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MULTI-DIMENSIONAL ASYMMETRIC AND NON-UNIFORM GAMBLER'S RUIN PROBLEM WITH TWO PLAYERS

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Abstract

In this paper, we find the expected duration of multi - dimensional gambler's ruin problem for the case of non-uniform and symmetric game. We also show that our result is a generalization of Kmet and Petkovsek's result where they considered uniform and symmetric game. Numerical Results are also provided.

1 Introduction

In the classical one-dimensional two-player gambler's ruin problem, we consider the game which total value of fortune is N and two players start out with i and j dollars where $1 \leqslant i, j \leqslant N-1$ and $i+j=N$. In each round, one of the two gamblers wins one dollar with probability p where $p \in (0,1)$ or loses one dollar to the adversary with probability $1-p$. The play continues until one of the two players goes broke, so that the winning player ends up with N dollars. The expected duration of the game is ij [4].

There are several aspects of the gambler's ruin problems that brought researcher's attentions. For example, in the aspect of multiple player games

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and multidimensional games. In the n-player games, the n players start with initial fortunes of $c_1, c_2, \dots, c_n$, respectively. When one of the n players wins, each of other players must pay one dollar to the winner, so the winner has a total gain of $n-1$ dollars for the round. Play continues until one of the players is ruined. In each round, player i^{th} $(i = 1, \dots, n)$ either wins with probability p_i or loses with probability $1 - p_i$ where $p_1 + p_2 + \cdots + p_n = 1$. When $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, the game is said to be *symmetric*; otherwise it is *asymmetric*.

Extensions of the classical two-player gambler's ruin problem to n-player symmetric games were studied by Sandell [13], Chang [1], and Cho [3] for 3 players, 4 players and n players, respectively. Rocha and Stern [11] considered the asymmetric n-player game for $n \geqslant 3$ with equal initial fortunes of d dollars, $1 \leq d \leq n+1$, and then generalized their previous studies to the games with equal initial fortunes of $n+c$ dollars for fixed c, $c \in \mathbb{N} \cup \{0\}$ and $n \geq \max(c, 2)$ in Rocha and Stern [12]. The case when ties may occur in each round was studied by Hashemiparast and Sabzevari [6]. They derived a closed-form formula for the expected time until ruin and computed individual ruin probability and proved the independence of the ruin time and which player is ruined.

In the multidimensional games, the players randomly select one of the m currencies to gambling in each round and the game is over when one of the players ruins one currency. In each round, the gamblers select the j^{th} currency $(j = 1, \dots, m)$ with probability r_j where $r_1 + r_2 + \cdots + r_m = 1$. When $r_1 = r_2 = \cdots = r_m = \frac{1}{m}$, the game is said to be *uniform*; otherwise it is *non-uniform*. Orr and Zeilberger [10] considered the two-dimensional symmetric and uniform game with two players and derived a formula for the expected duration of the game by using generating function. Kmet and Petkovsek [7] generalized Orr and Zeilberger's results to the case of the m-dimensional symmetric and uniform two-player game. Hashemiparast and Sabzevari [6] extended Orr and Zeilberger's results to the two-dimensional asymmetric and non-uniform n-player game with equal initial fortunes of d where $1 \leq d \leq n+1$. The result was then extended to the three-dimensional game in Chanpana et al. [2].

In this paper, we consider the m-dimensional two-player asymmetric and non-uniform gambler's ruin problem. Our main result is provided in Section 2 where we find a solution of the expected duration of the asymmetric and non-uniform with two players and m currencies. Corresponding numerical results with discussion can be found in Section 3.

2 Main Results

Kmet and Petkovsek [7] gave a solution of the expected duration of the symmetric and uniform two-player game in m dimensions that is the two players have equal chances of winning and the probabilities of choosing currencies are equal. However, the assumption could be too restricted. Therefore, in this paper, we extend their studies to find a solution for a more general case of asymmetric and non-uniform game in m dimensions. We first find a solution of the expected duration of the two-dimensional game and then generalize our technique to give a solution in the general m-dimensional game which results are presented in Theorem 1 and Theorem 2, respectively.

2.1 A solution of expected duration of the two-dimensional game

We now consider the two-dimensional game with two players. The players use two different currencies and start with the initial fortunes of (i, j) and $(N-i, N-j)$, respectively where N is the total value of fortunes in each currency. In each round, they toss a coin to decide the currency to gambling. The play continues until one of the two players goes broke one currency.

