ON \((\sigma, \tau)-\ast\)-DERIVATION AND COMMUTATIVITY OF \ast\-PRIME RING

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Abstract

In this paper we study the notion of \((\sigma, \tau)-\ast\)-derivation and prove the following result: Let \(R\) be a \ast\-prime ring with characteristic different from two and \(Z(R)\) be the center of \(R\). If \(R\) admits a non-zero \((\sigma, \tau)-\ast\)-derivation \(d\) of \(R\), with associated automorphisms \(\sigma\) and \(\tau\) of \(R\), such that \(\sigma, \tau\) and \(d\) commute with \ast\ satisfying \([d(U), d(U)]_{\sigma, \tau} = \{0\}\), then \(R\) is commutative, where \(U\) is an ideal of \(R\) such that \(U^\ast = U\).

1 Introduction

Throughout, \(R\) will denote an associative ring with center \(Z(R)\). An additive mapping \(d : R \rightarrow R\) is said to be a derivation of \(R\) if \(d(xy) = d(x)y + xd(y)\) holds for all \(x, y \in R\). For a fixed \(a \in R\), the mapping \(I_a : R \rightarrow R\) given by \(I_a(x) = [a, x] = ax - xa\) is a derivation which is said to be an inner derivation. Recall that \(R\) is said to be prime if \(aRb = \{0\}\) implies \(a = 0\) or \(b = 0\). A ring \(R\) is said to be 2-torsion free, if \(2x = 0\) implies \(x = 0\).

For any two endomorphisms \(\sigma\) and \(\tau\) of \(R\), we call an additive mapping \(d : R \rightarrow R\) a \((\sigma, \tau)\)-derivation of \(R\) if \(d(xy) = d(x)\sigma(y) + \tau(x)d(y)\) for all \(x, y \in R\). Of course, a \((1, 1)\)-derivation is a derivation on \(R\), where 1 is the

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identity mapping on $R$. We set $[x, y]_{\sigma, \tau} = x\sigma(y) - \tau(y)x$. In particular $[x, y]_{1, 1} = [x, y] = xy - yx$, is the usual Lie product.

An additive mapping $x \mapsto x^*$ on a ring $R$ is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $\ast$-ring. A ring $R$ equipped with an involution $\ast$ is said to be $\ast$-prime if $aRb = aRb^* = \{0\}$ (or, equivalently $aRb = a^*Rb = \{0\}$) implies $a = 0$ or $b = 0$. It is important to note that, a prime ring is $\ast$-prime, but the converse is in general not true. An example due to Shulaing [13] justifies this fact. If $R^\circ$ denotes the opposite ring of a prime ring $R$, then $S = R \times R^\circ$ equipped with the exchange involution $\ast_{ex}$ defined by $\ast_{ex}(x, y) = (y, x)$ is $\ast_{ex}$-prime, but not a prime ring because of the fact that $(1, 0)S(0, 1) = 0$. In all that follows, $S_\ast(R)$ will denote the set of symmetric and skew symmetric elements of $R$, i.e., $S_\ast(R) = \{x \in R | x^* = \pm x\}$. An ideal $U$ of $R$ is said to be a $\ast$-ideal of $R$ if $U^* = U$. It can also be noted that an ideal of a ring $R$ may not be $\ast$-ideal of $R$. As an example, let $R = \mathbb{Z} \times \mathbb{Z}$, and consider an involution $\ast$ on $R$ such that $(a, b)^* = (b, a)$ for all $(a, b) \in R$. The subset $U = \mathbb{Z} \times \{0\}$ of $R$ is an ideal of $R$ but it is not a $\ast$-ideal of $R$, because $U^* = \{0\} \times \mathbb{Z} \neq U$.

Let $R$ be a ring with involution $\ast$. An additive mapping $d : R \to R$ is said to be a $\ast$-derivation if $d(xy) = d(x)y^* + xd(y)$ holds for all $x, y \in R$. The concept of $\ast$-derivation was introduced by Brešar and Vukman in [8]. In [1], Shakir and Fošner introduced $(\sigma, \tau)$-$\ast$-derivation as follows: Let $\sigma$ and $\tau$ be two endomorphism of $R$. An additive mapping $d : R \to R$ is said to be $(\sigma, \tau)$-$\ast$-derivation if $d(xy) = d(x)\sigma(y^*) + \tau(x)d(y)$, holds for all $x, y \in R$. In [8], Brešar and Vukman studied some algebraic properties of $\ast$-derivations.

