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RATIONAL ARITHMETICAL FUNCTIONS RELATED TO CERTAIN UNITARY ANALOGS OF GCD TYPE MATRICES

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Abstract

An arithmetical function f is a rational arithmetical function of order (r, s) if it can be written as the Dirichlet convolution of r completely multiplicative functions and s inverses of completely multiplicative functions. In this paper we show that pseudo-unitarily semimultiplicative functions and a related generalization of the unitary analog of Euler's totient function are rational arithmetical functions of orders $(1, 2)$ and $(2, 3)$. These functions arise from the theory the so-called pseudo-LCUM and GCUD reciprocal pseudo-LCUM matrices, where GCUD and pseudo-LCUM stand for the greatest common unitary divisor and an extension of the least common unitary multiple.

1 Introduction

An arithmetical function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}_+$ with $(m, n) = 1$. A multiplicative function is totally determined by its values at prime powers. A multiplicative function f is said to be completely multiplicative if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}_+$. A completely multiplicative function is totally determined by its values at primes. In fact, a multiplicative function f is completely multiplicative if and only if $f(p^k) = f(p)^k$ for all primes p and integers $k \geq 2$.

Key words: GCD matrix, unitary divisor, semimultiplicative function, Euler's totient function, rational arithmetical function.
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The Dirichlet convolution of arithmetical functions f and g is defined as

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

The function δ, defined as $\delta(1) = 1$ and $\delta(n) = 0$ otherwise, serves as the identity under the Dirichlet convolution. The Dirichlet inverse of f exists if and only if $f(1) \neq 0$ and it is denoted as f^{-1}.

An arithmetical function f is said to be a rational arithmetical function of order (r, s) if

$$f = g_1 \star g_2 \star \cdots \star g_r \star h_1^{-1} \star h_2^{-1} \star \cdots \star h_s^{-1}, \tag{1.1}$$

where $g_1, g_2, \ldots, g_r, h_1, h_2, \ldots, h_s$ are completely multiplicative functions. This notion originates from Vaidyanathaswamy [33]. For further papers on rational arithmetical functions we refer to [3, 5, 7, 8, 12, 18, 19, 20, 25]. General material on arithmetical functions can be found from the books [2, 22, 26, 29, 30].

Rational arithmetical functions of order $(1, 1)$ are called totients. Euler's ϕ-function is a typical example of a totient. Rational arithmetical functions of order $(2, 0)$ are said to be quadratics or specially multiplicative functions. The divisor-number ja divisor-sum functions τ and σ belong to this class. Completely multiplicative functions are rational arithmetical functions of order $(1, 0)$. For example, Liouville's function λ is completely multiplicative. The Möbius function μ is an example of a rational arithmetical function of order $(0, 1)$, and Pillai's function [9, 32] is an example of a rational arithmetical function of order $(2, 1)$. If f is a rational arithmetical function of order (r, s), then f is a rational arithmetical function of order (t, u) for all $t \geq r$ and $u \geq s$, since the identity function δ is completely multiplicative. Each rational arithmetical function is multiplicative.

The purpose of this paper is to show that certain arithmetical functions associated with the so-called pseudo-LCUM and GCUD reciprocal pseudo-LCUM matrices are rational arithmetical functions of orders $(1, 2)$ and $(2, 3)$. Here GCUD refers to the greatest common unitary divisor of two positive integers and pseudo-LCUM refers to an extension of the least common unitary multiple (LCUM) of two positive integers. The arithmetical functions studied here are pseudo-unitarily semimultiplicative functions (being an analog of semimultiplicative functions), completely multiplicative functions and a generalization of the unitary analog of Euler's totient function ϕ. We also study strongly multiplicative functions, which are also related with certain GCUD reciprocal LCUM type matrices and appear to be totients, that is, rational arithmetical functions of order $(1, 1)$.

The paper is organized as follows. In Section 2.1 we discuss unitary divisors as well as GCUD and pseudo-LCUM. In Section 2.2 we consider semimultiplicative functions. The pseudo-LCUM and the GCUD reciprocal pseudo-LCUM matrices are introduced in Section 2.3, and the Bell series of a multiplicative function is introduced in Section 2.4. Properties of pseudo-unitarily semimultiplicative functions are presented in Section 3, and properties of our generalization of the unitary analog of Euler's totient function are presented in Section 4. Some results on strongly multiplicative functions are given in Section 5.