Denote by game(i, j) the game with first player's initial assets equal to (i, j) where $1 \leq i, j \leq N-1$. Assume that the first gambler wins with probability $\frac{p}{q}$ and looses with probability $\frac{q-p}{q}$ where $p < q$ and the probabilities in choosing currencies are r_1 and r_2, respectively where $0 < r_1, r_2 < 1$ and $r_1 + r_2 = 1$. Then, after the first step; game(i, j) turns into game $(i+1, j)$ with probability $\frac{p}{q}r_1$; game$(i, j+1)$ with probability $\frac{p}{q}r_2$; game$(i-1, j)$ with probability $\frac{(q-p)}{q}r_1$; or game$(i, j-1)$ with probability $\frac{(q-p)}{q}r_2$.
Let $a_{i,j}$(for $1 \leq i, j \leq N-1$) be the expected duration of game(i, j) satisfying the recurrence equation

$$a_{i,j} = \frac{pr_1 a_{i+1,j} + (q-p)r_1 a_{i-1,j} + pr_2 a_{i,j+1} + (q-p)r_2 a_{i,j-1}}{q} + 1 \qquad (1)$$

with the boundary conditions

$$a_{0,j} = a_{N,j} = a_{i,0} = a_{i,N} = 0 \;\; \text{for } 0 \leq i, j \leq N. \qquad (2)$$

Let $\mathbf{A}_1 = [a_{i,j}]_{i,j=1}^{N-1}$ be the matrix of unknown values $a_{i,j}$. By writing (1) in the form

$$((q-p)r_2 a_{i,j-1} - \frac{q}{2}a_{i,j} + pr_2 a_{i,j+1}) + ((q-p)r_1 a_{i-1,j} - \frac{q}{2}a_{i,j} + pr_1 a_{i+1,j}) = -q, \qquad (3)$$

for $1 \leq i, j \leq N-1$, and using the boundary conditions (2), we can show that $\mathbf{A}_1$ satisfies the matrix equation

$$\mathbf{A}_1 \mathbf{D}_{r_2}^T + \mathbf{D}_{r_1} \mathbf{A}_1 = -q\mathbf{J}_1, \tag{4}$$

where

$$\mathbf{D}_{r_1} = \begin{bmatrix} -\frac{q}{2} & pr_1 & 0 & 0 & \dots & 0 \\ (q-p)r_1 & -\frac{q}{2} & pr_1 & 0 & \dots & 0 \\ 0 & (q-p)r_1 & -\frac{q}{2} & pr_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & (q-p)r_1 & -\frac{q}{2} & pr_1 \\ 0 & 0 & \dots & 0 & (q-p)r_1 & -\frac{q}{2} \end{bmatrix} \tag{5}$$

and

$$\mathbf{D}_{r_2} = \begin{bmatrix} -\frac{q}{2} & pr_2 & 0 & 0 & \dots & 0 \\ (q-p)r_2 & -\frac{q}{2} & pr_2 & 0 & \dots & 0 \\ 0 & (q-p)r_2 & -\frac{q}{2} & pr_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & (q-p)r_2 & -\frac{q}{2} & pr_2 \\ 0 & 0 & \dots & 0 & (q-p)r_2 & -\frac{q}{2} \end{bmatrix} \tag{6}$$

are $(N-1)\times(N-1)$ tridiagonal Toeplitz matrices and $\mathbf{J}_1$ is the $(N-1)\times(N-1)$ matrix of ones.

From Laub ([8], p.145), we can rewrite (4) as a vector equation

$$(\mathbf{I}_{N-1} \otimes \mathbf{D}_{r_1} + \mathbf{D}_{r_2} \otimes \mathbf{I}_{N-1})\mathbf{a}_1 = -q\mathbf{j}_1, \tag{7}$$

where $\mathbf{a}_1$ and $\mathbf{j}_1$ are the vectors obtained by stacking the columns of $\mathbf{A}_1$ and $\mathbf{J}_1$, respectively.

