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EXISTENCE RESULTS AND ITERATIVE METHOD FOR SOLVING SYSTEMS OF BEAMS EQUATIONS

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Abstract

In this paper, we propose a method for investigating the solvability and iterative solution of coupled beams equations with fully nonlinear terms. Differently from other authors, we reduce the problem to an operator equation for the right-hand side functions. The advantage of the proposed method is that it does not require any Nagumo-type conditions for the nonlinear terms. Some examples, where exact solution of the problem are known or not, demonstrate the effectiveness of the obtained theoretical results.

1. Introduction

In the beginning of the 2017 Minhós and Coxe [7] for the first time considered the fully fourth order coupled system

$$\begin{cases} u^{(4)}(t) &= f(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)), \\ v^{(4)}(t) &= h(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)) \end{cases} \quad (1)$$

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with the boundary conditions

$$\begin{cases} u(0) &= u'(0) = u''(0) = u''(1) = 0, \\ v(0) &= v'(0) = v''(0) = v''(1) = 0. \end{cases} \tag{2}$$

They gave sufficient conditions for the solvability of the system by using the lower and upper solutions method and the Schauder fixed point theorem. The proof of this result is very cumbersome and complicated. It requires Nagumo-type conditions for the sum of the functions f and h. Furthermore, it contained some errors due to the use of non-correct definition of the norm of the space $C^3 \times C^3$. The necessary corrections are made in the Corrigendum in [8].

Motivated by the above fact, in this paper we study the system (1)-(2) by another method, namely by reducing it to an operator equation for the pair of nonlinear terms but not for the pair of the functions to be sought (u, v). Without any Nagumo-type conditions and under some easily verified conditions we establish the existence and uniqueness of a solution of the system (1)-(2). Besides, we also prove the property of sign preserving of the solution and the convergence of an iterative method for finding the solution. Some examples, where exact solutions of the problem are known or not, demonstrate the effectiveness of the obtained theoretical results. The method used here is a further development of the method proposed in our recent works [1, 2, 3, 4].

Note that some particular cases of the system (1) were studied before, namely, in [5, 10] the authors considered the equations containing only even order derivatives associated with the boundary conditions different from (2). Under very complicated conditions, by using a fixed point index theorem on cones, the authors obtained the existence of positive solutions. But it should be emphasized that the obtained results are of pure theoretical character because no examples of existing solutions are shown.

The paper is organized as follows. In Section 2 we consider the existence and uniqueness of a solution of the problem (1)-(2) and its sign preservation. In Section 3 we study an iterative method for solving the problem, where the convergence of iterations is proved. Section 4 is devoted to some examples for demonstrating the applicability and efficiency of our approach. Finally, Section 5 is Conclusion.

2. Existence of a solution

To investigate the problem (1)-(2), for $u, v \in C^4[0, 1]$ we set

$$\begin{aligned} \varphi(t) &= f(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)), \\ \psi(t) &= h(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)), \\ w &= (\varphi, \psi)^T. \end{aligned} \tag{3}$$

Then the problem becomes

$$\begin{cases} u^{(4)}(t) = \varphi(t), & 0 < t < 1, \\ v^{(4)}(t) = \psi(t), & 0 < t < 1 \end{cases} \tag{4}$$

with the boundary conditions

$$\begin{cases} u(0) = u'(0) = u''(0) = u''(1) = 0, \\ v(0) = v'(0) = v''(0) = v''(1) = 0. \end{cases} \tag{5}$$

The problem

$$\begin{cases} u^{(4)}(t) = \varphi(t), & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u''(1) = 0 \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)\varphi(s)ds, \tag{6}$$

where $G(t,s)$ is the Green function

$$G(t,s) = \begin{cases} -\dfrac{s^3}{6} + \dfrac{s^2 t}{2} - \dfrac{st^2}{2} + \dfrac{st^3}{6}, & 0 \le s \le t \le 1, \\ \dfrac{st^3}{6} - \dfrac{t^3}{6}, & 0 \le t \le s \le 1. \end{cases}$$

