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DECOMPOSITION FORMULAS FOR SOME HYPERGEOMETRIC FUNCTIONS IN THREE VARIABLES

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Abstract

Recently, many authors have established several decomposition formulas associated with the hypergeometric functions in two and more variablese. In this paper, we obtain some decomposition formulas for Gaussian triple hypergeometric functions by using certain inverse pairs of symbolic operators introduced by Choi and Hasanov in 2011. Certain transformation formulas for these triple functions have also been obtained.

1 Introduction

The use of many mathematical operations goes beyond the class of elementary functions. Calculation of integrals, summation of series, solution of algebraic, transcendental, difference and differential equations and their systems require expanding the class of functions studied. The development of the concept of a

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function, going in parallel with the development of the concepts of number and space, led to the emergence of new hypergeometric functions of many complex variables.

The great success of the theory of hypergeometric functions in a single variable has stimulated the development of the theory of hypergeometric functions in several variables by the fact that the solutions of partial differential equations arising in many applied problems of mathematical physics are given in terms of such hypergeometric functions (see e.g., [9, 11, 13, 18, 19]). Multiple Hypergeometric functions occur in numerous problems in hydrodynamics, control theory, electrical current, heat conduction, classical and quantum mechanics (see, for details, [2, 8, 15, 17] and the references cited therein). In view of theory and applications, a large number of hypergeometric functions have been developed, for example, as many as 205 hypergeometric functions are recorded in the monograph [21]. For our purpose, we begin by recalling the following Gaussian hypergeometric functions in three variables (see [21, pp. 80-81]):

$$F_{26b}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_n(b)_{2p-m}}{(c_1)_n(c_2)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.1}$$

$$F_{26c}\left(a_1, a_2, b_1, b_2; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_n(b_1)_{2p-m}(b_2)_{m+n-p}}{(c)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \tag{1.12}$$

$$F_{26d}\left(a_1, a_2, b_1, b_2; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_n(b_1)_{2p-m}(b_2)_{m-p}}{(c)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \tag{1.3}$$

$$F_{29b}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_n(a_2)_p(a_3)_p(b)_{2m+n-p}}{(c_1)_m(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.4}$$

$$F_{29d}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(b)_{p-n}}{(c_1)_m(c_2)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.5}$$

$$F_{29e}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(b)_{p-m}}{(c_1)_n(c_2)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.6}$$

$$F_{29g}\left(a_1, a_2, b_1, b_2; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_n(a_2)_p(b_1)_{2m+n-p}(b_2)_{p-m}}{(c)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.7}$$

$$F_{29k}\left(a_1, a_2, b_1, b_2; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_p(b_1)_{n-m}(b_2)_{p-n}}{(c)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.8}$$

where $(a)_n$ is the Pochhammer symbol defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0), \\ a(a+1)...(a+n-1) & (n \in N := \{1,2,...\}). \end{cases}$$

Burchnall and Chaundy presented the inverse pairs of symbolic operators ∇ and Δ ([3, 4]; also see [5]) by means of which they established several decomposition formulas for Appells double hypergeometric functions in terms of the Gaussian hypergeometric functions in one variable. Recently, Hasanov and Srivastava [10, 12] introduced multivariable analogues of the BurchnallChaundy's symbolic operators and with the help of these operators, the authors obtained a number of decomposition formulas associated with multiple Lauricella hypergeometric functions $F_A^{(r)}, F_B^{(r)}, F_C^{(r)}$ and $F_D^{(r)}$. Choi and Hasanov (see [6]) established the following multivariable symbolic operators:

$$H_{x_1,\ldots,x_r}(\alpha,\beta) = \frac{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(\beta)_{k_1+\cdots+k_r}k_1!\cdots k_r!}, \tag{1.9}$$

$$\bar{H}_{x_1,\ldots,x_r}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(1-\alpha-\delta_1-\cdots-\delta_r)_{k_1+\cdots+k_r}k_1!\cdots k_r!} \tag{1.10}$$

$$\left(\delta_j := x_j\frac{\partial}{\partial x_j}, j=1,...,r; r\in\mathbb{N}:=\{1,2,3,...\}\right).$$

Based on the operators (1.9) and (1.10), we aim in this paper to derive certain decomposition formulas for the Gaussian hypergeometric functions in three variable (1.1)-(1.8), which are used to obtain some transformation formulas for these functions.

