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REGULARITY OF RINGS WITH INVOLUTION

Dedicated to Professor Richard Wiegandt on his 88th birthday.

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Abstract

The compact methodology of studying $*$-rings, is to study them in its own independent category. In this paper we continue the study of $*$-rings and get the involutive definition of regularity of elements which is compatible with the definition of *-regular *-rigs given by Kaplansky and Berberian. We introduce also both strongly $*$-regular $*$-rings and the concept of $*$-regular pairs which works as a weak definition of invertibility of element.

1 Introduction

Throughout this paper, all rings are associative with identity. A $*$-ring R is a ring with involution $*$. $*$-rings are objects of the category of rings with involution with morphisms also preserving involution. Therefore the consistent way of investigating $*$-rings is to study them within this category, as done in a series of papers (for instance [1, 2].

A self-adjoint idempotent element e (i.e., $e^* = e = e^2$) is called a *projection*. A $*$-ring R is said to be *Abelian* (resp. $*$-*Abelian*) if every idempotent (resp. projection) of R is central. An involution $*$ is called *proper* if $aa^* \neq 0$ for every nonzero element $a \in R$. A nonempty subset S of a $*$-ring R is said to be *self-adjoint* or $*$-*subset* if it is closed under involution (i.e., $S^* = S$).

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In a $*$-ring R, an element a is called *$*$-nilpotent* if $a^n = (aa^*)^m = 0$ for some positive integers n and m (see [3]). A $*$-ring without nonzero nilpotent (resp. $*$-nilpotent) elements is called *reduced* (resp. *$*$-reduced*).

In 1936, von Neumann introduced his sense of regularity for the elements of a ring during his study of von Neumann algebras and continuous geometry. Von Neumann's study of the projection lattices of certain operator algebras led him to introduce continuous geometries and regular rings.

According to [11], an element a of a ring R is said to be *von Neumann regular* (or simply *regular*) if there exists an element x, which is not necessary depends on a such that $a = axa$ and the ring R is *regular* if all its elements are regular.

In [9], Kaplansky introduced the involutive version of regularity of rings to use it as a basic tool in his work about the projection lattice of $AW*$-algebra. He showed that these lattice and others are continuous geometries. He called a $*$-ring R *$*$-regular* if it is regular and $*$ is proper.

Latter, Berberian [6] proved that a $*$-ring R is $*$-regular if and only if for each $a \in R$ there exists a projection e such that $Ra = Re$ if and only if R is regular and is a Rickart $*$-ring (A *Rickert $*$-ring* is the $*$-ring in which the right annihilator of every element is generated by a projection as a right ideal).

Following [4], A ring R is called *strongly regular* if for every element a of R there exists at least one element x in R such that $a = a^2x$. One can see that every strongly regular is regular (see [8]).

Azumaya in [5], gave a compact definition of strong regularity for elements. An element a of a ring R is called *right* (resp. *left*) *regular* if $a \in a^2R$ (resp. $a \in Ra^2$)). Moreover, a is *strongly regular* if it is both right and left regular.

2 $*$-Regular elements

In this section we introduce the definition of $*-$-regular elements which is compatible with the definition of $*$-regular $*$-rings given by Berberian [6]; that is a $*$-ring is $*$-regular if and only if all its elements are $*$-regular and every $*$-regular element is regular.

Definition 2.1 ([6]). A *$*$-regular* $*$-ring is a regular $*$-ring with proper involution.

Proposition 2.1 ([6], Proposition 3). *For a $*$-ring R, the following conditions are equivalent:*

(a) R is $*$-regular.

(b) for each $a \in R$, there exists a projection e such that $Ra = Re$.

(c) R is regular and is a Rickart $*$-ring.

Now, we introduce the definition of $*$-regular elements which has to be compatible with the above definition.

Definition 2.2. An element a of a $*$-ring R is said to be $*$-*regular* if and only if $a \in aa^*R \cap Ra^*a$.