By Theorem 7, $\mathbf{D}_{r_1}$ and $\mathbf{D}_{r_2}$ are tridiagonal Toeplitz matrices having $N-1$ linearly independent eigenvectors, so they are diagonalizable. Then, $\mathbf{D}_{r_1} = \mathbf{P}_{r_1}\Lambda_{r_1}\mathbf{P}_{r_1}^{-1}$ and $\mathbf{D}_{r_2} = \mathbf{P}_{r_2}\Lambda_{r_2}\mathbf{P}_{r_2}^{-1}$ where Λ_{r_1} and Λ_{r_2} are diagonal matrices of the eigenvalues of $\mathbf{D}_{r_1}$ and $\mathbf{D}_{r_2}$, respectively and $\mathbf{P}_{r_1}$ and $\mathbf{P}_{r_2}$ are invertible matrices whose columns are eigenvectors corresponding to the eigenvalues of $\mathbf{D}_{r_1}$ and $\mathbf{D}_{r_2}$, respectively. From Laub ([8], p.140), we can rewrite (7) as

$$(\mathbf{P}_{r_2} \otimes \mathbf{P}_{r_1})(\Lambda_{r_2} \otimes \mathbf{I}_{N-1} + \mathbf{I}_{N-1} \otimes \Lambda_{r_1})(\mathbf{P}_{r_2} \otimes \mathbf{P}_{r_1})^{-1}\mathbf{a}_1 = -q\mathbf{j}_1.$$

By Theorem 7, all eigenvalues of $\mathbf{D}_{r_2}^T$ and $\mathbf{D}_{r_1}$ are $[-\frac{q}{2} + 2r_2\sqrt{p(q-p)}\cos\frac{i\pi}{N}]$ and $\mathbf{D}_{r_1}$ are $[-\frac{q}{2} + 2r_1\sqrt{p(q-p)}\cos\frac{j\pi}{N}]$, respectively, where $i, j \in \{1, \dots, N-1\}$. Assume that there exist $i, j \in \{1, \dots, N-1\}$ such that $\lambda_{r_1,i} + \lambda'_{r_2,j} = 0$

where $\lambda_{r_1,i}$ is an eigenvalue of $\mathbf{D}_{r_1}$ and $\lambda'_{r_2,j}$ is an eigenvalue of $\mathbf{D}^T_{r_2}$. Thus

$$\begin{aligned} 0 &= \lambda_{r_1,i} + \lambda'_{r_2,j} \qquad (8)\\ &= -q + 2\sqrt{p(q-p)}[r_1 \cos\frac{i\pi}{N} + r_2 \cos\frac{j\pi}{N}] \\ &< -q + 2\sqrt{p(q-p)} \qquad (-1 < \cos\frac{x\pi}{N} < 1 \text{ for all } x \in \{1, \ldots, N-1\}). \end{aligned}$$

This implies that $q < 2\sqrt{p(q-p)}$ and so $(q-2p)^2 < 0$ which is not true. Then $-\mathbf{D}^T_{r_2}$ and $\mathbf{D}_{r_1}$ have no common eigenvalues. Then $\Lambda_{r_2} \otimes \mathbf{I}_{N-1} + \mathbf{I}_{N-1} \otimes \Lambda_{r_1}$ is invertible matrix. Therefore the vector of expected duration time is given as follows.

$$\mathbf{a}_1 = -q(\mathbf{P}_{r_2} \otimes \mathbf{P}_{r_1})(\Lambda_{r_2} \otimes \mathbf{I}_{N-1} + \mathbf{I}_{N-1} \otimes \Lambda_{r_1})^{-1}(\mathbf{P}_{r_2} \otimes \mathbf{P}_{r_1})^{-1}\mathbf{j}_1.$$

Thus, we have proven the following theorem.

Theorem 1. *For a two - dimensional asymmetric and non-uniform game with two players where the first gambler wins with probability $\frac{p}{q}$ and looses with probability $\frac{q-p}{q}$ where $0 < p < q$ and the probabilities in choosing currencies are r_1 and r_2, respectively where $0 < r_1, r_2 < 1$ and $r_1 + r_2 = 1$. The solution of the expected duration of the game is given as*

$$\mathbf{a}_1 = -q(\mathbf{P}_{r_2} \otimes \mathbf{P}_{r_1})(\Lambda_{r_2} \otimes \mathbf{I}_{N-1} + \mathbf{I}_{N-1} \otimes \Lambda_{r_1})^{-1}(\mathbf{P}_{r_2} \otimes \mathbf{P}_{r_1})^{-1}\mathbf{j}_1,$$

where for $i = 1, 2$, P_{r_i} is a matrix whose columns are eigenvectors and Λ_{r_i} is a diagonal matrix of the eigenvalues of $\mathbf{D}_{r_i}$ defined in (5) and (6), respectively.