2 A set of operator identities

Applying the symbolic operators in (1.9) and (1.10), we establish the following operator identities involving the Horn functions in two variables H_4, H_5, H_6, H_7 (see [7, 14]) and the hypergeometric functions (1.1)-(1.8):

$$F_{26b}(a_1,a_2,a_3,b;c_1,c_2;x,y,z)$$

$$= H_y(a_3,c_1)(1-y)^{-a_1}H_7\left(b,a_1,a_2;c_2;z,\frac{x}{1-y}\right), \tag{2.1}$$

$$(1-y)^{-a_1}H_7\left(b,a_1,a_2;c_2;z,\frac{x}{1-y}\right)$$

$$= \bar{H}_y(a_3,c_1)F_{26b}(a_1,a_2,a_3,b;c_1,c_2;x,y,z), \tag{2.2}$$

$$F_{26c}(a_1,a_2,b_1,b_2;c;x,y,z)$$

$$= H_y(a_2,c)(1-y)^{-b_2}H_6\left(b_1,b_2,a_1;(1-y)z,\frac{x}{1-y}\right), \tag{2.3}$$

$$(1-y)^{-b_2}H_6\left(b_1,b_2,a_1;(1-y)z,\frac{x}{1-y}\right)$$

$$= \bar{H}_y(a_2,c)F_{26c}(a_1,a_2,b_1,b_2;c;x,y,z), \tag{2.4}$$

$$F_{26d}\left(a_1,a_2,b_1,b_2;c;x,y,z\right) = H_y(a_2,c)\,(1-y)^{-a_1}\,H_6\left(b_1,b_2,a_1;z,\frac{x}{1-y}\right), \tag{2.5}$$

$$(1-y)^{-a_1}\,H_6\left(b_1,b_2,a_1;z,\frac{x}{1-y}\right) = \bar{H}_y(a_2,c)F_{26d}\left(a_1,a_2,b_1,b_2;c;x,y,z\right), \tag{2.6}$$

$$F_{29b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right) = H_y(a_1,c_2)\,(1-y)^{-b}\,H_7\left(b,a_2,a_3;c_1;\frac{x}{(1-y)^2},(1-y)z\right), \tag{2.7}$$

$$(1-y)^{-b}\,H_7\left(b,a_2,a_3;c_1;\frac{x}{(1-y)^2},(1-y)z\right) = \bar{H}_y(a_1,c_2)F_{29b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right), \tag{2.8}$$

$$F_{29d}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right) = H_z(a_3,c_2)\,(1-z)^{-b}\,H_4\left(a_1,a_2;c_1,1-b;x,-y(1-z)\right), \tag{2.9}$$

$$(1-z)^{-b}\,H_4\left(a_1,a_2;c_1,1-b;x,-y(1-z)\right) = \bar{H}_z(a_3,c_2)F_{29d}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right), \tag{2.10}$$

$$F_{29e}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right) = H_z(a_3,c_2)\,(1-z)^{-b}\,H_4\left(a_1,a_2;1-b,c_1;-x(1-z),y\right), \tag{2.11}$$

$$(1-z)^{-b}\,H_4\left(a_1,a_2;1-b,c_1;-x(1-z),y\right) = \bar{H}_z(a_3,c_2)F_{29e}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right), \tag{2.12}$$

$$F_{29g}\left(a_1,a_2,b_1,b_2;c;x,y,z\right) = H_y(a_1,c)\,(1-y)^{-b_1}\,H_6\left(b_1,b_2,a_2;\frac{x}{(1-y)^2},(1-y)z\right), \tag{2.13}$$

$$(1-y)^{-b_1}\,H_6\left(b_1,b_2,a_2;\frac{x}{(1-y)^2},(1-y)z\right) = \bar{H}_y(a_1,c)F_{29g}\left(a_1,a_2,b_1,b_2;c;x,y,z\right), \tag{2.14}$$

$$F_{29k}\left(a_1,a_2,b_1,b_2;c;x,y,z\right) = H_z(a_2,c)\,(1-z)^{-b_2}\,H_5\left(a_1,b_1;1-b_2;x,-y(1-z)\right), \tag{2.15}$$

$$(1-z)^{-b_2}\,H_5\left(a_1,b_1;1-b_2;x,-y(1-z)\right) = \bar{H}_z(a_2,c)F_{29k}\left(a_1,a_2,b_1,b_2;c;x,y,z\right). \tag{2.16}$$

The operator identities (2.1)-(2.16) can be proved by using the Mellin and the inverse Mellin transformations (see [1, 16, 21]). The proofs of the operator identities are omitted here.