All projections and invertible elements of a $*$-ring are $*$-regular.

Every $*$-regular element a of a $*$-ring R is clearly regular. Indeed, $a \in aa^*R \cap Ra^*a$ implies $a = aa^*x$ for some $x \in R$. Hence $(x^*a)(x^*a)^* = x^*aa^*x = x^*a$ and so x^*a is a projection. Thus $x^*a = a^*x$ and $a = ax^*a \in aRa$. However, the converse is not necessary true as shown by the following example.

Example 2.3. Let e be a nonzero idempotent of an Abelian ring A. In the $*$-ring $R = A \bigoplus A$, with the exchange involution $*$ defined as $(a, b)^* = (b, a)$, the element $(e, 0)$ is regular but not $*$-regular.

Proposition 2.2. *The only $*$-regular $*$-nilpotent element of a $*$-ring R is* 0.

Proof. Let a be a $*$-regular element of a $*$-ring R. Hence, $a \in aa^*R$ and $a \in Ra^*a$. So that, $a \in aa^*R \subseteq a(Ra^*a)^*R = aa^*aR \subseteq aa^*(aa^*R)R = (aa^*)^2R$. Continuing this procedure, we get $a \in (aa^*)^nR$ for every positive integer n. If a is $*$-nilpotent, then $a = 0$. □

Proposition 2.3. *A nonzero element a of $*$-ring R is $*$-regular if and only if there are two projections e, f and an element b of R such that $ab = e$, $ba = f$ and $a = ea = af$.*

Proof. First, let a be a $*$-regular element of the $*$-ring R. Then $a = aa^*x = ya^*a$ for some x and y in R. Choose $e = ya^*$ to get $ee^* = ya^*ay^* = ay^* = e^*$ which means that e is a projection and $ea = a$. Similarly, the choice $f = a^*x$ makes f a projection and $af = a$. Moreover $e = ya^* = ay^* = (aa^*x)y^* = a(a^*xy^*)$ and $f = x^*a = x^*ea = x^*ay^*a = (a^*xy^*)a$. Choose $b = a^*xy^*$ to get the result.

Conversely, let the condition be satisfied. Hence, $a = ea = e^*a = (ab)^*a \in Ra^*a$. Similarly, $a = af = af^* = a(ba)^* \in aa^*R$. Hence $a \in aa^*R \cap Ra^*R$ and a is $*$-regular. □

Proposition 2.4. *Let R be a $*$-ring. Then the following conditions are equivalent:*

(i) R *is $*$-regular.*

(ii) $a \in Ra^*a$, *for every* $a \in R$.

(iii) $a \in aa^*R$, *for every* $a \in R$.

(iv) $a \in aa^*R \cap Ra^*a$, *for every* $a \in R$.

Proof. (i)⇒(ii): Let R be $*$-regular, then for every $a \in R$, $aR = eR$ for some projection e of R. Hence $a = ea$ and $e = ar$ for some $r \in R$. Thus $a = e^*a = r^*a^*a \in Ra^*a$.

(ii)⇒(iii): Direct by applying the condition on the element $a^* \in R$.

(iii)⇒(iv): Obvious.

(iv)⇒(i): From $a \in aa^*R \cap Ra^*a$, we get $a = ra^*a$, for some $r \in R$. Hence $(ra^*)(ra^*)^* = (ra^*a)r^* = ar^* = (ra^*)^*$ and $e = ra^*$ is a projection. Finally, $e = ar^* \in aR$ and $a = ea \in eR$ imply $aR = eR$ and R is $*$-regular. □

Corollary 2.4. *A $*$-ring R is $*$-regular if and only if all its elements are $*$-regular.*

Corollary 2.5. *Every $*$-regular $*$-ring is $*$-reduced.*

Corollary 2.6. *Every ideal I of a $*$-regular $*$-ring R satisfies*

$$I^*I = II^* = I \cap I^* = I = I^2$$

A $*$-regular element a of a $*$-subring S of a $*$-ring R is clearly $*$-regular in R. The converse is not necessary true by the next example.

