{curl}(\boldsymbol{v} - \boldsymbol{h})dx \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{h})dx \tag{1.5}$$

for all $\boldsymbol{v}$ in an appropriate space such that $|\operatorname{curl}\boldsymbol{v}| \leq j(|\boldsymbol{h}|)$ a.e. in Ω.

When Ω is multiply-connected and has holes, it is insufficient to show the existence of solution to the system (1.5) under only the boundary condition (1.3). To do so, in addition to (1.3), we impose that

$$\langle \boldsymbol{h} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 \text{ for } i = 1, \ldots, I,$$

where $\Gamma_i\,(i = 0, 1, \ldots, I)$ are connected components of the boundary Γ, Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$ and $\langle \cdot, \cdot \rangle_{\Gamma_i}$ is a duality bracket.

In this paper, taking a generalization into consideration, we consider the following quasi-variational inequality: to find $\boldsymbol{h} \in \mathbb{K}_{\boldsymbol{h}}$ such that

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{h}|^2)\mathrm{curl}\, \boldsymbol{h} \cdot \mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{h})dx \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{h} \rangle_\Omega \tag{1.6}$$

for all $\boldsymbol{v} \in \mathbb{K}_{\boldsymbol{h}}$. Here a function $S(x, t)$ satisfies some structural conditions and $\mathbb{K}_{\boldsymbol{h}}$ is a convex subset satisfying a constrained condition, and $\langle \cdot, \cdot \rangle_\Omega$ denotes some duality bracket. All the definitions of the spaces and the properties are stated in details in section 2.

The paper is organized as follows. Section 2 consists of two subsections. In subsection 2.1, since we allow that the domain $\Omega \subset \mathbb{R}^3$ is multiply-connected and has holes, we define the geometry of the domain and a basic space of functions. In subsection 2.2, we give the main theorem (Theorem 2.4). In section 3, we consider the associated variational problem for which we show the existence of a unique solution and an estimate of the solution. In section 4, we give a proof of the main theorem (Theorem 2.4).

2 Preliminaries and the main theorem

This section consists of two subsections. In subsection 2.1, we give a Carathéodory function $S(x, t)$ on $\Omega \times [0, +\infty)$ satisfying some structural conditions, and introduce some spaces of functions. In subsection 2.2, we state the main theorem.

2.1 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$ boundary Γ. Since we allow Ω to be a multiply-connected domain with holes in $\mathbb{R}^3$, we assume that Ω satisfies the following conditions as in Amrouche and Seloula [2] (cf. Amrouche and Seloula [1], Dautray and Lions [5, vol. 3] and Girault and Raviart [8]). Ω is locally situated on one side of Γ and satisfies the following (O1) and (O2).

(O1) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_I$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(O2) There exist J connected open surfaces Σ_j, $(j = 1, \ldots, J)$, called cuts, contained in Ω such that

- (a) each surface Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$,
- (b) $\partial\Sigma_j \subset \Gamma$ $(j = 1, \ldots, J)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ,

(c) $\overline{\Sigma_j} \cap \overline{\Sigma_k} = \emptyset\,(j \neq k)$,

(d) the open set $\Omega^\circ = \Omega \setminus (\cup_{j=1}^J \Sigma_j)$ is simply connected and has a pseudo-$C^{1,1}$ boundary.

The number J is called the first Betti number and I the second Betti number. We say that Ω is simply connected if $J = 0$ and Ω has no holes if $I = 0$. If we define

$$\mathbb{K}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

and

$$\mathbb{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},$$

then it is well known that $\dim \mathbb{K}_T^p(\Omega) = J$ and $\dim \mathbb{K}_N^p(\Omega) = I$.

Throughout this paper, let $1 < p < \infty$ and we denote the conjugate exponent of p by p', i.e., $(1/p) + (1/p') = 1$. From now on we use $L^p(\Omega)$, $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ for the standard L^p and Sobolev spaces of functions. For any Banach space B, we denote $B \times B \times B$ by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vector and vector-valued functions, and we denote the standard Euclidean inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. For the dual space $\boldsymbol{B}'$ of $\boldsymbol{B}$, we write $\langle \cdot, \cdot \rangle_{\boldsymbol{B}', \boldsymbol{B}}$ for the duality bracket.