Similarly, the problem

$$\begin{cases} v^{(4)}(t) = \psi(t), & 0 < t < 1, \\ v(0) = v'(0) = v''(0) = v''(1) = 0 \end{cases}$$

has a unique solution

$$v(t) = \int_0^1 G(t,s)\psi(s)ds. \tag{7}$$

Therefore, the solution of the problem (1)-(2) can be represented in the form

$$\begin{cases} u(t) = \int_0^1 G(t,s)\varphi(s)ds, \\ v(t) = \int_0^1 G(t,s)\psi(s)ds, \end{cases} \tag{8}$$

where $\varphi(t), \psi(t)$ are defined by (3) and the pair of functions $(u(t), v(t)) \in E$ with $E = C^4[0,1] \times C^4[0,1])$.
From (8) it follows

$$\begin{cases} u'(t) = \int_0^1 G_1(t,s)\varphi(s)ds, \\ v'(t) = \int_0^1 G_1(t,s)\psi(s)ds, \end{cases} \tag{9}$$

where we denote

$$G_1(t,s) = \begin{cases} \dfrac{s^2}{2} - st + \dfrac{st^2}{2}, & 0 \le s \le t \le 1 \\ \dfrac{st^2}{2} - \dfrac{t^2}{2}, & 0 \le t \le s \le 1. \end{cases}$$

It is easy to verify that

$$\max_{0\le t\le 1} \int_0^1 |G(t,s)|ds = \frac{1}{24}, \quad \max_{0\le t\le 1} \int_0^1 |G_1(t,s)|ds = \frac{1}{12}, \tag{10}$$

Now we set

$$\begin{aligned} &u_1(t) = u'(t), \quad u_2(t) = u''(t), \quad u_3(t) = u'''(t), \\ &v_1(t) = v'(t), \quad v_2(t) = v''(t), \quad v_3(t) = v'''(t). \\ &U(t) = (u(t), u_1(t), u_2(t), u_3(t)), \quad V(t) = (v(t), v_1(t), v_2(t), v_3(t)). \end{aligned} \tag{11}$$

Then the problem (4)-(5) is reduced to a sequence of the problems

$$\begin{cases} u_2''(t) = \varphi(t), \quad 0 < t < 1, \\ u_2(0) = u_2(1) = 0, \end{cases} \tag{12}$$

$$\begin{cases} u''(t) = u_2(t), \quad 0 < t < 1, \\ u(0) = u'(0) = 0. \end{cases} \tag{13}$$

$$\begin{cases} v_2''(t) = \psi(t), \quad 0 < t < 1, \\ v_2(0) = v_2(1) = 0, \end{cases} \tag{14}$$

$$\begin{cases} v''(t) = v_2(t), \quad 0 < t < 1, \\ v(0) = v'(0) = 0. \end{cases} \tag{15}$$

Clearly, the solutions u_2 and u of the problems (12)-(13) depend on φ, that is, $u_2 = u_{2\varphi}(t)$, $u = u_\varphi(t)$. Similarly, the solutions v_2 and v of the problems (14)-(15) depend on ψ, that is, $v_2 = v_{2\psi}(t)$, $v = v_\psi(t)$. Therefore, φ and ψ must satisfy equations

$$\begin{cases} \varphi = Aw, \\ \psi = Bw, \end{cases} \tag{16}$$

where A and B are nonlinear operators defined by

$$\begin{cases} (Aw)(t) = f(t, U_\varphi(t), V_\psi(t)), \\ (Bw)(t) = h(t, U_\varphi(t), V_\psi(t)), \end{cases} \tag{17}$$

U_φ, V_ψ being defined by (11) with the corresponding subscripts for each components. Then, for w we have the equation

$$w = Tw, \tag{18}$$

where T is defined by

$$Tw = \begin{pmatrix} Aw \\ Bw \end{pmatrix}. \tag{19}$$

Now, for each number $M > 0$ denote

$$\mathcal{D}_M = \{(t, u, u_1, u_2, u_3, v, v_1, v_2, v_3)\},$$

where

$$\begin{aligned} 0 \le t \le 1, \quad |u| \le \frac{M}{24}, \quad |u_1| \le \frac{M}{12}, \quad |u_2| \le \frac{M}{8}, \quad |u_3| \le \frac{M}{2}, \\ |v| \le \frac{M}{24}, \quad |v_1| \le \frac{M}{12}, \quad |v_2| \le \frac{M}{8}, \quad |v_3| \le \frac{M}{2} \end{aligned}$$

and by $B[O, M]$ we denote the closed ball centered at O with the radius M in the space $F = (C[0, 1])^2$, i.e.,

$$B[0, M] = \{w \in F : \|w\|_F \le M\}$$

with the norms

$$\|w\|_F = \max\{\|\varphi\|, \|\psi\|\},$$

$$\|\varphi\| = \max_{0\le t\le 1} |\varphi(t)|, \quad \|\psi\| = \max_{0\le t\le 1} |\psi(t)|.$$

Theorem 1. *Suppose that there exists a number $M > 0$ such that the functions $f(t, U, V)$ and $h(t, U, V)$ are continuous and*

$$\max\{|f(t, U, V)|, |h(t, U, V)|\} \le M \tag{20}$$

for any $(t, U, V) \in \mathcal{D}_M$.