2.2 A generalization of the multi-dimensional game

In m-dimensional games, the two players use m different currencies. Their initial fortunes of the m currencies are $(i_1, \ldots, i_m)$ and $(N - i_1, \ldots, N - i_m)$, respectively where N is the total value of fortune of each currency. In each round, players randomly select 1 of the m currencies to gambling. The play continues until one of the two players goes broke one currency. In this section, we generalize our study in the previous section to find a solution expected duration to m-dimensional games.

Assume that the probability in choosing the i^{th} currency is r_i $(0 < r_i < 1)$ where $1 \leq i \leq m$ and $\sum_{i=1}^{m} r_i = 1$. Denote by game $(i_1, \ldots, i_m)$ the game with first player's initial assets equal to $(i_1, \ldots, i_m)$ where $1 \leq i_1, \ldots, i_m \leq N-1$. Let $a_{i_1,\ldots,i_m}$ be the expected duration of the game$(i_1, \ldots, i_m)$. We define $\mathbf{A}_k$ be the $(N-1)^k \times (N-1)$ matrix whose the l^{th} column $a_{\cdot,\cdot,\ldots,l}$ is obtained by stacking the columns of the $(N-1)^{k-1} \times (N-1)$ matrix $[a_{i_1,i_2,\ldots,i_{k-1},l}]_{i_1,i_2,\ldots,i_{k-1}=1}^{N-1}$

one above another and, for $k = 1, 2, \ldots, m$,

$$\mathbf{D}_{r_k} = \begin{bmatrix} -\frac{q}{m} & pr_k & 0 & 0 & \ldots & 0 \\ (q-p)r_k & -\frac{q}{m} & pr_k & 0 & \ldots & 0 \\ 0 & (q-p)r_k & -\frac{q}{m} & pr_k & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & (q-p)r_k & -\frac{q}{m} & pr_k \\ 0 & 0 & \ldots & 0 & (q-p)r_k & -\frac{q}{m} \end{bmatrix}. \tag{9}$$

Consider the m-dimensional case, we have the recurrence

$$\begin{aligned} -q = {} & ((q-p)r_m a_{i_1,i_2,\ldots,i_m-1} - \frac{q}{m} a_{i_1,i_2,\ldots,i_m} + pr_m a_{i_1,i_2,\ldots,i_m+1}) \\ & + \cdots + ((q-p)r_1 a_{i_1-1,i_2,\ldots,i_m} - \frac{q}{m} a_{i_1,i_2,\ldots,i_m} + pr_1 a_{i_1+1,i_2,\ldots,i_m}), \end{aligned}$$

for $1 \leq i_1, i_2, \ldots, i_m \leq N-1$. The equation is equivalent to

$$\mathbf{A}_{m-1}\mathbf{D}_{r_m}^T + \mathbf{D}_{m-2}\mathbf{A}_{m-1} = -q\mathbf{J}_{m-1}, \tag{10}$$

where

$$\mathbf{D}_{m-2} = \mathbf{D}_{r_{m-1}} \otimes \mathbf{I}_{(N-1)^{m-2}} + \mathbf{I}_{N-1} \otimes \mathbf{D}_{m-3} = \mathbf{D}_{m-3} \oplus \mathbf{D}_{r_{m-1}}$$

for $m \geq 3$, $\mathbf{D}_0 = \mathbf{D}_{r_1}$ and $\mathbf{J}_{m-1}$ is the $(N-1)^{m-1} \times (N-1)$ matrix of ones.

To show that (10) has a unique solution for all $m \geq 3$, by Theorem 6, it is sufficient to show that $\mathbf{D}_{m-2}$ and $-\mathbf{D}_{r_m}^T$ have no eigenvalues in common. The proof is given as follows.