2 Preliminaries

2.1 Unitary divisors

A divisor $d \in \mathbb{Z}_+$ of $n \in \mathbb{Z}_+$ is said to be a unitary divisor of n and is denoted by $d \parallel n$ if $(d, n/d) = 1$. For example, the unitary divisors of 72 ($= 2^3 3^2$) are $1, 8, 9, 72$. If $d \parallel n$, we also say that n is a unitary multiple of d. The greatest common unitary divisor (GCUD) of m and n exists for all $m, n \in \mathbb{Z}_+$ but, unfortunately, the least common unitary multiple (LCUM) of m and n does not always exist. For example, the LCUM of 2 and 4 does not exist. The GCUD of m and n is denoted by $(m, n)^{**}$ and the LCUM is denoted by $[m, n]^{**}$ when it exists.

Hansen and Swanson [6] overcame the difficulty of the nonexistence of the LCUM by defining

$$[m, n]^{**} = \frac{mn}{(m, n)^{**}}. \tag{2.1}$$

It is easy to see that $mn/(m, n)^{**}$ exists for all $m, n \in \mathbb{Z}_+$ and is equal to the usual LCUM of m and n when the usual LCUM exists. Therefore $[m, n]^{**}$ in (2.1) is well-defined. For example, $[2, 4]^{**} = 8$ but it is not reasonable to say that 8 is the LCUM of 2 and 4. We say that $[m, n]^{**}$ in (2.1) is the pseudo-LCUM of m and n. If the LCUM exists, then it is equal to the pseudo-LCUM. There exist also extensions of the LCUM other than the pseudo-LCUM in the literature [11]. Note that there is some inconsistency in the notations of GCUD and (pseudo-)LCUM in the literature. It might look natural to denote the GCUD as $(m, n)^*$ but $(m, n)^*$ usually stands for the semi-unitary greatest common divisor [13].

The unitary convolution of arithmetical functions f and g is defined as

$$(f \oplus g)(n) = \sum_{d \| n} f(d)g(n/d).$$

The function δ also serves as the identity under the unitary convolution. The unitary analog of the Möbius function is the inverse of the constant function 1 under the unitary convolution and it is denoted by μ^*. The function μ^* is the multiplicative function such that $\mu^*(p^k) = -1$ for all prime powers p^k ($k \geq 1$).

The concept of a unitary divisor and the unitary convolution originates from Vaidyanathaswamy [33] and was further developed notably by Cohen [4].

2.2 Semimultiplicative functions

An arithmetical function f is said to be semimultiplicative [10, 30] if

$$f((m,n))f([m,n]) = f(m)f(n) \tag{2.2}$$

for all $m, n \in \mathbb{Z}_+$. An arithmetical function f is said to be quasimultiplicative if $f(1) \neq 0$ and

$$f(1)f(mn) = f(m)f(n) \tag{2.3}$$

for all $m, n \in \mathbb{Z}_+$ with $(m,n) = 1$. Quasimultiplicative functions f may also be characterized as semimultiplicative functions with $f(1) \neq 0$, see (2.2). A quasimultiplicative function f is multiplicative if and only if $f(1) = 1$. An arithmetical function f with $f(1) \neq 0$ is quasimultiplicative if and only if $f/f(1)$ is multiplicative. The concept of a semimultiplicative function is due to Rearick [24] and Selberg [28].

An arithmetical function f is pseudo-unitarily semimultiplicative if

$$f((m,n)^{**})f([m,n]^{**}) = f(m)f(n) \tag{2.4}$$

for all $m, n \in \mathbb{Z}_+$, where "pseudo" refers to the pseudo-LCUM. This concept was developed for the purpose of certain matrix formulas in [11]. In general, semimultiplicative functions and their analogs are useful in the theory of the so-called GCD type matrices, see e.g. [21].

A multiplicative function f is said to be strongly multiplicative if $f(p^k) = f(p)$ for all prime powers p^k with $k \geq 1$. These functions also have connections to GCD type matrices (see [11, Theorem 4.4]) and rational arithmetical functions.