Then, the problem (1)-(2) has a solution satisfying the estimates

$$\begin{aligned} |u(t)| \le \frac{M}{24}, \quad |u'(t)| \le \frac{M}{12}, \quad |u''(t)| \le \frac{M}{8}, \quad |u'''(t)| \le \frac{M}{2}, \\ |v(t)| \le \frac{M}{24}, \quad |v'(t)| \le \frac{M}{12}, \quad |v''(t)| \le \frac{M}{8}, \quad |v'''(t)| \le \frac{M}{2}. \end{aligned}$$

for any $0 \le t \le 1$.

Proof. Since the problem (4) is reduced to the operator equation (18), the theorem will be proved if we show that this operator equation has a solution. For this purpose, first we show that the operator T defined by (19) maps the closed ball $B[0, M]$ into itself.

Let w be an element in $B[O, M]$. Then, from (8)-(10) it is easy to obtain

$$\|u\| \le \frac{1}{24}\|\varphi\|, \quad \|u'\| \le \frac{1}{12}\|\varphi\|, \quad \|v\| \le \frac{1}{24}\|\psi\|, \quad \|v'\| \le \frac{1}{12}\|\psi\|. \tag{21}$$

For estimating $\|u''\|$ and $\|u'''\|$ we notice that the solutions of the problem (12), (14) can be represented in the form

$$\begin{cases} u_2(t) = \int_0^1 G_2(t,s)\varphi(s)ds, \\ v_2(t) = \int_0^1 G_2(t,s)\psi(s)ds, \end{cases} \tag{22}$$

where $G_2(t,s)$ is the Green function

$$G_2(t,s) = \begin{cases} -s+st, & 0 \le s \le t \le 1, \\ st-t, & 0 \le t \le s \le 1. \end{cases}$$

It is easy to verify that

$$\max_{0\le t\le 1} \int_0^1 |G_2(t,s)|ds = \frac{1}{8}. \tag{23}$$

Therefore, taking into account (22) we have

$$\|u''\| = \|u_2\| \le \frac{1}{8}\|\varphi\|, \quad \|v''\| = \|v_2\| \le \frac{1}{8}\|\psi\|. \tag{24}$$

Now, rewrite (22) in the form

$$\begin{cases} u_2(t) = \int_0^t(-s+st)\varphi(s)ds + \int_t^1(st-t)\varphi(s)ds, \\ v_2(t) = \int_0^t(-s+st)\psi(s)ds + \int_t^1(st-t)\psi(s)ds. \end{cases} \tag{25}$$

From here we obtain

$$\begin{cases} u_3(t) = u_2'(t) = \int_0^t s\varphi(s)ds + \int_t^1(s-1)\varphi(s)ds = \int_0^1 G_3(t,s)\varphi(s)ds, \\ v_3(t) = v_2'(t) = \int_0^t s\psi(s)ds + \int_t^1(s-1)\psi(s)ds = \int_0^1 G_3(t,s)\psi(s)ds, \end{cases} \tag{26}$$

where $G_3(t,s)$ is the function continuous in the square $[0,1]^2$ except for the line $s=t$

$$G_3(t,s) = \begin{cases} s, & 0 \le s < t \le 1, \\ s-1, & 0 \le t < s \le 1. \end{cases}$$

Hence,

$$\|u'''\| = \|u_3\| \le \frac{M}{2}\|\varphi\|, \quad \|v'''\| = \|v_3\| \le \frac{M}{2}\|\psi\|. \tag{27}$$

Taking into account (21), (24), (27) and $\|w\| = \max\{\|\varphi\|, \|\psi\|\} \le M$ we have

$$\begin{aligned} &\|u\| \le \frac{M}{24}, \quad \|u_1\| \le \frac{M}{12}, \quad \|u_2\| \le \frac{M}{8}, \quad \|u_3\| \le \frac{M}{2}, \\ &\|v\| \le \frac{M}{24}, \quad \|v_1\| \le \frac{M}{12}, \quad \|v_2\| \le \frac{M}{8}, \quad \|v_3\| \le \frac{M}{2}. \end{aligned} \tag{28}$$

Therefore, $(t,U,V)\in\mathcal{D}_M$ for $t\in[0,1]$. From the definition of T by (19), (17) and the condition (20), we have $Tw\in B[0,M]$, i.e., the operator T maps the ball $B[0,M]$ into itself.