Let $\Lambda_{r_k} = \operatorname{diag}(\lambda_{r_k,i})_{i=1}^{N-1}$ be the diagonal matrix of the eigenvalues of D_{r_k} for all $k = 1, 2, \ldots, m$ and $i = 1, \ldots, N-1$.
Since D_{r_k} is a tridiagonal Toeplitz matrix for all $k = 1, 2, \ldots, m$, by Theorem 7,

$$\lambda_{r_k,i} = -\frac{q}{m} + 2r_k\sqrt{p(q-p)}\cos\frac{i\pi}{N}.$$

Since $\mathbf{D}_{m-k} = \mathbf{D}_{r_{m-k+1}} \oplus \mathbf{D}_{m-k-1}$ for all $1 \leq k < m$,

$$\mathbf{D}_{m-2} = (((\mathbf{D}_{r_1} \oplus \mathbf{D}_{r_2}) \cdots \oplus \mathbf{D}_{r_{m-3}}) \oplus \mathbf{D}_{r_{m-2}}) \oplus \mathbf{D}_{r_{m-1}}.$$

By Theorem 5, the set of all eigenvalues of $\mathbf{D}_{m-2}$ is

$$\{\lambda_{r_1,j_1} + \cdots + \lambda_{r_{m-1},j_{m-1}} | \text{ for } j_1, \ldots, j_{m-1} = 1, \ldots, N-1\}.$$

Assume that there exists $i \in \{1, \ldots, N-1\}$ and $j \in \{1, \ldots, (N-1)^{m-1}\}$ be such that $\lambda'_{r_m,i} + \mu_j = 0$ where $\lambda'_{r_m,i}$ is an eigenvalue of $\mathbf{D}_{r_m}^T$ and μ_j is an eigenvalue of $\mathbf{D}_{m-2}$. We can generalize the equation (8) to the m-dimensional games where $\mu_j = \sum_{l=1}^{m-1} \lambda_{r_l, j_l}$. Then $\lambda_{r_m,i} + \mu_j \neq 0$ for all $i \in \{1, \ldots, N-1\}$, $j \in \{1, \ldots, (N-1)^{m-1}\}$, by Theorem 6, (10) has a unique solution for all $m \geq 3$.

Then for all $m \geq 3$ we have

$$\mathbf{D}_{m-1}\mathbf{a}_{m-1} = -q\mathbf{j}_{m-1},$$

where $\mathbf{D}_{m-1} = \mathbf{D}_{r_m} \otimes \mathbf{I}_{(N-1)^{m-1}} + \mathbf{I}_{N-1} \otimes \mathbf{D}_{m-2} = \mathbf{D}_{m-2} \oplus \mathbf{D}_{r_m}$.

Thus,

$$\mathbf{a}_{m-1} = -q\mathbf{D}_{m-1}^{-1}\mathbf{j}_{m-1},$$

where $\mathbf{a}_{m-1}$ is the vector with components $a_{i_1,\ldots,i_m}$ and $\mathbf{j}_{m-1}$ is the vector with all components equal to 1.
Therefore, we have proven the following theorem.

Theorem 2. *For an m-dimensional asymmetric and non-uniform game with two players where the first gambler wins with probability $\frac{p}{q}$ and looses with probability $\frac{q-p}{q}$ where $0 < p < q$ and the probabilities in choosing currencies are r_i $(i = 1, 2, \ldots, m)$ where $0 < r_i < 1$ and $\sum_{i-1}^{m} r_i = 1$. The solution of the expected duration of the game is given by an iterative form as follows.*

$$\mathbf{a}_{m-1} = -q\mathbf{D}_{m-1}^{-1}\mathbf{j}_{m-1}, \tag{11}$$

where $\mathbf{D}_{m-1} = \mathbf{D}_{r_m} \otimes \mathbf{I}_{(N-1)^{m-1}} + \mathbf{I}_{N-1} \otimes \mathbf{D}_{m-2} = \mathbf{D}_{m-2} \oplus \mathbf{D}_{r_m}$ and $D_{r_k}(k = 1, 2, \ldots, m)$ are given in (9).