We assume that a Carathéodory function $S(x, t)$ in $\Omega \times [0, \infty)$ satisfies the following structural conditions. For a.e. $x \in \Omega$, $S(x, t) \in C^2((0, \infty)) \cap C^0([0, \infty))$, and positive constants $0 < \lambda \leq \Lambda < \infty$ such that for a.e. $x \in \Omega$,

$$S(x, 0) = 0 \text{ and } \quad \lambda t^{(p-2)/2} \leq S_t(x, t) \leq \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1a}$$

$$\lambda t^{(p-2)/2} \leq S_t(x, t) + 2t S_{tt}(x, t) \leq \Lambda t^{(p-2)/2} \text{ for } t > 0, \tag{2.1b}$$

$$\text{If } 1 < p < 2, S_{tt}(x, t) < 0, \text{ and if } p \geq 2, S_{tt}(x, t) \geq 0 \text{ for } t > 0, \tag{2.1c}$$

where $S_t = \partial S / \partial t$ and $S_{tt} = \partial^2 S / \partial t^2$. We note that from (2.1a), we have

$$\frac{2}{p} \lambda t^{p/2} \leq S(x, t) \leq \frac{2}{p} \Lambda t^{p/2} \text{ for } t \geq 0. \tag{2.2}$$

Example 2.1. If $S(x, t) = \nu(x) g(t) t^{p/2}$, where ν is a measurable function in Ω and satisfies $0 < \nu_* \leq \nu(x) \leq \nu^* < \infty$ for a.e. $x \in \Omega$ for some constants ν_* and ν^*, and $g \in C^\infty([0, \infty))$,

When $g(t) \equiv 1$, it follows from elementary calculations that (2.1a)-(2.1c) hold.

As an another example, we can take

$$g(t) = \begin{cases} a(e^{-1/t} + 1) & \text{if } t > 0, \\ a & \text{if } t = 0 \end{cases}$$

with a constant $a > 0$. Then $S(x, t) = \nu(x) g(t) t^{p/2}$ satisfies (2.1a)-(2.1c) if $p \geq 2$. (cf. Aramaki [4, Example 3.2]).

We give a monotonic property of S_t.

Lemma 2.2. There exists a constant $c>0$ such that for all $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$,

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big)\cdot(\boldsymbol{a}-\boldsymbol{b}) \geq \begin{cases} c|\boldsymbol{a}-\boldsymbol{b}|^p & \text{if } p\geq 2,\\ c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \text{if } 1<p<2.\end{cases}$$

In particular, if $\boldsymbol{a}\neq\boldsymbol{b}$, we have

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big)\cdot(\boldsymbol{a}-\boldsymbol{b})>0.$$

For the proof, see Aramaki [3, Lemma 3.6].

We can see that the convexity of $S(x,t)$ in the following sense.

Lemma 2.3. If $S(x,t)$ satisfies (2.1a) and (2.1b), then for a.e. $x\in\Omega$, the function $\mathbb{R}\ni t\mapsto g[t]=S(x,t^2)$ is strictly convex.

For the proof, see [4, Lemma 2.3].

The following inequality is used frequently (cf. [2]). If Ω is a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$ boundary Γ, and if $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ satisfies $\operatorname{curl}\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$, $\operatorname{div}\boldsymbol{u}\in L^p(\Omega)$ and $\boldsymbol{u}\times\boldsymbol{n}\in\boldsymbol{W}^{1-1/p,p}(\Gamma)$, then $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant $C>0$ depending only on p and Ω such that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C(\|\operatorname{curl}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}+\|\operatorname{div}\boldsymbol{u}\|_{L^p(\Omega)}+\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{u}\times\boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma)}). \quad (2.3)$$