Example 2.7. Consider the $*$-ring of complex numbers $\mathbb{C}$ with conjugate as involution and S be the set of Gaussian integers. Clearly, $a = 1 + i \in S$ is $*$-regular in $\mathbb{C}$ since $a = (1 + i)(1 - i)(\frac{1}{2} + \frac{1}{2}i) \in aa^*R \cap Ra^*a$ and a can not be $*$-regular in S.

3 $*$-Regular Pairs

Definition 3.1. A pair (a, b) of elements of a $*$-ring R satisfying $ab = e$ and $ba = f$ for some projections e and f of R such that $a = ea = af$ and $b = be = fb$, is called a *$*$-regular pair* and b is called the *$*$-regular conjugate* of a and vice versa.

In the $*$-ring R, $(0, 0)$ and $(1, 1)$ are the improper $*$-regular pairs.

Proposition 3.1. *If (a, b) is a $*$-regular pair, then both a and b are $*$-regular.*

Proof. $a = ea$ and $e = ab$ imply $a = ea = e^*a = b^*a^*a \in Ra^*a$. Also $a \in aa^*R$ and a is $*$-regular. Similarly, its conjugate b is also $*$-regular. □

The converse of the previous proposition is not true as clear from the next example which shows also that a $*$-ring which is not $*$-regular may contains $*$-regular elements.

Example 3.2. The $*$-ring $R = \mathbb{M}_2(\mathbb{R})$ of all 2×2 real matrices with transpose involution is not $*$-regular since $\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ satisfies $\alpha \notin R\alpha^*\alpha$. Moreover, the elements $a = \begin{pmatrix} 4 & 0 \\ 3 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} \frac{4}{25} & \frac{3}{25} \\ 0 & 0 \end{pmatrix}$ form a $*$-regular pair with the corresponding projections $e = \begin{pmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Furthermore, the element $\beta = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ is $*$-regular and can not form a $*$-regular pair with any element of R.

The next result claims the uniqueness of the $*$-regular conjugate.

Proposition 3.2. *The $*$-regular conjugate is unique.*

Proof. Assume that b and c are tow $*$-regular conjugates of $a \in R$. So that

$$ab = e, ba = f, a = ea = af, b = be = fb$$

and

$$ac = e', ca = f', a = e'a = af', c = ce' = f'c,$$

for some projections e, f, e' and f' of R. So that $ab = e'ab = (ac)(ab) = (ac)^*(ab)^* = c^*a^*b^*a^* = c^*(aba)^* = c^*(ea)^* = c^*a^* = (ac)^* = (e')^* = e' = ac$. Similarly, $ba = ca$. Now,

$$b = fb = bab = bac = cac = f'c = c$$

□

Now, we give a compact definition for $*$-regular pairs depends only on the conjugate elements.

Proposition 3.3. *A pair (a, b) of a $*$-ring R is $*$-regular if and only if $a = (ab)^*a = b^*a^*a$ and $b = (ba)^*b = a^*b^*b$.*

Proof. Let (a, b) be a $*$-regular pair. Then $a = ea = af$, $b = be = fb$, $ab = e$ and $ba = f$ for some projections e and f of R. Hence $a = ea = e^*a = b^*a^*a$ and $b = fb = a^*b^*b$. Conversely, assume that $a = b^*a^*a$ and $b = a^*b^*b$. Let $e = ab$, then $e^*e = b^*a^*ab = ab = e$ implies e is a projection. Similarly, $f = ba$ is also a projection. Obviously, $ea = e^*a = b^*a^*a = a$. Similarly, $a = af$ and $b = be = fb$. Hence (a, b) is a $*$-regular pair. □

Note that the previous proposition is still valid if we interchange the first element by the third one; that is $a = aa^*b^*$ and $b = bb^*a^*$.