Next, we prove that the operator T is a compact one in the space F.

Providing the subscript φ for u and ψ for v in the formulas (8), (9), (22) and (26) we have

$$\begin{cases} u_\varphi(t)=\int_0^1 G(t,s)\varphi(s)ds,\\ v_\psi(t)=\int_0^1 G(t,s)\psi(s)ds,\end{cases} \tag{29}$$

$$\begin{cases} u'_\varphi(t)=\int_0^1 G_1(t,s)\varphi(s)ds,\\ v'_\psi(t)=\int_0^1 G_1(t,s)\psi(s)ds,\end{cases} \tag{30}$$

$$\begin{cases} u''_\varphi(t)=\int_0^1 G_2(t,s)\varphi(s)ds,\\ v''_\psi(t)=\int_0^1 G_2(t,s)\psi(s)ds.\end{cases} \tag{31}$$

$$\begin{cases} u'''_\varphi(t)=\int_0^1 G_3(t,s)\varphi(s)ds,\\ v'''_\psi(t)=\int_0^1 G_3(t,s)\psi(s)ds.\end{cases} \tag{32}$$

According to [6, Sec. 31] the integral operators in (29)-(32) which put each pair of functions $(\varphi,\psi)\in F$ in correspondence to the pairs of functions $(u_\varphi,v_\psi),(u'_\varphi,v'_\psi),(u''_\varphi,v''_\psi),(u'''_\varphi,v'''_\psi)$, are compact operators. Therefore, in view of the continuity of the functions $f(t,U,V),h(t,U,V)$ it is easy to deduce that the operator T defined by (19) is compact operator in the space F. Thus, T is a compact operator from the closed ball $B[0,M])$ into itself. By the Schauder Fixed Point Theorem [9] the operator equation (18) has a solution. The theorem is proved. □

We now denote

$$\mathcal{D}_M^{++}=\{(t,u,u_1,u_2,u_3,v,v_1,v_2,v_3)\},$$

where

$$\begin{aligned} 0\le t\le 1,\ 0\le u\le \frac{M}{24},\ 0\le u_1\le\frac{M}{12},\ 0\le u_2\le\frac{M}{8},\ |u_3|\le\frac{M}{2},\\ 0\le v\le \frac{M}{24},\ 0\le v_1\le\frac{M}{12},\ 0\le v_2\le\frac{M}{8},\ |v_3|\le\frac{M}{2},\end{aligned}$$

and

$$S_M^{--}=\{w\in F\mid -M\le\varphi(t)\le 0,\ -M\le\psi(t)\le 0\}.$$

Similarly, we introduce the notations $D_M^{--},\ S_M^{++},\ D_M^{+-},\ S_M^{-+},\ D_M^{-+},\ S_M^{+-}$ as follows

$$\mathcal{D}_M^{--}=\{(t,u,u_1,u_2,u_3,v,v_1,v_2,v_3)\},$$

where

$$0 \le t \le 1,\ -\frac{M}{24} \le u \le 0, -\frac{M}{12} \le u_1 \le 0,\ -\frac{M}{8} \le u_2 \le 0,\ |u_3| \le \frac{M}{2},$$
$$-\frac{M}{24} \le v \le 0, -\frac{M}{12} \le v_1 \le 0,\ -\frac{M}{8} \le v_2 \le 0,\ |v_3| \le \frac{M}{2},$$

and

$$S_M^{++} = \{w \in F \mid 0 \le \varphi(t) \le M,\ 0 \le \psi(t) \le M\};$$

$$\mathcal{D}_M^{+-} = \{(t, u, u_1, u_2, u_3, v, v_1, v_2, v_3)\},$$

where

$$0 \le t \le 1,\ 0 \le u \le \frac{M}{24}, 0 \le u_1 \le \frac{M}{12},\ 0 \le u_2 \le \frac{M}{8},\ |u_3| \le \frac{M}{2},$$
$$-\frac{M}{24} \le v \le 0, -\frac{M}{12} \le v_1 \le 0,\ -\frac{M}{8} \le v_2 \le 0,\ |v_3| \le \frac{M}{2},$$