Moreover, if $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ satisfies $\operatorname{curl}\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$, then $\boldsymbol{u}\times\boldsymbol{n}\in\boldsymbol{W}^{-1/p,p}(\Gamma)$ is well defined, and if $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ satisfies $\operatorname{div}\boldsymbol{u}\in L^p(\Omega)$, then $\boldsymbol{u}\cdot\boldsymbol{n}\in W^{-1/p,p}(\Gamma)$ is well defined by

$$\langle\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{\phi}\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma),\boldsymbol{W}^{1-1/p',p'}(\Gamma)}=\int_\Omega\boldsymbol{u}\cdot\operatorname{curl}\boldsymbol{\phi}dx-\int_\Omega\operatorname{curl}\boldsymbol{u}\cdot\boldsymbol{\phi}dx$$

for all $\boldsymbol{\phi}\in\boldsymbol{W}^{1,p'}(\Omega)$ and

$$\langle\boldsymbol{u}\cdot\boldsymbol{n},\phi\rangle_{W^{-1/p,p}(\Gamma),W^{1-1/p',p'}(\Gamma)}=\int_\Omega\boldsymbol{u}\cdot\nabla\phi dx+\int_\Omega(\operatorname{div}\boldsymbol{u})\phi dx$$

for all $\phi\in W^{1,p'}(\Omega)$. Furthermore, if $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ satisfies $\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ, then there exists a constant $C>0$ depending only on p and Ω such that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\leq C(\|\operatorname{curl}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}+\|\operatorname{div}\boldsymbol{u}\|_{L^p(\Omega)}+\sum_{i=1}^I|\langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|$$

where $\langle\cdot,\cdot\rangle_{\Gamma_i}=\langle\cdot,\cdot\rangle_{W^{-1/p,p}(\Gamma_i),W^{1-1/p',p'}(\Gamma_i)}$.

Define a space

$$\mathbb{V}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathrm{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p(\Omega), \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \text{ for } i = 1, \dots, I\}.$$

with the norm

$$\|\boldsymbol{v}\|_{\mathbb{V}_N^p(\Omega)} = \|\mathrm{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

We note that $\|\boldsymbol{v}\|_{\mathbb{V}_N^p(\Omega)}$ is equivalent to $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ for $\boldsymbol{v} \in \mathbb{V}_N^p(\Omega)$ (cf. [2]). Since $\mathbb{V}_N^p(\Omega)$ is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega)$, we can see that $\mathbb{V}_N^p(\Omega)$ is a reflexive Banach space.

2.2 The main theorem

Let $F : [0, \infty) \to \mathbb{R}$ be a given continuous function such that there exists a constant $\nu > 0$ such that

$$F(s) \geq \nu \text{ for all } s \geq 0 \tag{2.4}$$

and for any $\boldsymbol{h} \in \mathbb{V}_N^p(\Omega)$, define a closed convex subset

$$\mathbb{K}_{\boldsymbol{h}} = \{\boldsymbol{v} \in \mathbb{V}_N^p(\Omega); |\mathrm{curl}\, \boldsymbol{v}| \leq F(|\boldsymbol{h}|) \text{ a.e. in } \Omega\}. \tag{2.5}$$

For given $\boldsymbol{f} \in \mathbb{V}_N^p(\Omega)'$, we consider the following quasi-variational inequality: to find $\boldsymbol{h} \in \mathbb{K}_{\boldsymbol{h}}$ such that

$$\int_\Omega S_t(x, |\mathrm{curl}\, \boldsymbol{h}|^2) \mathrm{curl}\, \boldsymbol{h} \cdot \mathrm{curl}\, (\boldsymbol{v} - \boldsymbol{h}) dx \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{h} \rangle_{\mathbb{V}_N^p(\Omega)', \mathbb{V}_N^p(\Omega)} \tag{2.6}$$

for all $\boldsymbol{v} \in \mathbb{K}_{\boldsymbol{h}}$.

We are in a position to state the main theorem.