2.3 GCD type matrices

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. The GCD matrix on S associated with f is defined as the $n \times n$ matrix having f evaluated at the greatest common divisor of x_i and x_j as its ij entry. Various analogs and generalizations of these matrices have been presented in the literature, see e.g. [1, 14, 15, 16, 17].

The $n \times n$ matrix having $f((x_i, x_j)^{**})$ as its ij entry is denoted as $(S^{**})_f$, and similarly the $n \times n$ matrix having $f([x_i, x_j]^{**})$ as its ij entry is denoted as $[S^{**}]_f$. We say that these matrices are the GCUD and the pseudo-LCUM matrices on S with respect to f.

Now, assume that $f(x) \neq 0$ for all $x \in \mathbb{Z}_+$. Let $(S^{**})_f/[S^{**}]_f$ denote the $n \times n$ matrix having $f((x_i, x_j)^{**})/f([x_i, x_j]^{**})$ as its ij entry, that is, $(S^{**})_f/[S^{**}]_f$ is the Hadamard quotient of the matrices $(S^{**})_f$ and $[S^{**}]_f$. The matrix $(S^{**})_f/[S^{**}]_f$ is referred to as the GCUD reciprocal pseudo-LCUM matrix on S with respect to f in [11].

Let

$$B_f^*(x_i) = \sum_{\substack{d \| x_i \\ d \nparallel x_t \\ t<i}} \phi_f^*(d) \tag{2.5}$$

for all $i = 1, 2, \ldots, n$, where

$$\phi_f^*(x) = \sum_{d \| x} f(d)\mu^*(x/d) = (f \oplus \mu^*)(x). \tag{2.6}$$

By the unitary analog of the Möbius inversion formula [4] we have

$$f(x) = \sum_{d \| x} \phi_f^*(d).$$

If $f(x) = x$ for all x, then $\phi_f^* = \phi^*$, the unitary analog of Euler's totient function ϕ, see [4, 23]. If $f(x) = x^k$ for all x, then $\phi_f^* = J_k^*$, the unitary analog of Jordan's totient function J_k, see [23, 27].

If the set S is unitary divisor closed, then $B_f^*(x_i) = \phi_f^*(x_i)$. The function B_f^* plays an important role in the theory of pseudo-LCUM, GCUD reciprocal pseudo-LCUM and related matrices. In [13], it is shown, for example, that if S is GCUD-closed and f is completely multiplicative with $f(x) \neq 0$ for all x, then

$$\det\,[[S^{**}]_f] = \prod_{k=1}^{n} f^2(x_k) B_{1/f}^*(x_k), \tag{2.7}$$

where $f^2(x) = (f(x))^2$ and $(1/f)(x) = 1/(f(x))$ for all x. In [11], it is shown, for example, that if S is GCUD-closed and f is pseudo-unitarily semimultiplicative, then

$$\det\,[(S^{**})_f/[S^{**}]_f] = \prod_{k=1}^{n} \frac{B_{f^2}^*(x_k)}{f^2(x_k)}. \tag{2.8}$$

Note that the notations here are slightly different from those in [11].

In Sections 4 and 5 we analyse the properties of the function ϕ_f^* and show that it is a rational arithmetical function in the case when f is a completely multiplicative function, a pseudo-unitarily semimultiplicative function or a strongly multiplicative function.

2.4 Bell series

The Bell series of a multiplicative function f to the base p is defined as the formal power series

$$f_p(x) = \sum_{n=0}^{\infty} f(p^n)x^n.$$

A multiplicative function is totally determined by its Bell series. It is known [25] that a multiplicative function f is a rational arithmetical function of order (r, s) if and only if for each prime p, $f_p(x)$ is of the form

$$f_p(x) = \frac{1 + a_1(p)x + a_2(p)x^2 + \cdots + a_s(p)x^s}{1 + b_1(p)x + b_2(p)x^2 + \cdots + b_r(p)x^r}, \tag{2.9}$$

where $a_1(p), a_2(p), \ldots, a_s(p), b_1(p), b_2(p), \ldots, b_r(p)$ are complex numbers. The Bell series of a rational arithmetical function of order (r, s) given as in (1.1) can be written as

$$f_p(x) = \frac{(1 - h_1(p)x)(1 - h_2(p)x) \cdots (1 - h_s(p)x)}{(1 - g_1(p)x)(1 - g_2(p)x) \cdots (1 - g_r(p)x)}. \tag{2.10}$$