and

$$S_M^{-+} = \{w \in F \mid -M \le \varphi(t) \le 0,\ 0 \le \psi(t) \le M\};$$

$$\mathcal{D}_M^{-+} = \{(t, u, u_1, u_2, u_3, v, v_1, v_2, v_3)\},$$

where

$$0 \le t \le 1,\ -\frac{M}{24} \le u \le 0,\ -\frac{M}{12} \le u_1 \le 0,\ -\frac{M}{8} \le u_2 \le 0,\ |u_3| \le \frac{M}{2},$$
$$0 \le v \le \frac{M}{24},\ 0 \le v_1 \le \frac{M}{12},\ 0 \le v_2 \le \frac{M}{8},\ |v_3| \le \frac{M}{2},$$

and

$$S_M^{+-} = \{w \in F \mid 0 \le \varphi(t) \le M,\ -M \le \psi(t) \le 0\}.$$

Now consider some particular cases of Theorem 1.

Theorem 2. *(Positivity or negativity of solution)*
(i) Suppose that in $\mathcal{D}_M^{++}$ the functions f, h are continuous and

$$-M \le f(t, U, V) \le 0, \quad -M \le h(t, U, V) \le 0. \tag{33}$$

Then, the problem (1)-(2) has a solution $(u(t), v(t))$ with the properties $u(t) \ge 0, u'(t) \ge 0, u''(t) \ge 0, v(t) \ge 0, v'(t) \ge 0, v''(t) \ge 0$.
(ii) Suppose that in $\mathcal{D}_M^{--}$ the functions f, h are continuous and

$$0 \le f(t, U, V) \le M, \quad 0 \le h(t, U, V) \le M. \tag{34}$$

Then, the problem (1)-(2) *has a solution* $(u(t), v(t))$ *with the properties* $u(t) \leq 0, u'(t) \leq 0, u''(t) \leq 0, v(t) \leq 0, v'(t) \leq 0, v''(t) \leq 0$.
(iii) Suppose that in $\mathcal{D}_M^{+-}$ *the functions* f, h *are continuous and*

$$-M \leq f(t, U, V) \leq 0, \quad 0 \leq h(t, U, V) \leq M. \tag{35}$$

Then, the problem (1)-(2) *has a solution* $(u(t), v(t))$ *with the properties* $u(t) \geq 0, u'(t) \geq 0, u''(t) \geq 0, v(t) \leq 0, v'(t) \leq 0, v''(t) \leq 0$.
(iv) Suppose that in $\mathcal{D}_M^{-+}$ *the functions* f, h *are continuous and*

$$0 \leq f(t, U, V) \leq M, \quad -M \leq h(t, U, V) \leq 0. \tag{36}$$

Then, the problem (1)-(2) *has a solution* $(u(t), v(t))$ *with the properties* $u(t) \leq 0, u'(t) \leq 0, u''(t) \leq 0, v(t) \geq 0, v'(t) \geq 0, v''(t) \geq 0$.

Proof.

The existence of a solution $(u(t), v(t))$ of the problem in the case (i) is proved in a similar way as in Theorem 1, where instead of $\mathcal{D}_M$ and $B[0, M]$ there stand $\mathcal{D}_M^{++}$ and S_M^{--}. The sign of $u(t), v(t)$ and their derivatives are deduced from the representations (8), (9), (22) if taking into account the sign of $\varphi(s), \psi(s)$ and that $G(t, s), G_1(t, s), G_2(t, s)$ are nonpositive functions.

The proof of the cases (ii), (iii) and (iv) is similar to that of (i), where instead of the pair $(\mathcal{D}_M^{++},\ S_M^{--})$ there stand the pairs $(\mathcal{D}_M^{--},\ S_M^{++})$, $(\mathcal{D}_M^{+-},\ S_M^{-+})$ and $(\mathcal{D}_M^{-+},\ S_M^{+-})$, respectively.□

Now we denote

$$\begin{aligned}
&u_1^i = (u^i)', \quad u_2^i = (u^i)'', \quad u_3^i = (u^i)''';\\
&v_1^i = (v^i)', \quad v_2^i = (v^i)'', \quad v_3^i = (v^i)''';\\
&U^i = (u^i, u_1^i, u_2^i, u_3^i), \quad V^i = (v^i, v_1^i, v_2^i, v_3^i);\\
&\varphi_i = f(t, U^i, V^i), \quad \psi_i = h(t, U^i, V^i); (i = 1.2).
\end{aligned}$$