Theorem 2.4. Let Ω be a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$ boundary Γ satisfying (O1) and (O2), and assume that a Carathéodory function $S(x, t)$ satisfies the structural conditions (2.1a)-(2.1c), and a function $F : [0, \infty) \to \mathbb{R}$ satisfies (2.4), and if $1 < p \leq 3$,

$$F(s) \leq c_0 + c_1 s^\alpha, \tag{2.7}$$

where $\alpha \geq 0$ if $p = 3$ and $0 \leq \alpha < p/(3 - p)$ if $1 < p < 3$. Then for any $\boldsymbol{f} \in \mathbb{V}_N^p(\Omega)'$, the quasi-variational inequality (2.6) has a solution $\boldsymbol{h} \in \mathbb{K}_{\boldsymbol{h}}$ and there exists a constant $C > 0$ such that

$$\|\boldsymbol{h}\|^p_{\mathbb{V}_N^p(\Omega)} \leq C \|\boldsymbol{f}\|^{p'}_{\mathbb{V}_N^p(\Omega)'}. \tag{2.8}$$

3 Associate variational inequality

In this section we consider an associate variational inequality. For any given function $\varphi \in L^\infty(\Omega)$, we define

$$\mathbb{K}_\varphi = \{\boldsymbol{v} \in \mathbb{V}^p_N(\Omega); |\mathrm{curl}\,\boldsymbol{v}| \le F(|\varphi|) \text{ for a.e. in } \Omega\}.$$

We consider the following variational inequality: to find $\boldsymbol{h} \in \mathbb{K}_\varphi$ such that

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{h}|^2)\mathrm{curl}\,\boldsymbol{h} \cdot \mathrm{curl}\,(\boldsymbol{v} - \boldsymbol{h})dx \\ \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{h}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)} \text{ for all } \boldsymbol{v} \in \mathbb{K}_\varphi. \quad (3.1)$$

We prove the following proposition.

Proposition 3.1. Let $\varphi \in L^\infty(\Omega)$ and $\boldsymbol{f} \in \mathbb{V}^p_N(\Omega)'$. Then the variational inequality (3.1) has a unique solution $\boldsymbol{h} \in \mathbb{K}_\varphi$ and there exists a constant depending only on λ and p such that

$$\|\boldsymbol{h}\|^p_{\mathbb{V}^p_N(\Omega)} \le C\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p_N(\Omega)'}. \quad (3.2)$$

Proof. Define a functional on $\mathbb{K}_\varphi$ by

$$E[\boldsymbol{v}] = \frac{1}{2}\int_\Omega S(x, |\mathrm{curl}\,\boldsymbol{v}|^2)dx - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)}. \quad (3.3)$$

We derive the following minimization problem: to find $\boldsymbol{h} \in \mathbb{K}_\varphi$ such that

$$E[\boldsymbol{h}] = \inf_{\boldsymbol{v} \in \mathbb{K}_\varphi} E[\boldsymbol{v}]. \quad (3.4)$$

We call such a function $\boldsymbol{h}$ a minimizer of (3.4).

Lemma 3.2. The minimization problem (3.4) has a unique minimizer $\boldsymbol{h} \in \mathbb{K}_\varphi$.

Proof. We remember that the space $\mathbb{K}_\varphi$ is a closed convex subset of $\mathbb{V}^p_N(\Omega)$. The functional E is proper, strictly convex functional from Lemma 2.3 (cf. [4]). We show that E is coercive on $\mathbb{K}_\varphi$. Using the Young inequality,

$$\begin{aligned} E[\boldsymbol{v}] &\ge \frac{\lambda}{p}\|\mathrm{curl}\,\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} - \|\boldsymbol{f}\|_{\mathbb{V}^p_N(\Omega)'}\|\boldsymbol{v}\|_{\mathbb{V}^p_N(\Omega)} \\ &\ge \frac{\lambda}{p}\|\boldsymbol{v}\|^p_{\mathbb{V}^p_N(\Omega)} - C(\varepsilon)\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p_N(\Omega)'} - \varepsilon\|\boldsymbol{v}\|^p_{\mathbb{V}^p_N(\Omega)} \end{aligned}$$