Recently many authors have studied commutativity of prime and semiprime rings with involution admitting suitably constrained derivations (for reference see [2, 12, 16, 20] etc). A lot of work have been done by L. Okhtite and his co-authors on rings with involution (see for reference [17, 18, 19], where further references can be found).

In [15], Lee and Lee proved that if a prime ring of characteristic different from 2 admits a derivation $d$ such that $[d(R), d(R)] \subseteq Z(R)$, then $R$ is commutative. On the other hand in [11] for $a \in R$, Herstein proved that if $[a, d(R)] = \{0\}$, then $a \in Z(R)$. Further in the year 1992, Aydin together with Kaya [7] extended the theorems mentioned above by replacing derivation by $(\sigma, \tau)$-derivation and in some of those, $R$ by a non-zero ideal of $R$. Recently, in [4] we investigated the commutativity of $\ast$-prime ring $R$ equipped with an involution $\ast$ admitting a $(\sigma, \tau)$-$\ast$-derivation $d$ satisfying $[d(U), d(U)]_{\sigma, \tau} = \{0\}$, where $U$ is a nonzero $\ast$-ideal of $R$. In this paper we prove the above mentioned theorem in case of $(\sigma, \tau)$-$\ast$-derivation. In fact, it is shown that if a $\ast$-prime ring admits a nonzero $(\sigma, \tau)$-$\ast$-derivation $d$ satisfying $[d(U), d(U)]_{\sigma, \tau} = \{0\}$, then $R$ is commutative.
2 The Results

In the remaining part of the paper, $R$ will represent a $\ast$-prime ring which admits a nonzero $(\sigma, \tau)$-$\ast$-derivation $d$ with automorphisms $\sigma$ and $\tau$ such that $\ast$ commutes with $d$, $\sigma$ and $\tau$. We shall use the following relations frequently without specific mention:

\[
[x, y, z]_{\sigma, \tau} = x[y, z]_{\sigma, \tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma, \tau}y,
\]

\[
[x, yz]_{\sigma, \tau} = \tau(y)[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau}z,
\]

and

\[
[x, yz]_{\sigma, \tau} + [[x, z]_{\sigma, \tau}, y]_{\sigma, \tau} - [[x, y]_{\sigma, \tau}, z]_{\sigma, \tau} = 0.
\]

Remark 2.1. We find that if $R$ is a $\ast$-prime ring with characteristic different from 2, then $R$ is a 2-torsion free. In fact, if $2x = 0$ for all $x \in R$, then $xR(2s) = 0$ for all $r, s \in R$. But since char $R \neq 2$, there exists a nonzero $l \in R$ such that $2l \neq 0$ and hence by the above $xR2l = \{0\}$. This also gives that $xR(2l)^{\ast} = \{0\}$ and $\ast$-primeness of $R$ yields that $x = 0$, i.e., $R$ is 2-torsion free.

The main result of the present paper states as follows:

**Theorem 2.2.** Let $R$ be a $\ast$-prime ring with characteristic different from two and $\sigma, \tau$ be automorphisms of $R$, and $U$ a $\ast$-ideal of $R$. If $R$ admits a nonzero $(\sigma, \tau)$-$\ast$-derivation $d : R \to R$ such that $[d(U), d(U)]_{\sigma, \tau} = \{0\}$, then $R$ is commutative.

We facilitate our discussion with the following lemmas which are required for developing the proof of our main result.

Since every $\ast$-prime ring is semiprime and every $\ast$-right ideal is right ideal. Hence Lemma 1.1.5 of [9] can be rewritten in case of $\ast$-prime ring as follows:

**Lemma 2.3.** Let $R$ be a $\ast$-prime ring and $U$ a non-zero $\ast$-right ideal of $R$. Then $Z(U) \subseteq Z(R)$.

**Corollary 2.4.** Let $R$ be a $\ast$-prime ring and $U$ a non-zero $\ast$-right ideal of $R$. If $U$ is commutative then $R$ is commutative.