The following corollary shows that each invertible element is the $*$-regular conjugate of its inverse.

Corollary 3.3. *If a is an invertible element in a $*$-ring R, then (a, a^{-1}) is a $*$-regular pair.*

Proposition 3.4. *The following statements hold for a $*$-regular pair (a, b) of a $*$-ring R.*

1. $(-a, -b)$ *is a pair* $*$*-regular.*
2. (b, a) *is a* $*$*-regular pair.*
3. (a^*, b^*) *is a* $*$*-regular pair.*

Proof. The proof is direct. □

Proposition 3.5. *The* $*$*-regular conjugate of a projection is also a projection.*

Proof. Assume that (e, b) is a $*$-regular pair and e is a projection. Hence, $e = b^*e^*e = b^*e$ and $b^*b = b^*(e^*b^*b) = eb^*b = e^*b^*b = b$ and b is a projection. □

The next corollary shows that $*$-regular pair , as a relation, is reflexive only for projections.

Corollary 3.4. *In a* $*$*-ring* R*,* (a, a) *is* $*$*-regular pair if and only if* a *is a projection, for every* $a \in R$*.*

Corollary 3.5. *Let* e *and* f *be projections of a* $*$*-ring* R*, then* $e = f$ *if and only if* (e, f) *is a* $*$*-regular pair.*

Proposition 3.6. *Let* R *be a* $*$*-Abelian* $*$*-ring and* (a, b)*,* (c, d) *be two* $*$*-regular pairs. Then* (ac, db) *is a* $*$*-regular pair.*

Proof. (a, b) and (c, d) are $*$-regular pairs imply $a = e_1a = af_1$, $b = be_1 = f_1b$, $ab = e_1$, $ba = f_1$,$c = e_2c = cf_2$, $d = de_2 = f_2d$, $cd = e_2$ and $dc = f_2$ for some projections e_1,e_2, f_1 and f_2 of R. Now $(ac)(db) = a(cd)b = ae_2b = (ab)e_2 = e_1e_2$ and similarly, $(db)(ac) = f_1f_2$. Since e_1e_2 and f_1f_2 are projections and $(e_1e_2)(ac) = (ac)(f_1f_2) = ac$ and $(f_1f_2)(db) = (db)(e_1e_2) = db$, then (ac, db) is a $*$-regular pair. □

Now, if we define the mapping $\dagger : \mathcal{P}(R) \to \mathcal{P}(R)$ which takes each element in $\mathcal{P}(R)$ to its $*$-regular con1jugate, where $\mathcal{P}(R)$ is the set of all $*$-regular conjugate elements, then we have the following:

1. $\dagger$ is bijective of order 2; that is $(a^\dagger)^\dagger = a$.
2. $\dagger$ is an odd mappping; that is $(-a)^\dagger = -a^\dagger$.
3. $\dagger$ commutes with $*$. that is $(a^\dagger)^* = (a^*)^\dagger$.

We call this mapping a *$*$-regular conjugate mapping*, briefly $* - RC$. The following are additional properties for $\dagger$.

Proposition 3.7. *Let* R *be a* $*$*-ring, then*

(i) $aa^\dagger$ *and* $a^\dagger a$ *are projections.*

(ii) $aa^\dagger a = a$

(iii) $a^\dagger a a^\dagger = a^\dagger$

Proof. (i) From Proposition 3.3, we have $a = (a^\dagger)^* a^* a$ and $a^\dagger = a^*(a^\dagger)^* a^\dagger$. Hence $(aa^\dagger)^*(aa^\dagger) = (a^\dagger)^* a^* a a^\dagger = aa^\dagger$ and so $aa^\dagger$ is a projection. The second part is proved similarly.

(ii) $a = (a^\dagger)^* a^* a = (aa^\dagger)^* a = aa^\dagger a$.