for any $\varepsilon > 0$ and for some constant $C(\varepsilon)$. We choose $\varepsilon = \lambda/(2p)$, we have

$$E[\boldsymbol{v}] \ge \frac{\lambda}{2p}\|\boldsymbol{v}\|^p_{\mathbb{V}^p_N(\Omega)} - C\big(\frac{\lambda}{2p}\big)\|\boldsymbol{f}\|^{p'}_{\mathbb{V}^p_N(\Omega)'}.$$

Hence E is coercive on $\mathbb{K}_\varphi$. Finally, we show that E is lower semi-continuous. Let $\boldsymbol{v}_n, \boldsymbol{v} \in \mathbb{K}_\varphi$ and $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\mathbb{V}^p_N(\Omega)$. Then $\operatorname{curl} \boldsymbol{v}_n \to \operatorname{curl} \boldsymbol{v}$ strongly in $\boldsymbol{L}^p(\Omega)$. According to Aramaki [3], we have

$$\int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2)dx \le \liminf_{n\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_n|^2)dx.$$

This implies that E is lower semi-continuous. By Ekeland and Témam [6, Chapter II, Proposition 1.2], the minimization problem (3.4) has a unique minimizer $\boldsymbol{h} \in \mathbb{K}_\varphi$. □

Let $\boldsymbol{h} \in \mathbb{K}_\varphi$ be the minimizer of (3.4). For any $\boldsymbol{v} \in \mathbb{K}_\varphi$, $(1-\mu)\boldsymbol{h} + \mu\boldsymbol{v} = \boldsymbol{h} + \mu(\boldsymbol{v} - \boldsymbol{h}) \in \mathbb{K}_\varphi$ for $0 < \mu < 1$. Thus $E[\boldsymbol{h}] \le E[\boldsymbol{h} + \mu(\boldsymbol{v} - \boldsymbol{h})]$. Hence

$$\frac{d}{d\mu} E[\boldsymbol{h} + \mu(\boldsymbol{v} - \boldsymbol{h})]\Big|_{\mu=+0} \ge 0.$$

That is,

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{h}|^2)\operatorname{curl} \boldsymbol{h} \cdot \operatorname{curl}(\boldsymbol{v} - \boldsymbol{h})dx \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{h}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)}$$

for all $\boldsymbol{v} \in \mathbb{K}_\varphi$, so $\boldsymbol{h}$ is a solution of the variational inequality (3.1).

We show the uniqueness of solution. Let $\boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathbb{K}_\varphi$ be two solutions of (3.1). Then we have

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{h}_1|^2)\operatorname{curl} \boldsymbol{h}_1 \cdot \operatorname{curl}(\boldsymbol{h}_2 - \boldsymbol{h}_1)dx \ge \langle \boldsymbol{f}, \boldsymbol{h}_2 - \boldsymbol{h}_1\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)}$$

and

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{h}_2|^2)\operatorname{curl} \boldsymbol{h}_2 \cdot \operatorname{curl}(\boldsymbol{h}_1 - \boldsymbol{h}_2)dx \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{h}\rangle_{\mathbb{V}^p_N(\Omega)', \mathbb{V}^p_N(\Omega)}.$$

Therefore, we have

$$\int_\Omega \Big(S_t(x, |\operatorname{curl} \boldsymbol{h}_1|^2)\operatorname{curl} \boldsymbol{h}_1 - S_t(x, |\operatorname{curl} \boldsymbol{h}_2|^2)\operatorname{curl} \boldsymbol{h}_2 \Big) \cdot \operatorname{curl}(\boldsymbol{h}_1 - \boldsymbol{h}_2)dx \le 0.$$

Using Lemma 2.2, we have

$$\int_\Omega |\operatorname{curl}(\boldsymbol{h}_1 - \boldsymbol{h}_2)|^p dx = 0,$$

if $p \ge 2$ and

$$\int_\Omega (|\operatorname{curl} \boldsymbol{h}_1| + |\operatorname{curl} \boldsymbol{h}_2|)^{p-2} \operatorname{curl}(\boldsymbol{h}_1 - \boldsymbol{h}_2)|^2 dx = 0,$$

if $1 < p < 2$. Hence we have $\boldsymbol{h}_1 = \boldsymbol{h}_2$ in $\mathbb{V}_N^p(\Omega)$ in each case.