3 Decomposition formulas

In [20, p. 93], it is proved that, for every analytic function $f(\xi)$, the following formulas holds true:

$$(-\delta)_n\{f(\xi)\} = (-1)^n \xi^n \frac{d^n}{d\xi^n}\{f(\xi)\} \tag{3.1}$$

and

$$(\alpha+\delta)_n\{f(\xi)\}=\xi^{1-\alpha}\frac{d^n}{d\xi^n}\{\xi^{\alpha+n-1}f(\xi)\}, \tag{3.2}$$

where

$$\delta:=\xi\frac{d}{d\xi};\alpha\in\mathbb{C};n\in\mathbb{N}_0:=\{0,1,2,...\}.$$

In view of formulas (3.1) and (3.2), and taking into account the differentiation formula for hypergeometric functions, from operator identities (2.1) to (2.16), we obtain the following expansion formulas:

$$F_{26b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=(1-y)^{-a_1}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c_1-a_3)_i}{(c_1)_i i!}\left(\frac{y}{1-y}\right)^i H_7\left(b,a_1+i,a_2;c_2;z,\frac{x}{1-y}\right), \tag{3.3}$$

$$(1-y)^{-a_1}H_7\left(b,a_1,a_2;c_2;z,\frac{x}{1-y}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(a_1)_i(c_1-a_3)_i}{(c_1)_i i!}y^iF_{26b}\left(a_1+i,a_2,a_3,b;c_1+i,c_2;x,y,z\right), \tag{3.4}$$

$$F_{26c}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)=(1-y)^{-b_2}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_2)_i(b_2)_i}{(c)_i i!}\left(\frac{y}{1-y}\right)^i H_6\left(b_1,b_2+i,a_1;(1-y)z,\frac{x}{1-y}\right), \tag{3.5}$$

$$(1-y)^{-b_2}H_6\left(b_1,b_2,a_1;(1-y)z,\frac{x}{1-y}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c-a_2)_i(b_2)_i}{(c)_i i!}y^iF_{26c}\left(a_1,a_2,b_1,b_2+i;c+i;x,y,z\right), \tag{3.6}$$

$$F_{26d}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)=(1-y)^{-a_1}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c-a_2)_i}{(c)_i i!}\left(\frac{y}{1-y}\right)^i H_6\left(b_1,b_2,a_1+i;z,\frac{x}{1-y}\right), \tag{3.7}$$

$$(1-y)^{-a_1}H_6\left(b_1,b_2,a_1;z,\frac{x}{1-y}\right)$$
$$=\sum_{i=0}^{\infty}\frac{(a_1)_i(c-a_2)_i}{(c)_i i!}y^iF_{26d}\left(a_1+i,a_2,b_1,b_2;c+i;x,y,z\right), \tag{3.8}$$

$$F_{29b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=(1-y)^{-b}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(c_2-a_1)_i(b)_i}{(c_2)_i i!}\left(\frac{y}{1-y}\right)^i H_7\left(b+i,a_2,a_3;c_1;\frac{x}{(1-y)^2},(1-y)z\right), \tag{3.9}$$

$$(1-y)^{-b}H_7\left(b,a_2,a_3;c_1;\frac{x}{(1-y)^2},(1-y)z\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c_2-a_1)_i(b)_i}{(c_2)_i i!}y^iF_{29b}\left(a_1,a_2,a_3,b+i;c_1,c_2+i;x,y,z\right), \tag{3.10}$$

$$F_{29d}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=(1-z)^{-b}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(c_2-a_3)_i(b)_i}{(c_2)_i i!}\left(\frac{z}{1-z}\right)^i H_4\left(a_1,a_2;c_1,1-b-i;x,-y(1-z)\right), \tag{3.11}$$