Remark 1. *Note that in the case $N = 3$, the matrices D_{r_k} $(k = 1, \ldots, m)$ appeared in Equations (4) and (10) are not tridiagonal. Then the techniques in showing the existence and uniqueness of the solutions used previously are not hold. We give an alternative technique in this remark as follows.*
By direct calculations, eigenvalues of $\mathbf{D}_{r_k}$ are $-\frac{q}{m} \pm r_k\sqrt{pq - p^2}$. If $\mathbf{D}_{r_m}$ and $\mathbf{D}_{m-2}$ in Equations (4) and (10) have common eigenvalues, then

$$-\frac{q}{m} \pm r_m\sqrt{pq - p^2} + \sum_{k=1}^{m-1} \Big[-\frac{q}{m} \pm r_k\sqrt{pq - p^2)}\Big] = 0.$$

Then $q^2 = \Big[\sum_{k=1}^{m} \pm r_k\sqrt{pq - p^2}\Big]^2 = (pq - p^2)(\sum_{k=1}^{m} \pm r_k)^2 = pq - p^2 < q^2$ which is not true. By Theorem 6, the Equations (4) and (10) have unique solutions.

Remark 2. *In this section, we give a formula of the expected duration for the m-dimensional asymmetric and non-uniform game. For a special case of our result, symmetric and uniform game, the formula (11) in Theorem 2 reduces to Kmet and Petkovsek's result [7] given as follows.*

$$\mathbf{a}_{m-1} = -2m\mathbf{D}_{m-1}^{-1}\mathbf{j}_{m-1},$$

where $\mathbf{D}_{m-1} = \mathbf{D} \otimes \mathbf{I}_{(N-1)^{m-1}} + \mathbf{I}_{N-1} \otimes \mathbf{D}_{m-2}$ *and*

$$\mathbf{D} = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & \dots & 0 & 1 & -2 \end{bmatrix}.$$

3 Numerical results

In this section, we present some numerical results of Theorem 2 in calculating the expected duration time of the three-dimensional game which we will consider in four cases: *symmetric and uniform* game ($p/q = 1/2$ and $r_1 = r_2 = r_3$), *symmetric and non-uniform* game ($p/q = 1/2$ and $r_1 = 1/3, r_2 = 2/9$), *asymmetric and uniform* game ($p/q = 1/3$ and $r_1 = r_2 = r_3$) and *asymmetric and non-uniform* game ($p/q = 1/3$ and $r_1 = 1/3, r_2 = 2/9$).

TABLE I
Numerical results of expected duration in the three-dimensional two-player game with $N = 4$.

Game (i, j, k)	Expected duration of game ($a_{i,j,k}$)			
	$p/q = 1/2$ and $r_1 = r_2 = r_3$	$p/q = 1/2$ and $r_1 = \frac{1}{3}, r_2 = \frac{2}{9}$	$p/q = 1/3$ and $r_1 = r_2 = r_3$	$p/q = 1/3$ and $r_1 = \frac{1}{3}, r_2 = \frac{2}{9}$
(1,1,1)	2.588	2.601	1.784	1.798
(2,1,1)	3.176	3.193	2.352	2.374
(3,1,1)	2.588	2.601	2.182	2.202
(1,2,1)	3.176	3.086	2.352	2.230
(2,2,1)	3.941	3.829	3.209	3.035
(3,2,1)	3.176	3.086	2.969	2.810
(1,3,1)	2.588	2.601	2.182	2.081
(2,3,1)	3.176	3.193	2.969	2.818
(3,3,1)	2.588	2.601	2.749	2.612
(1,1,2)	3.176	3.267	2.352	2.489
(2,1,2)	3.941	4.053	3.209	3.407
(3,1,2)	3.176	3.267	2.969	3.150
(1,2,2)	3.941	3.916	3.209	3.186
(2,2,2)	4.941	4.908	4.535	4.499
(3,2,2)	3.941	3.916	4.187	4.155
(1,3,2)	3.176	3.267	2.969	2.957
(2,3,2)	3.941	4.053	4.187	4.156
(3,3,2)	3.176	3.267	3.870	3.841
(1,1,3)	2.588	2.601	2.182	2.307
(2,1,3)	3.176	3.193	2.969	3.156
(3,1,3)	2.588	2.601	2.749	2.920
(1,2,3)	3.176	3.086	2.969	2.952
(2,2,3)	3.941	3.829	4.187	4.170
(3,2,3)	3.176	3.086	3.870	3.854
(1,3,3)	2.588	2.601	2.749	2.742
(2,3,3)	3.176	3.193	3.870	3.855
(3,3,3)	2.588	2.601	3.580	3.566

From Table I, we can see that $a_{i,j,k} = a_{N-i,j,k} = a_{i,N-j,k} = a_{i,j,N-k} = a_{N-i,N-j,k} = a_{N-i,j,N-k} = a_{i,N-j,N-k} = a_{N-i,N-j,N-k}$ in symmetric and non-uniform game and clearly that $a_{i,j,k} = a_{i,k,j} = a_{j,i,k} = a_{j,k,i} = a_{k,i,j} = a_{k,j,i}$ in asymmetric and uniform game but $a_{i,j,k} = a_{N-i,j,k}$ is not true. These observations are also true for general m-dimensional games.