**Proof.** Since $U$, is commutative, by the Lemma 2.3, we have $U = Z(U) \subseteq Z(R)$. If for any $x, y \in R$, $a \in U$ we have $ax \in U$ and hence $ax \in Z(R)$ and hence $(ax)y = y(ax) = ayx$. This further yields $U(xy - yx) = \{0\}$. Since $U$ is a non-zero $\ast$-right ideal of $R$, we have $UR(xy - yx) = \{0\} = U^{\ast}R(xy - yx)$. Also, since $U \neq \{0\}$ right ideal, $\ast$-primeness of $R$ gives $xy - yx = 0$, for all $x, y \in R$. Hence $R$ is commutative. \qed
Lemma 2.5. Let $\mathcal{R}$ be a $*$-prime ring and $U$ a non-zero $*$-right ideal of $\mathcal{R}$. Suppose that $a \in \mathcal{R}$ centralizes $U$. Then $a \in Z(\mathcal{R})$.

Proof. Since $a$ centralizes $U$, for all $u \in U$ and $x \in \mathcal{R}$, $au = uxa$. But $au = ua$, therefore $uax = uxa$, i.e., $u[a, x] = 0$. On replacing $u$ by $uy$ for any $y \in \mathcal{R}$, we get $uR[a, x] = \{0\}$ for all $u \in U$, $x \in \mathcal{R}$. Also, since $U$ is $*$-right ideal, we get $u^*R[a, x] = \{0\}$. Again since $U \neq \{0\}$, $*$-primeness of $\mathcal{R}$ yields that $[a, x] = 0$ for all $x \in \mathcal{R}$. Therefore, $a \in Z(\mathcal{R})$.

Lemma 2.6. Let $\mathcal{R}$ be a $*$-prime ring and $U$ a $*$-right ideal of $\mathcal{R}$. Suppose $d$ is a $(\sigma, \tau)$-$*$-derivation of $\mathcal{R}$ satisfying $d(U) = \{0\}$, then $d = 0$.

Proof. For all $u \in U$ and $x \in \mathcal{R}$, $0 = d(ux) = d(u)\sigma(x^*) + \tau(u)d(x) = \tau(u)d(x)$. On replacing $x$ by $xy$ for any $y \in \mathcal{R}$, we get $\tau(u)d(x)\sigma(y^*) + \tau(u)\tau(x)d(y) = 0$, or, $\tau(u)d(x)d(y) = 0$, i.e., $\tau(u)Rd(y) = \{0\}$ for all $u \in U$ and $y \in \mathcal{R}$. Also since $U$ is a $*$-right ideal, we get $\tau(u)^*Rd(y) = \{0\}$. Also, $*$-primeness of $\mathcal{R}$ yields that $\tau(u) = 0$ for all $u \in U$ or $d = 0$. Since $U \neq \{0\}$, we get $d = 0$.

Lemma 2.7. Let $\mathcal{R}$ be a $*$-prime ring, $U$ a non-zero $*$-ideal of $\mathcal{R}$ and $a \in \mathcal{R}$. Suppose $d$ is a $(\sigma, \tau)$-$*$-derivation of $\mathcal{R}$ satisfying $ad(U) = \{0\}$ (or, $d(U)a = \{0\}$), then $a = 0$ or $d = 0$.

Proof. For $u \in U$, $x \in \mathcal{R}$, $0 = ad(ux) = ad(u)\sigma(x^*) + a\tau(u)d(x)$. By assumption, we have $a\tau(u)d(x) = 0$, for all $x \in \mathcal{R}$. On replacing $u$ by $uy$ for any $y \in \mathcal{R}$, we obtain $a\tau(u)Rd(x) = \{0\}$ for all $u \in U$, $x \in \mathcal{R}$. Also, $a\tau(u)Rd(x)^* = \{0\}$. Since $\mathcal{R}$ is $*$-prime, we find that either $a\tau(u) = 0$ or $d(x) = 0$. If $a\tau(u) = 0$ for all $u \in U$, then $\tau^{-1}(a)U = \{0\}$. Now since $U$ is $*$-ideal, we can write $\tau^{-1}(a)U^* = \{0\}$. This implies that $\tau^{-1}(a)RU = \{0\} = \tau^{-1}(a)RU^*$. By the $*$-primeness of $\mathcal{R}$, we obtain $\tau^{-1}(a) = 0$, since $U \neq \{0\}$. In conclusion, we get either $a = 0$ or $d = 0$. Similarly, $d(U)a = \{0\}$ implies $a = 0$ or $d = 0$.