(iii) As in (ii). □

Example 3.6. Let R be ring of complex numbers with conjugate involution. Define † as $a^\dagger = \begin{cases} 0, & \text{if } a = 0 \\ \frac{1}{x} & \text{if } a \neq 0 \end{cases}$. Clearly † is a ∗-RC mapping .

Proposition 3.8. *If R is a ∗-Abelian ∗-ring, then $\mathcal{P}(R)$ is a †-semigroup with zero.*

Proof. From the property $(a^\dagger)^\dagger = a$ and Proposition 3.6, we see that † is an involution and $\mathcal{P}(R)$ is a semigroup with 0. □

4 Strongly ∗-regular ∗-ring

According to [5], an element a of a ring R is said to be *right* (resp. *left*) *regular* if $a \in a^2R$ (resp. $(a \in Ra^2)$ and is called *strongly regular* if it is both right and left regular. R is called *strongly regular* if every element is strongly regular. For ∗-rings, the condition of strongly regularity will be only $a \in a^2R$ (or $a \in a^2R$). Here, we give the involutive version of strongly regularity; that is strongly ∗-regularity.

Definition 4.1. An element a of a ∗-ring R is said to be *strongly ∗-regular* if and only if $a \in a^*Ra \cap aRa^*$ and R is *strongly ∗-regular* if every element of R is strongly ∗-regular.

The zero and all invertible elements of ∗-rings are strongly ∗-regular. The condition $a \in a^*Ra \cap aRa^*$ in the previous definition can not be reduced to $a \in aRa^*$ or $a \in a^*Ra$ as clear from the following example.

Example 4.2. Consider the ∗-ring $\mathbb{M}_n(F)$ of all $n \times n$ matrices over a field F with the transpose involution. The element $a = e_{11} + e_{12}$ is not strongly ∗-regular because $a \notin aRa^* \cap a^*Ra$, while the element $b = e_{11} + 2e_{12} + 3e_{21}$ is non-invertible but strongly ∗-regular, where e_{ij} it the matrix with zero entries everywhere and 1 in the ij-position. Moreover the element $c = e_{11} + e_{21} + \cdots + e_{n1}$ satisfies $c \in cRc^*$ but $c \notin c^*Rc$.

However, the condition of strongly ∗-regularity for elements is reduced for strongly ∗-regular ∗-rings as obvious from the next result.

Proposition 4.1. *For a ∗-ring R, the following conditions are equivalent:*

(i) R is strongly $*$-regular.

(ii) $a \in aRa^*$ for every $a \in R$.

(iii) $a \in a^*Ra$ for every $a \in R$.

Proof. (i)$\Rightarrow$(ii) is direct.

(ii)$\Rightarrow$(iii): By assumption, $a^* \in a^*R(a^*)^* = a^*Ra$ and consequently $a \in a^*Ra$.

(iii)$\Rightarrow$(i): As in (ii)$\Rightarrow$(iii). □

Lemma 4.3. *Every idempotent in a strongly $*$-regular $*$-ring is projection.*

Proof. Let e be an idempotent of a strongly $*$-regular $*$-ring R. Hence $e = exe^*$ for some $x \in R$ implies $ee^* = (exe^*)e^* = exe^* = e$ and e is a projection. □

Proposition 4.2. *Every strongly $*$-regular $*$-ring R is reduced.*

Proof. If R is strongly $*$-regular, then for every $0 \neq a \in R$, $a = a^*xa = aya^*$ for some $x, y \in R$. Now, $a = a^*xa = (aya^*)^*xa = ay^*(a^*xa) = ay^*a$. Set $e = ay^*$, hence $e^2 = (ay^*)^2 = (ay^*a)y^* = ay^* = e$ and so that e is an idempotent and consequently a projection by the previous lemma. Hence $ay^* = ya^*$ implies $a = a^2y^*$, so a can not be nilpotent and R is reduced. □

Since every reduced ring is Abelian, we have the following corollary.