Finally we show the estimate (3.2). If we take $\boldsymbol{v} = \boldsymbol{0}$ as a test function of (3.1), then we have

$$\int_\Omega S_t(x, |\mathrm{curl}\,\boldsymbol{h}|^2)\mathrm{curl}\,\boldsymbol{h}\cdot\mathrm{curl}\,\boldsymbol{h}dx \le \langle \boldsymbol{f}, \boldsymbol{h}\rangle_{\mathbb{V}_N^p(\Omega)', \mathbb{V}_N^p(\Omega)}.$$

By the structural condition (2.1a), we can see that

$$\lambda\|\mathrm{curl}\,\boldsymbol{h}\|_{\boldsymbol{L}^p(\Omega)} \le \|\boldsymbol{f}\|_{\mathbb{V}_N^p(\Omega)'}\|\boldsymbol{h}\|_{\mathbb{V}_N^p(\Omega)}.$$

This implies the estimate (3.2). This completes the proof of Lemma 3.2. □

We show that the solution of (3.1) is continuously depending on $\varphi \in L^\infty(\Omega)$.

Lemma 3.3. Assume that $\varphi_n, \varphi \in L^\infty(\Omega)$ and $\varphi_n \to \varphi$ in $L^\infty(\Omega)$ as $n \to \infty$, and let $\boldsymbol{h}_n \in \mathbb{K}_{\varphi_n}$ and $\boldsymbol{h} \in \mathbb{K}_\varphi$ be solutions of (3.1), respectively. Then $\boldsymbol{h}_n \to \boldsymbol{h}$ in $\mathbb{V}_N^p(\Omega)$ as $n \to \infty$.

Proof. First we prove that $\mathrm{Lim}\,\mathbb{K}_{\varphi_n} = \mathbb{K}_\varphi$ in the sense of Mosco (cf. [10]). In order to do so, we must first show that if $\boldsymbol{v}_n \in \mathbb{K}_{\varphi_n}$ and $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\mathbb{V}_N^p(\Omega)$, then $\boldsymbol{v} \in \mathbb{K}_\varphi$. In fact, since $|\mathrm{curl}\,\boldsymbol{v}_n| \le F(|\varphi_n|)$ a.e. in Ω, for any measurable subset $\omega \subset \Omega$,

$$\int_\omega |\mathrm{curl}\,\boldsymbol{v}|dx \le \liminf_{n\to\infty}\int_\omega |\mathrm{curl}\,\boldsymbol{v}_n|dx \le \liminf_{n\to\infty}\int_\omega F(|\varphi_n|)dx = \int_\omega F(|\varphi|)dx.$$

Hence $|\mathrm{curl}\,\boldsymbol{v}| \le F(|\varphi|)$. a.e. in Ω, so $\boldsymbol{v} \in \mathbb{K}_\varphi$.

Next we must show that for any $\boldsymbol{v} \in \mathbb{K}_\varphi$, there exists $\boldsymbol{v}_n \in \mathbb{K}_{\varphi_n}$ such that $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\mathbb{V}_N^p(\Omega)$ as $n \to \infty$. Indeed, put

$$\lambda_n = \|F(|\varphi_n|) - F(|\varphi|)\|_{L^\infty(\Omega)}.$$

Then $\lambda_n \to 0$ as $n \to \infty$ by the hypothesis. Define

$$\boldsymbol{v}_n = \frac{1}{\mu_n}\boldsymbol{v} \text{ with } \mu_n = 1 + \frac{\lambda_n}{\nu},$$

where ν is a constant of (2.4). Then $\boldsymbol{v}_n \in \mathbb{V}_N^p(\Omega)$ and

$$|\mathrm{curl}\,\boldsymbol{v}_n| \le \frac{1}{\mu_n}|\mathrm{curl}\,\boldsymbol{v}| \le \frac{1}{\mu_n}F(|\varphi|).$$