Lemma 2.8. Let $d$ be a non-zero $(\sigma, \tau)$-$*$-derivation of $*$-prime ring $\mathcal{R}$ and $U$ a $*$-right ideal of $\mathcal{R}$. If $d(U) \subseteq Z(\mathcal{R})$, then $\mathcal{R}$ is commutative.

Proof. Since $d(U) \subseteq Z(\mathcal{R})$, we have $[d(U), \mathcal{R}] = \{0\}$. For $u, v \in U$ and $x \in \mathcal{R}$,

\[
[x, d(uv)] = x, d(u)\sigma(v^*) + \tau(u)d(v) = d(u)[x, \sigma(v^*)] + d(v)[x, \tau(u)] = 0.
\]

Replacing $x$ by $x\sigma(v^*)$, $v \in U$ in (1), we have

\[
0 = d(u)[x\sigma(v^*)], \sigma(v^*)] + d(v)[x\sigma(v^*)], \tau(u)] = d(u)[x, \sigma(v^*)] \sigma(v^*) + d(v)[x[\sigma(v^*)], \tau(u)] + [x, \tau(u)]\sigma(v^*)].
\]

By using (1), we get

\[
d(v)R(\sigma(v^*), \tau(u)] = \{0\}, \text{ for all } u, v \in U.
\]

Let $v \in U \cap Sa_*(\mathcal{R})$. From (2), it follows that

\[
d(v)^*R(\sigma(v^*), \tau(u)] = \{0\}, \text{ for all } u \in U.
\]
By (2) and (3), the \( \ast \)-primeness of \( \mathcal{R} \) yields that \( d(v) = 0 \) or \( [\sigma(v^\ast), \tau(u)] = 0 \) for all \( u \in U \). Let \( w \in U \), since \( w - w^\ast \in U \cap Sa_\ast(\mathcal{R}) \), then
\[
d(w - w^\ast) = 0 \text{ or } [\sigma(w - w^\ast)^\ast, \tau(u)] = 0.
\]

Assume that \( d(w - w^\ast) = 0 \). Then \( d(w) = d(w^\ast) \). Replacing \( v \) by \( w^\ast \) in (2) and since \( U \) is \( \ast \)-right ideal, we get \( d(w^\ast)\mathcal{R}[\sigma(w^\ast)^\ast, \tau(u)] = \{0\} \) for all \( u \in U \). Consequently,
\[
d(w)\mathcal{R}[\sigma(w^\ast)^\ast, \tau(u)] = \{0\}, \text{ for all } u, w \in U. \tag{4}
\]

Also by (2), we get \( d(w)\mathcal{R}[\sigma(w^\ast), \tau(u)] = \{0\} \), on using \( \ast \)-primeness of \( \mathcal{R} \) together with (4), we find that for each \( w \in U \) either \( d(w) = 0 \) or \( [\sigma(w^\ast), \tau(u)] = 0 \), for all \( u \in U \). Now suppose the remaining case that \( [\sigma(v^\ast), \tau(u)] = 0 \), for all \( u \in U \). Then we have \( [\sigma(w - w^\ast)^\ast, \tau(u)] = 0 = [\sigma(w - w^\ast), \tau(u)] \), or \( [\sigma(w), \tau(u)] = [\sigma(w^\ast), \tau(u)] \). Replacing \( v \) by \( w^\ast \) in (2), we get \( d(w^\ast)\mathcal{R}[\sigma(w^\ast)^\ast, \tau(u)] = \{0\} \) for all \( u \in U \). Consequently, \( d(w^\ast)\mathcal{R}[\sigma(w), \tau(u)] = \{0\} \). This yields that
\[
or, \ d(w^\ast)\mathcal{R}[\sigma(w^\ast)^\ast, \tau(u)] = \{0\}, \text{ for all } u, w \in U. \tag{5}
\]