3 Pseudo-unitarily semimultiplicative functions

In this section we examine properties of pseudo-unitarily semimultiplicative functions defined by Equation (2.4). It is shown in [11] that if f is pseudo-unitarily semimultiplicative and $f(1) = 1$, then f is multiplicative and

$$f(p^k) = f(p)^{k-2} f(p^2) \tag{3.1}$$

for all primes p and integers $k \geq 2$. The converse does not hold. In fact, let f be a multiplicative function such that for some prime p,

$$f(p) = 1,\ f(p^k) = 2,\ k \geq 2.$$

Then (3.1) holds. However,

$$f((p^2, p^3)^{**}) f([p^2, p^3]^{**}) = f(1) f(p^5) = 1 \cdot 2 \neq 2 \cdot 2 = f(p^2) f(p^3)$$

and thus (2.4) does not hold, which means that f is not pseudo-unitarily semimultiplicative.

We next give a necessary and sufficient condition for an arithmetical function f with $f(1) = 1$ to be pseudo-unitarily semimultiplicative.

Theorem 3.1. *An arithmetical function f is pseudo-unitarily semimultiplicative with $f(1) = 1$ if and only if f is multiplicative and for each prime p, one of the following conditions holds:*

1) $f(p^k) = f(p)^k$ for all $k \geq 1$,

2) $f(p) = 0$ and $f(p^k) = 0$ for all $k \geq 3$,

3) $f(p^k) = 0$ for all $k \geq 2$.

Proof. We should prove that

$$f((p^a,p^b)^{**})f([p^a,p^b]^{**}) = f(p^a)f(p^b) \tag{3.2}$$

for all $b \geq a \geq 0$ if and only if one of the conditions 1, 2, 3 holds.

Consider (3.2). If $a = b \geq 0$ or $a = 0$, $b \geq 1$, then

$$(p^a,p^b)^{**} = p^a, \quad [p^a,p^b]^{**} = p^b$$

and thus (3.2) holds. Therefore we may confine ourselves to the case $b > a > 0$. Then

$$(p^a,p^b)^{**} = 1, \quad [p^a,p^b]^{**} = p^{a+b}.$$

Therefore (3.2) reduces to

$$f(p^{a+b}) = f(p^a)f(p^b), \quad b > a > 0. \tag{3.3}$$

For $a = 1$, $b \geq 2$ (3.3) becomes

$$f(p^{b+1}) = f(p)f(p^b)$$

or equivalently

$$f(p^b) = f(p)^{b-2}f(p^2). \tag{3.4}$$

Let $a \geq 2$, $b > a$. Then $a+b, a, b \geq 2$ and we may apply (3.4) to obtain

$$f(p^{a+b}) = f(p)^{a+b-2}f(p^2)$$

and

$$f(p^a)f(p^b) = f(p)^{a+b-4}f(p^2)^2;$$

hence

$$f(p)^{a+b-2}f(p^2) = f(p)^{a+b-4}f(p^2)^2.$$

Now, we have proved that (3.2) is equivalent to

$$\begin{aligned} f(p^b) &= f(p)^{b-2}f(p^2), b \geq 2, && (3.5)\\ f(p)^{a+b-2}f(p^2) &= f(p)^{a+b-4}f(p^2)^2, \ a \geq 2, b > a. && (3.6) \end{aligned}$$

It is easy to see that (3.5) and (3.6) hold if and only one of the conditions 1, 2, 3 holds. □

From the condition 1 of Theorem 3.1 we obtain the following corollary.

Corollary 3.1. *If f is a completely multiplicative function, then it is pseudo-unitarily semimultiplicative with $f(1) = 1$.*

Remark 3.1. There exist pseudo-unitarily semimultiplicative functions f with $f(1) = 1$ that are not completely multiplicative. For example, let $f(p) = 1$ and $f(p^k) = 0$ for all $k \geq 2$ and for all primes p. Then f is pseudo-unitarily semimultiplicative (satisfying the condition 3 of Theorem 3.1) but it is not completely multiplicative.