Since

$$\mu_n = 1 + \frac{\|F(|\varphi_n|) - F(|\varphi|)\|_{L^\infty(\Omega)}}{\nu} \ge 1 + \frac{F(|\varphi|) - F(|\varphi_n|)}{F(|\varphi_n|)},$$

we have $|\mathrm{curl}\,\boldsymbol{v}_n| \leq F(|\varphi_n|)$, so $\boldsymbol{v}_n \in \mathbb{K}_{\varphi_n}$. Since $\mu_n \to 1$ as $n \to \infty$, we have

$$\|\boldsymbol{v}_n - \boldsymbol{v}\|^p_{\mathbb{V}^p_N(\Omega)} = \int_\Omega |\mathrm{curl}\,(\boldsymbol{v}_n - \boldsymbol{v})|^p dx = \left(1 - \frac{1}{\mu_n}\right)^p \int_\Omega |\mathrm{curl}\,\boldsymbol{v}|^p dx \to 0$$

as $n \to \infty$. Thus $\mathbb{K}_\varphi = \text{ s -Lim}\,\mathbb{K}_{\varphi_n}$ in the sense of Mosco. By the well known result of Mosco (cf. [10]), we can see that $\boldsymbol{h}_n \to \boldsymbol{h}$ in $\mathbb{V}^p_N(\Omega)$. $\square$

4 Proof of Theorem 2.4

To prove Theorem 2.4, we use a fixed point argument. For any $\varphi \in C(\overline{\Omega})$, we denote the unique solution of the variational inequality (3.1) by $\boldsymbol{h}_\varphi \in \mathbb{K}_\varphi$. Define an operator $S : C(\overline{\Omega}) \ni \varphi \mapsto \boldsymbol{h}_\varphi \in \mathbb{V}^p_N(\Omega)$. From Lemma 3.3, S is continuous. When $p > 3$, it follows from Kondrachov theorem that the embedding mapping $\mathbb{V}^p_N(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact. In particular, there exists a constant $C_1 > 0$ independent of φ such that

$$\|\varphi\|_{C(\overline{\Omega})} \leq C_1 \|\boldsymbol{h}_\varphi\|_{\mathbb{V}^p_N(\Omega)}.$$

Therefore, the following nonlinear mapping

$$\begin{array}{cccccccc} \widetilde{S}: & C(\overline{\Omega}) & \to & \mathbb{V}^p_N(\Omega) & \hookrightarrow & C(\overline{\Omega}) & \to & C(\overline{\Omega}) \\ & \varphi & \mapsto & \boldsymbol{h}_\varphi & \mapsto & \boldsymbol{h}_\varphi & \mapsto & |\boldsymbol{h}_\varphi| \end{array}$$

is continuous and compact. On the other hand, since it follows from Proposition 3.1 that we have

$$\|\varphi\|_{C(\overline{\Omega})} \leq C_1 \|\boldsymbol{h}_\varphi\|_{\mathbb{V}^p_N(\Omega)} \leq C_2 \|\boldsymbol{f}\|^{p'-1}_{\mathbb{V}^p_N(\Omega)'} = C_3 \tag{4.1}$$

where C_3 is a constant independent of φ. Hence there exists $R > 0$ such that $\widetilde{S}(C(\overline{\Omega})) \subset D_R$, where

$$D_R = \{\varphi \in C(\overline{\Omega}); \|\varphi\|_{C(\overline{\Omega})} \leq R\}.$$

Thus since $\widetilde{S} : D_R \to D_R$ is continuous and compact, it follows from the Schauder fixed point theorem that $\widetilde{S}$ has a fixed point φ in D_R, that is, $\varphi = |\boldsymbol{h}_\varphi|$. Thus $\boldsymbol{h}_\varphi \in \mathbb{K}_\varphi$.