Corollary 4.4. *Every strongly $*$-regular $*$-ring is Abelian.*

Proposition 4.3. *Every one-sided principal ideal of a strongly $*$-regular $*$-ring is self-adjoint and so is a $*$-ideal.*

Proof. Let aR be a right principal ideal of R generated by a, hence $a = axa^*$ for some $x \in R$. As in the proof of Proposition 4.2 and Corollary 4.4, ax^* is central projection, hence $(aR)^* = Ra^* = Rax^*a^* = ax^*Ra^* \subseteq aR$. Thus aR is self-adjoint and consequently two-sided ideal. □

Next, we give a compact equivalent definition for strongly $*$-regular $*$-rings.

Proposition 4.4. *A $*$-ring R is strongly $*$-regular if and only if for every $a \in R$ there is a central projection e of R such that $aR = eR$.*

Proof. For sufficiency, let $aR = eR$ for some central projection e of R. Hence, $a = ea$ and $e = ax$ for some x in R, so that $e = x^*a^*$ implies $a = ea = ae = ax^*a^* \in aRa^*$. Similarly, $a \in a^*Ra$. Thus $a \in a^*Ra \cap aRa^*$ and a is strongly $*$-regular. Conversely, if R is strongly $*$-regular, then for every $a \in R$, $a = a^*xa = aya^*$ for some $x, y \in R$. Now, $a = a^*xa = (aya^*)^*xa = ay^*(a^*xa) = ay^*a$. Setting $e = ya^*$, we have $e^2 = ya^*ya^* = y(aya^*)^* = ya^* = e$ which implies that e is an idempotent and hence is central projection by Lemma 4.3 and Corollary 4.4. □

The next two propositions shows that every strongly $*$-regular $*$-ring is both $*$-regular and strongly regular.

Proposition 4.5. *Every strongly $*$-regular $*$-ring is $*$-regular.*

Proof. Let R be a strongly $*$-regular. For each a in R, $a \in a^*Ra \cap aRa^*$ implies $a = a^*xa = aya^*$ for some $x, y \in R$. Hence, $a = a^*xa = (aya^*)^*xa = ay^*(a^*xa) = ay^*a \in aRa$ and so R is regular. According to [10, Theorem 4.5], it is enough to show that $*$ is proper to prove that R is $*$-regular. Now, $a = ay^*a$ implies $(y^*a)^2 = y^*ay^*a = y^*a$ and so y^*a is an idempotent and consequently a projection, by Lemma 4.3. Hence $y^*a = (y^*a)^* = a^*y$ and so $a = aa^*y$ implies $*$ is proper. □

The converse of the previous proposition is not necessary true as clear from the next example.

Example 4.5. In the ring $R = \mathbb{M}_n(\mathbb{R})$ of all $n \times n$ real matrices, if r is the rank of $a \in R$, then there exist invertible matrices x and y such that $xay = \alpha$, where $\alpha = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Hence $a = x^{-1}\alpha y^{-1} = x^{-1}\alpha^2 y^{-1} = x^{-1}\alpha(y^{-1}yxx^{-1})\alpha y^{-1} = (x^{-1}\alpha y^{-1})yx(x^{-1}\alpha y^{-1}) = ayxa \in aRa$ and R is regular . If the involution $*$ on R is the transpose, then it is proper and R is $*$-regular from Definition 1. On the other hand R is not strongly $*$-regular, by Corollary 4.4, since the projections $e_{ii} \in R$, $i = 1, \cdots n$, are all non-central.

Proposition 4.6. *Every strongly $*$-regular $*$-ring is strongly regular.*

Proof. Let R be a strongly $*$-regular $*$-ring and $a \in R$, hence $a = axa^* = a^*ya$ for some $x, y \in R$. As in the proof of Proposition 4.5, $a = aa^*y$ and since $a^* = ax^*a^*$, we get $a = a^2x^*a^*y$ which gives $a \in a^2R$ and so R is strongly regular. □

However, there is a strongly regular $*$-ring which is not strongly $*$-regular.