Since \( d(w)\mathcal{R}[\sigma(w^\ast), \tau(u)] = \{0\} \), by (2), the \( \ast \)-primeness of \( \mathcal{R} \) together with (5) assure that for each \( w \in U \) either \( d(w) = 0 \) or \( [\sigma(w^\ast), \tau(u)] = 0 \), for all \( u \in U \). In conclusion, for each fixed \( w \in U \), we have
\[
either \ d(w) = 0 \text{ or } [\sigma(w^\ast), \tau(u)] = 0 \text{ for all } u \in U.
\]

Now, define
\[
K = \{ w \in U \mid d(w) = 0 \} \text{ and } L = \{ w \in U \mid [\sigma(w^\ast), \tau(u)] = 0 \text{ for all } u \in U \}.
\]

Clearly both \( K \) and \( L \) are additive subgroups of \( U \) whose union is \( U \). But a group cannot be a set theoretic union of two of its proper subgroups and hence either \( K = U \) or \( L = U \). If \( K = U \), then \( d(U) = \{0\} \) and hence by Lemma 2.6, \( d = 0 \), a contradiction, therefore now assume that \( L = U \), i.e.,
\[
[\sigma(w^\ast), \tau(u)] = 0 \text{ for all } u, w \in U. \tag{6}
\]

Replacing \( w^\ast \) by \( w^\prime \sigma^{-1}(\tau(v)) \), \( u \in U \), in (6) and using (6), we get \( \sigma(w^\prime)\mathcal{R}[\tau(v), u] = 0 \), for all \( u, v, w^\prime \in U \). On replacing \( w^\prime \) by \( w^\prime x \) for any \( x \in \mathcal{R} \), we get \( \sigma(w^\prime)\mathcal{R}[\tau(v), u] = \{0\} \), for all \( u, v, w^\prime \in U \). Also, since \( U \) is \( \ast \)-right ideal, we get \( \sigma(w^\prime)^\ast \mathcal{R}\tau(v, u) = \{0\} \), for all \( u, v, w^\prime \in U \). Since \( \mathcal{R} \) is \( \ast \)-prime, we find that \( \sigma(w^\prime) = 0 \) or \( \tau(v, u) = 0 \) for all \( u, v, w^\prime \in U \). Since \( U \neq \{0\} \), we have \( U \) is commutative. In view of Corollary 2.4, we obtain the commutativity of \( \mathcal{R} \).

We are now well equipped to prove our main theorem:
**Proof of Theorem 2.2.** First we will show that for any \( a \in Sa_*(\mathcal{R}) \) such that \([d(U), a]_{\sigma, \tau} = \{0\}\), then \( a \in Z(\mathcal{R}) \). For any \( v \in U \), using the hypothesis, we have

\[
0 = [d(uv^*), a]_{\sigma, \tau} = [d(u)\sigma(v) + \tau(u)d(v^*), a]_{\sigma, \tau} = d(u)\sigma(v)\sigma(a) + \tau(u)d(v^*)\sigma(a) - \tau(a)d(u)\sigma(v) - \tau(a)\tau(u)d(v^*).
\]

In view of the hypothesis the above relation yields

\[
d(u)\sigma([v, a]) + \tau([u, a])d(v^*) = 0 \quad \text{for all} \quad u, v \in U. \tag{7}
\]

Replace \( u \) by \( au \) in (7) and use (7) to get

\[
0 = d(au)\sigma([v, a]) + \tau([au, a])d(v^*) = \{d(a)\sigma(u^*) + \tau(a)d(u)\}\sigma([v, a]) + \tau(a)\tau([u, a])d(v^*).
\]

We have \( d(a)\sigma(u^*)\sigma([v, a]) = 0 \), for all \( u, v \in U \). Replace \( u^* \) by \( xw \) for any \( x \in \mathcal{R}, \; w \in U \) we find that \( d(a)\mathcal{R}\sigma(w)\sigma([v, a]) = \{0\} \), for all \( w, v \in U \). Since \( a \in Sa_*(\mathcal{R}) \), the above expression can be rewritten as \( d(a)^*\mathcal{R}\sigma(w)\sigma([v, a]) = \{0\} \), for all \( u, v \in U \). On using \( * \)-primeness of \( \mathcal{R} \), we obtain that for all \( u, v \in U \)

\[
\sigma(w)\sigma([v, a]) = 0 \text{ or } d(a) = 0. \tag{8}
\]