Remark 3.2. Let f be a multiplicative function such that the condition 2 of Theorem 3.1 holds for all primes p, that is, $f(p) = 0$ and $f(p^k) = 0$ for all $k \geq 3$ and all primes p. Then

$$f_p(x) = 1 + f(p^2)x^2,$$

which means that f is a rational arithmetical function of order $(0, 2)$. In fact, $f = h_1^{-1} * h_2^{-1}$, where h_1 and h_2 are completely multiplicative functions such that $h_1(p) = \sqrt{-f(p^2)}$ and $h_2(p) = -\sqrt{-f(p^2)}$.

Remark 3.3. Let f be a multiplicative function such that the condition 3 of Theorem 3.1 holds for all primes p, that is, $f(p^k) = 0$ for all $k \geq 2$ and all primes p. Then

$$f_p(x) = 1 + f(p)x,$$

which means that f is a rational arithmetical function of order $(0, 1)$. In fact, $f = h_1^{-1}$, where h_1 is the completely multiplicative function with $h_1(p) = -f(p)$.

Theorem 3.2. *If f is a pseudo-unitarily semimultiplicative function with $f(1) = 1$, then f is a rational arithmetical function of order $(1, 0)$ or $(0, 2)$.*

Proof. The conditions 1, 2 and 3 of Theorem 3.1 imply that the Bell series of f is one of the following:

$$f_p(x) = \frac{1}{1 - f(p)x}, \quad f_p(x) = 1 + f(p^2)x^2, \quad f_p(x) = 1 + f(p)x.$$

Thus f is a rational arithmetical function of order $(1, 0)$, $(0, 2)$ or $(0, 1)$. Each rational arithmetical function of order $(0, 1)$ is also of order $(0, 2)$. This completes the proof. $\square$

Remark 3.4. It is shown in [11] that if f is a multiplicative function satisfying (3.1) for all primes p and integers $k \geq 2$, then f is a rational arithmetical function of order $(1, 2)$. The converse does not hold. For example, let f be a multiplicative function such that for some prime p,

$$f(p) = f(p^2) = 1, \; f(p^k) = 0, \; k \geq 3.$$

Then

$$f_p(x) = 1 + x + x^2,$$

which means that f is a rational arithmetical function of order $(0, 2)$ and therefore f is a rational arithmetical function of order $(1, 2)$. However, f does not satisfy (3.1). This function also serves as a counterexample to show that the converse of Theorem 3.2 does not hold.

Remark 3.5. In the study of the GCUD reciprocal pseudo-LCUM matrices we assume that f is always nonzero. It follows from Theorem 3.1 that under this condition f is a pseudo-unitarily semimultiplicative function with $f(1) = 1$ if and only if f is a completely multiplicative function.

Remark 3.6. Replacing the condition $f(1) = 1$ with the condition $f(1) \neq 0$ would lead to quasimultiplicative functions in Theorems 3.1 and 3.2. We do not present the details.

4 A generalization of the unitary analog of Euler's totient function

In this section we examine properties of the function ϕ_f^* defined in Equation (2.6). It should be noted that that the functions $\phi_{1/f}^*$ and $\phi_{f^2}^*$ appearing in the matrix formulas (2.7) and (2.8) also possess the properties presented below.

Theorem 4.1. *If f is a multiplicative function satisfying* (3.1) *for all primes p and integers $k \geq 2$, then ϕ_f^* is a rational arithmetical function of order* $(2, 3)$.