Example 4.6. The $*$-ring $R = S \oplus S$, where S is a strongly regular ring, with the exchange involution is strongly regular but not strongly $*$-regular.

Next, we give sufficient conditions for strongly regular $*$-rings and $*$-regular $*$-rings to be strongly $*$-regular.

Proposition 4.7. *For a $*$-ring R, the following conditions are equivalent:*

(i) R is strongly $*$-regular.

(ii) R is $*$-regular and reduced.

(iii) R is $*$-regular and Abelian.

(iv) R is $*$-regular and $*$-Abelian.

Proof. (i)⇒(ii) from Propositions 4.5 and 4.2.

(ii)⇒(iii)⇒(iv) are clear.

(iv)⇒(i): For every $a \in R$ we have $a = aa^*x = ya^*a$. But a^*x and ya^* are projections and hence central, from the assumption. Hence $a = a^*xa = aya^*$ and R is strongly $*$-regular. □

Proposition 4.8. *A $*$-ring R is strongly $*$-regular if and only if R is strongly regular and $*$ is proper.*

Proof. For necessity, R is strongly regular from Proposition 4.6. For any $0 \neq a \in R$, $a \in a^*Ra$ and $(a^*Ra)^2 = a^*Raa^*Ra \neq 0$, from the reduceness of R (Proposition 4.2) and so $*$ is proper.

Conversely, let R be strongly regular and $*$ be proper. According to [7][Theorems 3.2 and 3.5], every strongly regular ring is reduced and in particular Abelian. Now, to show that every idempotent is projection, let e be an idempotent of R, hence $(e - ee^*)(e - ee^*)^* = 0$ which implies $e = ee^*$. Next, let $a \in R$ which implies $a = a^2x = ya^2$ for some $x, y \in R$. Obviously ax is an idempotent, since $(ax)^2 = axax = ya^2xax = ya^2x = ax$, and therefore is a central projection. Hence $a = a^2x = a(ax)^* = ax^*a^* \in aRa^*$ and similarly $a \in a^*Ra$. Thus R is strongly $*$-regular. □

Proposition 4.9. *If R is $*$-central reduced and strongly $*$-regular, then R is a division $*$-ring.*

Proof. For every $0 \neq a \in R$, we have $a = aya^*$ and as in a previous proof, ay^* is a central projection. Since R is $*$-central reduced, either $ay^* = 0$ which implies $a = aya^* = a(ay^*)^* = 0$, contradicts our assumption, or $ay^* = 1$ and a is invertible. □

Proposition 4.10. *A strongly $*$-regular $*$-ring R is $*$-central reduced if and only if its center is $*$-field.*

Proof. First, if R is $*$-central reduced and strongly $*$-regular, then R is a division ring by Proposition 4.9 and consequently its center is a $*$-field. Conversely, let e be a central projection, hence $e(1 - e) = 0$. If $e = 0$, then it is done. If not, $e^{-1} \in R$ and then $1 - e = 0$ implies $e = 1$. Thus R is central $*$-reduced. □

5 Extending Strong Regularity

Lemma 5.1. *Every $*$-homomorphic image of strongly $*$-regular $*$-ring is strongly $*$-regular.*

Proof. The proof is routine. □

Proposition 5.1. *Let I be a $*$-ideal of a $*$-ring R. Then R is strongly $*$-regular if and only if I and R/I are strongly $*$-regular.*

Proof. First, let R be strongly $*$-regular and $a \in I$, hence $a = axa^* = a^*ya$ for some $x, y \in R$. As done in previous proofs, ax is a central projection and $a = ax^*a$. The element $z = xax^*$ is in I which satisfies $aza^* = axax^*a^* = ax^*axa^* = ax^*a = a$ shows that I is strongly $*$-regular. By Lemma 5.1,