When $1 < p \leq 3$, we apply the Leray-Schauder fixed point theorem. (cf. Gilbarg and Trudinger [7, Theorem 11.3]). For any $\varphi \in C(\overline{\Omega})$, the solution $\boldsymbol{h}_\varphi$ of (3.1) belongs to $\mathbb{V}^r_N(\Omega)$ for any $r > 3$, because $|\mathrm{curl}\,\boldsymbol{h}_\varphi| \leq F(|\varphi|) \leq C$. Since

$$\begin{aligned} \|\boldsymbol{h}_\varphi - \boldsymbol{h}_\psi\|^r_{\mathbb{V}^r_N(\Omega)} &= \int_\Omega |\mathrm{curl}\,(\boldsymbol{h}_\varphi - \boldsymbol{h}_\psi)|^r dx \\ &= \int_\Omega |\mathrm{curl}\,\boldsymbol{h}_\varphi - \mathrm{curl}\,\boldsymbol{h}_\psi|^{r-p} |\mathrm{curl}\,(\boldsymbol{h}_\varphi - \boldsymbol{h}_\psi|^p dx \\ &\leq 2^{r-p-1} \int_\Omega (F(|\varphi|)^{r-p} + F(|\psi|)^{r-p}) |\mathrm{curl}\,(\boldsymbol{h}_\varphi - \boldsymbol{h}_\psi)|^p dx. \end{aligned}$$

Thus

$$\begin{array}{rlllllll} \widetilde{S}: & C(\overline{\Omega}) & \to & \mathbb{V}^r_N(\Omega) & \hookrightarrow & C(\overline{\Omega}) & \to & C(\overline{\Omega}) \\ & \varphi & \mapsto & \boldsymbol{h}_\varphi & \mapsto & \boldsymbol{h}_\varphi & \mapsto & |\boldsymbol{h}_\varphi| \end{array}$$

is continuous and compact. We put

$$\mathcal{A} = \{\varphi \in C(\overline{\Omega}); \varphi = \lambda\widetilde{S}(\varphi) \text{ for some } \lambda \in [0,1]\}.$$

Let $\varphi \in \mathcal{A}$, that is, $\varphi = \lambda|\boldsymbol{h}_\varphi|$. Then we have

$$\begin{aligned} \|\varphi\|^r_{C(\overline{\Omega})} &\le \||\boldsymbol{h}_\varphi|\|^r_{C(\overline{\Omega})} \le C\int_\Omega |\mathrm{curl}\,\boldsymbol{h}_\varphi|^r dx \\ &\le \int_\Omega F(|\varphi|)^r dx \le \int_\Omega (c_0 + c_1|\varphi|^\alpha)^r dx \le \widetilde{c_0} + \widetilde{c_1}\lambda^{\alpha r}\int_\Omega |\boldsymbol{h}_\varphi|^{r\alpha} dx. \end{aligned}$$

From the hypotheses of the Theorem, there exists $r > 3$ such that $r\alpha \le 3p/(3-p)$. Therefore, it follows from Sobolev embedding theorem: $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{r\alpha}(\Omega)$ that we have

$$\int_\Omega |\boldsymbol{h}_\varphi|^{r\alpha} dx \le C\|\boldsymbol{h}_\varphi\|^{r\alpha}_{\boldsymbol{W}^{1,p}(\Omega)} \le C_1\|\boldsymbol{h}_\varphi\|^{r\alpha}_{\mathbb{V}^p_N(\Omega)} \le C_3.$$

Thus we have

$$\|\varphi\|^r_{C(\overline{\Omega})} \le C_4.$$

Hence $\widetilde{S}$ has a fixed point in $C(\overline{\Omega})$. This completes the proof of Theorem 2.4.

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