$$(1-z)^{-b}\,H_4\left(a_1,a_2;c_1,1-b;x,-y(1-z)\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c_2-a_3)_i(b)_i}{(c_2)_i i!}z^i F_{29d}\left(a_1,a_2,a_3,b+i;c_1,c_2+i;x,y,z\right), \tag{3.12}$$

$$F_{29e}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=(1-z)^{-b}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(c_2-a_3)_i(b)_i}{(c_2)_i i!}\left(\frac{z}{1-z}\right)^i H_4\left(a_1,a_2;1-b-i,c_1;-x(1-z),y\right), \tag{3.13}$$

$$(1-z)^{-b}\,H_4\left(a_1,a_2;1-b,c_1;-x(1-z),y\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c_2-a_3)_i(b)_i}{(c_2)_i i!}z^i F_{29e}\left(a_1,a_2,a_3,b+i;c_1,c_2+i;x,y,z\right), \tag{3.14}$$

$$F_{29g}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)=(1-y)^{-b_1}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_1)_i(b_1)_i}{(c)_i i!}\left(\frac{y}{1-y}\right)^i H_6\left(b_1+i,b_2,a_2;\frac{x}{(1-y)^2},(1-y)z\right), \tag{3.15}$$

$$(1-y)^{-b_1}\,H_6\left(b_1,b_2,a_2;\frac{x}{(1-y)^2},(1-y)z\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c-a_1)_i(b_1)_i}{(c)_i i!}y^i F_{29g}\left(a_1,a_2,b_1+i,b_2;c+i;x,y,z\right), \tag{3.16}$$

$$F_{29k}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)=(1-z)^{-b_2}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_2)_i(b_2)_i}{(c)_i i!}\left(\frac{z}{1-z}\right)^i H_5\left(a_1,b_1;1-b_2-i;x,-y(1-z)\right), \tag{3.17}$$

$$(1-z)^{-b_2}\,H_5\left(a_1,b_1;1-b_2;x,-y(1-z)\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c-a_2)_i(b_2)_i}{(c)_i i! z^i}F_{29k}\left(a_1,a_2,b_1,b_2+i;c+i;x,y,z\right). \tag{3.18}$$

4 Transformation formulas

The following transformation formulas of the Gaussian hypergeometric functions in three variables (1.1)-(1.8) follow from our main results in pervious Section.

$$F_{26b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)$$
$$=(1-y)^{-a_1}\,F_{26b}\left(a_1,a_2,c_1-a_3,b;c_1,c_2;\frac{x}{1-y},\frac{y}{y-1},z\right), \tag{4.1}$$

$$F_{26c}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)$$
$$=(1-y)^{-b_2}\,F_{26c}\left(a_1,c-a_2,b_1,b_2;c;\frac{x}{1-y},\frac{y}{y-1},(1-y)z\right), \tag{4.2}$$

$$F_{26d}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)$$
$$=(1-y)^{-a_1}\,F_{26d}\left(a_1,c-a_2,b_1,b_2;c;\frac{x}{1-y},\frac{y}{y-1},z\right), \tag{4.3}$$

$$F_{29b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)$$
$$=(1-y)^{-b}F_{29b}\left(c_2-a_1,a_2,a_3,b;c_1,c_2;\frac{x}{(1-y)^2},\frac{y}{y-1},(1-y)z\right), \quad (4.4)$$
$$F_{29d}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)$$
$$=(1-z)^{-b}F_{29d}\left(a_1,a_2,c_2-a_3,b;c_1,c_2;x,y(1-z),\frac{z}{z-1}\right), \quad (4.5)$$
$$F_{29e}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)$$
$$=(1-z)^{-b}F_{29e}\left(a_1,a_2,c_2-a_3,b;c_1,c_2;x(1-z),y,\frac{z}{z-1}\right), \quad (4.6)$$
$$F_{29g}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)$$
$$=(1-y)^{-b}F_{29g}\left(c-a_1,a_2,b_1,b_2;c;\frac{x}{(1-y)^2},\frac{y}{y-1},(1-y)z\right), \quad (4.7)$$
$$F_{29k}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)$$
$$=(1-z)^{-b_2}F_{29k}\left(a_1,c-a_2,b_1,b_2;c;x,y(1-z),\frac{z}{z-1}\right). \quad (4.8)$$

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