R/I is strongly $*$-regular. Conversely, let I and R/I be strongly $*$-regular, then for $a \in R$, we have $a + I = (a + I)(x + I)(a + I)^* = axa^* + I$ for some $x \in R$. Hence $a - axa^* \in I$ and since I is strongly $*$-regular, we get $a - axa^* = (a - axa^*)z(a - axa^*)^* = aza^* - axa^*za - azax^*a^* + axa^*zax^*a^* = aza^* - axa^*za - azax^*a^* + axa^*zax^*a^*$, for some $z \in I$. Again $a^*za = a^*z^*a^*$ and $a - axa^* = aza^* - axa^*z^*a^* - azax^*a^* + axa^*zax^*a^*$. Hence $a = a(x + z - xa^*z^* - zax^* + xa^*zax^*)a^* \in aRa^*$ which implies that R is strongly $*$-regular. □

Proposition 5.2. *A finite direct sum of strongly $*$-regular $*$-rings is strongly $*$-regular (under componentwise involution).*

Proof. It is enough to prove the result in case of two strongly $*$-regular $*$-rings A and B, that is $R = A \oplus B$. Since A and $R/A = (A + B)/A \cong B/(A \cap B)$ are strongly $*$-regular, by Lemma 5.1, then R is strongly $*$-regular, by Proposition 5.1. □

Proposition 5.3. *Every $*$-corner of strongly $*$-regular $*$-ring is strongly $*$-regular.*

Proof. Let R be a strongly $*-$regular $*$-ring and $e \in R$ is a projection. For every $a \in eRe$, there exists $y \in R$ such that $a = aya^* = (ae)y(ae)^* = a(eye)a^* \in a(eRe)a^*$ and eRe is strongly $*$-regular. □

We note that the converse of the previous proposition is not true even if all the non-trivial $*$-corners of R are strongly $*$-regular as clear from the next example.

Example 5.2. Let $R = \mathbb{M}_2(\mathbb{R})$ be the ring of all 2×2 matrices over the real numbers $\mathbb{R}$ with transpose involution. The nontrivial projections of R are $e_{11} =$ and e_{22} which satisfy $e_{11}Re_{11} = e_{22}Re_{22} = \mathbb{R}$ and these are strongly $*$-regular while R is not.

Proposition 5.4. *The center of a strongly $*$-regular $*$-ring is strongly $*$-regular.*

Proof. Let R be a strongly $*$-regular with center $\mathcal{Z}$ and let $a \in \mathcal{Z}$. Using Proposition 4.4, it is enough to show that $a\mathcal{Z}$ is generated by a central projection of R. Since R is strongly $*$-regular, then $aR = eR$ for some central projection e of R. Since $a = ea$, we have $a\mathcal{Z} \subseteq e\mathcal{Z}$. Moreover, there is $r \in R$ such that $e = ar = aer$. For every $s \in R$, we have $(er)s = rse = rsar = arsr = esr = s(er)$, whence $er \in \mathcal{Z}$ and $eZ \subseteq aZ$. Therefore, $aZ = eZ$ and Z is strongly $*$-regular. □

The converse of the previous proposition is not true. The $*$-ring in Example 5.2 is not strongly $*$-regular while its center is strongly $*$-regular.

The next proposition gives a sufficient and necessary condition for a $*$-ring with strongly $*$-regular center to be strongly $*$-regular, too.

Proposition 5.5. *A $*$-ring R is strongly $*$-regular if and only if its center $\mathcal{Z}(R)$ is strongly $*$-regular and R is strongly regular.*

Proof. Obviously, if R is strongly $*$-regular, then the center is strongly $*$-regular by Proposition 5.4 and R is strongly regular from Proposition 4.6. Conversely, let $a \in R$, then $aR = eR$ for some central idempotent e of R, because R is strongly regular. Since $e \in Z(R)$, which is strongly $*$-regular, then e is a projection form Lemma 4.3 and R is strongly $*$-regular. □

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