Let us suppose that \( d(a) = 0 \). Then for all \( u \in U \),

\[
d([u, a^*]) = d(ua^* - a^*u) = d(u)\sigma(a) + \tau(u)d(a^*) - d(a^*)\sigma(u^*) - \tau(a^*)d(u) = d(u)\sigma(a) - \tau(a^*)d(u) + \tau(a)d(u) = [d(u), a]_{\sigma, \tau} + \tau(a - a^*)d(u) = \tau(a - a^*)d(u).
\]

Hence the above yields that

\[
d([u, a^*]) - \tau(a - a^*)d(u) = 0. \tag{9}
\]

On replacing \( u \) by \( uv, \; v \in U \), in (9) and on using the same, we get

\[
\tau([u, a^*])d(v) + d(u)\sigma([v, a^*]) + \tau(u)d([v, a^*]) - \tau(a - a^*)\tau(u)d(v) = 0.
\]

By using (9), for all \( u, v, w \in U \) we have

\[
0 = \tau([u, a^*])d(v) + d(u)\sigma([v, a^*]) + \tau(u)d([v, a^*]) - \tau(a - a^*)\tau(u)d(v) = \tau([u, a])d(v) + d(u)\sigma([v, a^*]) + \tau([u, a - a^*])d(v) = \tau([u, a])d(v) + d(u)\sigma([v, a^*]).
\]
Again by using (7), we have
\[ 0 = -d(u)\sigma([v^*, a]) + d(u)\sigma([v, a^*]) = 2d(u)\sigma([v, a^*]). \]
Since \( \text{char} \, \mathcal{R} \neq 2 \), we get \( d(u)\sigma([a, v^*]) = 0 \) for all \( u, v \in U \). Replacing \( v^* \) by \( w \) in the above relation, we get \( d(u)\sigma([a, w]) = 0 \) for all \( u, w \in U \). Substituting \( w \) by \( ww' \) for any \( w' \in U \), reduces the above relation to \( d(u)U\sigma([a, w']) = \{0\} \) for all \( u, v, w \in U \), or \( \sigma^{-1}(d(u))U[a, w'] = \{0\} \) for all \( u, v, w \in U \). Therefore,
\[ \sigma^{-1}(d(u))\mathcal{R}U[a, w'] = \{0\} \] for all \( u, v, w \in U \).

Since \( U \) is a \( \ast \)-ideal, using \( \ast \)-primeness of \( \mathcal{R} \), we get either \( \sigma^{-1}(d(u)) = 0 \) for all \( u \in U \) or \( U[a, w'] = \{0\} \) for all \( w' \in U \). Since \( d(U) \neq \{0\} \), we have \( U[a, w'] = \{0\} = U\mathcal{R}[a, w'] \). Since \( U \) is a nonzero \( \ast \)-ideal, using \( \ast \)-primeness of \( \mathcal{R} \), we get \( [a, w'] = 0 \), for all \( w' \in U \). This reduces to \( [U, a] = \{0\} \). In view of Lemma 2.5, we find that \( a \in Z(\mathcal{R}) \). In view of (8) consider the remaining part \( \sigma(w)\sigma([v, a]) = 0 \) for all \( w, v \in U \), i.e., \( w[v, a] = 0 \) for all \( w, v \in U \). On replacing \( w \) by \( wx \) for any \( x \in \mathcal{R} \), the above equation reduces to \( w\mathcal{R}[v, a] = \{0\} \), for all \( w, v \in U \). Also, \( U \) being a \( \ast \)-ideal, we get \( w^*\mathcal{R}[v, a] = \{0\} \). Using the \( \ast \)-primeness of \( \mathcal{R} \) we find that either \( [v, a] = \{0\} \) or \( U = \{0\} \). Since \( U = \{0\} \) is not possible, it reduces to \( [U, a] = \{0\}. \) Hence again in view of Lemma 2.5, we find that \( a \in Z(\mathcal{R}) \), and by our hypothesis we obtain that \( d(U) \subseteq Z(\mathcal{R}) \). So by Lemma 2.8, \( \mathcal{R} \) is commutative.

\[ \square \]

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References

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