Proof. Since f is multiplicative, ϕ_f^* is also multiplicative. Therefore, it is enough to show that the Bell series of ϕ_f^* to the base p is of the form (2.9) with $r = 2$ and $s = 3$. It is easy to see that

$$\phi_f^*(p^k) = f(p^k) - 1, \ k \geq 1.$$

Thus

$$\begin{aligned} (\phi_f^*)_p(x) &= 1 + \sum_{k=1}^{\infty} (f(p^k) - 1) x^k \qquad (4.1) \\ &= 1 + \sum_{k=1}^{\infty} f(p^k) x^k - \sum_{k=1}^{\infty} x^k. \end{aligned}$$

On the basis of (3.1) we have

$$\begin{aligned}(\phi_f^*)_p(x) &= 1+f(p)x+\sum_{k=2}^{\infty}(f(p))^{k-2}f(p^2)x^k-\frac{x}{1-x}\\ &= 1+f(p)x+f(p^2)x^2\sum_{k=0}^{\infty}(f(p))^k x^k-\frac{x}{1-x}\\ &= 1+f(p)x+\frac{f(p^2)x^2}{1-f(p)x}-\frac{x}{1-x}.\end{aligned}$$

Thus

$$(\phi_f^*)_p(x)=\frac{(1-f^2(p)x^2)(1-x)+f(p^2)x^2(1-x)-(1-f(p)x)x}{(1-f(p)x)(1-x)}. \tag{4.2}$$

The nominator is a polynomial of order 3, and the denominator is a polynomial of order 2. Therefore ϕ_f^* is a rational arithmetical function of order $(2,3)$. □

Theorem 4.2. *If f is a completely multiplicative function, then ϕ_f^* is a rational arithmetical function of order $(2,2)$.*

Proof. Now, $f(p^2)=f(p)^2$, and thus (4.2) reduces to

$$(\phi_f^*)_p(x)=\frac{1-2x+f(p)x^2}{(1-f(p)x)(1-x)}, \tag{4.3}$$

which shows that ϕ_f^* is a rational arithmetical function of order $(2,2)$. □

Theorem 4.3. *If f is a pseudo-unitarily semimultiplicative function with $f(1)=1$, then ϕ_f^* is a rational arithmetical function of order $(2,3)$.*

Proof. If f is a pseudo-unitarily semimultiplicative function with $f(1)=1$, then it satisfies (3.1). Thus, Theorem 4.3 follows from Theorem 4.1. □

Remark 4.1. Assume that f is a pseudo-unitarily semimultiplicative function with $f(1)=1$. If f satisfies Condition 1 of Theorem 3.1, then (4.2) reduces to (4.3). If f satisfies Condition 2 of Theorem 3.1, then (4.2) reduces to

$$(\phi_f^*)_p(x)=\frac{(1-x)+f(p^2)x^2(1-x)-x}{1-x}.$$

If f satisfies Condition 3 of Theorem 3.1, then (4.2) reduces to

$$(\phi_f^*)_p(x)=\frac{(1-f^2(p)x^2)(1-x)-(1-f(p)x)x}{(1-f(p)x)(1-x)}.$$

In each case, the nominator is a polynomial of order ≤ 3, and the denominator is a polynomial of order ≤ 2. This also shows that if f is a pseudo-unitarily semimultiplicative function with $f(1)=1$, then ϕ_f^* is a rational arithmetical function of order $(2,3)$.

5 Strongly multiplicative functions

In this section we show that each strongly multiplicative function f and the related analog ϕ^*_f of Euler's totient function are rational arithmetical functions of order $(1,1)$, i.e., totients.

Theorem 5.1. *A strongly multiplicative function f is a rational arithmetical function of order $(1,1)$ with $g_1(p) = 1$ and $h_1(p) = 1 - f(p)$ for all primes p given in terms of Eq.* (1.1).

Proof. We have

$$f_p(x) = 1 + \sum_{k=1}^{\infty} f(p)x^k = 1 + \frac{f(p)x}{1-x} = \frac{1-(1-f(p))x}{1-x}.$$

This proves the theorem. □

Theorem 5.2. *If f is a strongly multiplicative function, then ϕ^*_f is a rational arithmetical function of order $(1,1)$ with $g_1(p) = 1$ and $h_1(p) = 2 - f(p)$ for all primes p given in terms of Eq.* (1.1).

Proof. From (4.1) we obtain

$$(\phi^*_f)_p(x) = 1 + \sum_{k=1}^{\infty}(f(p)-1)x^k = 1 + \frac{(f(p)-1)x}{1-x} = \frac{1-(2-f(p))x}{1-x}.$$

This proves the theorem. □

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