4 Conclusion

In this paper, we give formula of the expected duration time of the m-dimensional asymmetric and non - uniform games with two players. Some further extensions of our studies can be done. For example, the game with multiple players and the case where ties may occur.

5 Appendix

In this section, we state some definitions and theorems used in our paper. One main approach of our paper is to consider the *Sylvester's equation* of the form

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{m\times m}$, and $\mathbf{C} \in \mathbb{R}^{n\times m}$.

Definition 3. *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$. Then the Kronecker product (or tensor product) of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \otimes \mathbf{B}$, is the $mp \times nq$ block matrix:*

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix}$$

Definition 4. *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$. Then the Kronecker sum (or tensor sum) of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \oplus \mathbf{B}$, is the $mn \times mn$ matrix $(\mathbf{I}_m \otimes \mathbf{A}) + (\mathbf{B} \otimes \mathbf{I}_n)$. Note that, in general, $\mathbf{A} \oplus \mathbf{B} \neq \mathbf{B} \oplus \mathbf{A}$.*

Theorem 5. *(Laub [8]) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ have eigenvalues $\lambda_i, i = 1, \ldots, n$, and let $\mathbf{B} \in \mathbb{R}^{m\times m}$ have eigenvalues $\mu_j, j = 1, \ldots, m$. Then the Kronecker sum $\mathbf{A} \oplus \mathbf{B} = (\mathbf{I}_m \otimes \mathbf{A}) + (\mathbf{B} \otimes \mathbf{I}_n)$ has mn eigenvalues*

$$\lambda_1 + \mu_1, \ldots, \lambda_1 + \mu_m, \lambda_2 + \mu_1, \ldots, \lambda_2 + \mu_m, \ldots, \lambda_n + \mu_m.$$

Moreover, if $x_1, \ldots, x_p$ are linearly independent right eigenvectors of $\mathbf{A}$ corresponding to $\lambda_1, \ldots, \lambda_p$ ($p \leq n$), and $z_1, \ldots, z_q$ are linearly independent right eigenvectors of $\mathbf{B}$ corresponding to $\mu_1, \ldots, \mu_q$ ($q \leq m$), then $z_j \otimes x_i \in \mathbb{R}^{mn}$ are linearly independent right eigenvectors of $\mathbf{A} \oplus \mathbf{B}$ corresponding to $\lambda_i + \mu_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Theorem 6. *(Laub [8]) Let $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{m\times m}$ and $\mathbf{C} \in \mathbb{R}^{n\times m}$. Then the Sylvester equation*

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C}$$

has a unique solution if and only if $\mathbf{A}$ and $-\mathbf{B}$ have no eigenvalues in common.

Theorem 7. *(Mayer [9]) Let $\mathbf{A}$ be the $n \times n$ tridiagonal Toeplitz matrix be such that*

$$\mathbf{A} = \begin{bmatrix} b & a & 0 & 0 & \ldots & 0 \\ c & b & a & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & c & b & a \\ 0 & 0 & \ldots & 0 & c & b \end{bmatrix} \qquad \text{with } a, c \neq 0.$$

Then **A** *is diagonalizable with* n *different eigenvalues and eigenvectors, respectively; define by*

$$\lambda_k = b + 2a\sqrt{\frac{c}{a}}\cos\frac{k\pi}{n+1}$$

and

$$x_k = \begin{bmatrix} (c/a)^{1/2}\sin(1k\pi/(n+1)) \\ (c/a)^{2/2}\sin(2k\pi/(n+1)) \\ (c/a)^{3/2}\sin(3k\pi/(n+1)) \\ \vdots \\ (c/a)^{n/2}\sin(nk\pi/(n+1)) \end{bmatrix}$$

for $k = 1, \ldots, n$.

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