Theorem 3. *(Uniqueness of solution) Suppose that there exist numbers* $c_i, d_i \geq 0$ $(i = 0, ..., 7)$ *such that*

$$\begin{aligned}
&|f(t, U^2, V^2) - f(t, U^1, V^1)|\\
&\leq c_0|u^2 - u^1| + c_1|u_1^2 - u_1^1| + c_2|u_2^2 - u_2^1| + c_3|u_3^2 - u_3^1|\\
&+ c_4|v^2 - v^1| + c_5|v_1^2 - v_1^1| + c_6|v_2^2 - v_2^1| + c_7|v_3^2 - v_3^1|,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&|h(t, U^2, V^2) - h(t, U^1, V^1)|\\
&\leq d_0|u^2 - u^1| + d_1|u_1^2 - u_1^1| + d_2|u_2^2 - u_2^1| + d_3|u_3^2 - u_3^1|\\
&+ d_4|v^2 - v^1| + d_5|v_1^2 - v_1^1| + d_6|v_2^2 - v_2^1| + d_7|v_3^2 - v_3^1|,
\end{aligned} \tag{38}$$

for any $(t,U,V),(t,U^i,V^i)\in[0,1]\times\mathbb{R}^8$ $(i=1,2)$, *and*

$$q:=\max\{q_1,q_2\}<1 \tag{39}$$

with

$$\begin{aligned}q_1&:=\frac{c_0+c_4}{24}+\frac{c_1+c_5}{12}+\frac{c_2+c_6}{8}+\frac{c_3+c_7}{2},\\ q_2&:=\frac{d_0+d_4}{24}+\frac{d_1+d_5}{12}+\frac{d_2+d_6}{8}+\frac{d_3+d_7}{2}.\end{aligned}$$

Then the solution of the problem (1)-(2) *is unique if it exists.*

Proof. Suppose the problem has two solutions $(u^1(t),v^1(t))$ and $(u^2(t),v^2(t))$. Due to the estimates (28) we have

$$\begin{aligned}&\|u^2-u^1\|\le\frac{1}{24}\|\varphi_2-\varphi_1\|,\quad \|u_1^2-u_1^1\|\le\frac{1}{12}\|\varphi_2-\varphi_1\|,\\ &\|u_2^2-u_2^1\|\le\frac{1}{8}\|\varphi_2-\varphi_1\|,\quad \|u_3^2-u_3^1\|\le\frac{1}{2}\|\varphi_2-\varphi_1\|\\ &\|v^2-v^1\|\le\frac{1}{24}\|\psi_2-\psi_1\|,\quad \|v_1^2-v_1^1\|\le\frac{1}{12}\|\psi_2-\psi_1\|,\\ &\|v_2^2-v_2^1\|\le\frac{1}{8}\|\psi_2-\psi_1\|,\quad \|v_3^2-v_3^1\|\le\frac{1}{2}\|\psi_2-\psi_1\|.\end{aligned} \tag{40}$$

From (37), (38) and the above estimates we have

$$\begin{aligned}\|w_2-w_1\|&=\max\{\|f(t,U_2,V_2)-f(t,U_1,V_1)\|,\|h(t,U_2,V_2)-h(t,U_1,V_1)\|\}\\ &\le\max\{q_1\max\{\|\varphi_2-\varphi_1\|,\|\psi_2-\psi_1\|\},q_2\max\{\|\varphi_2-\varphi_1\|,\|\psi_2-\psi_1\|\}\}\\ &\le q\|w_2-w_1\|\end{aligned} \tag{41}$$

with

$$\begin{aligned}q_1&=\frac{c_0+c_4}{24}+\frac{c_1+c_5}{12}+\frac{c_2+c_6}{8}+\frac{c_3+c_7}{2},\\ q_2&=\frac{d_0+d_4}{24}+\frac{d_1+d_5}{12}+\frac{d_2+d_6}{8}+\frac{d_3+d_7}{2},\\ q&=\max\{q_1,q_2\}.\end{aligned}$$

Since $q<1$ the inequality (41) occurs only in the case $w_2=w_1$. This implies $u_2=u_1$ and $v_2=v_1$. Thus, the theorem is proved. □

Theorem 4. *Assume that there exist numbers* $M,c_i,d_i\ge 0$ $(i=0,..,7)$ *such that*

$$\max\{|f(t,U,V)|,|h(t,U,V)|\}\le M, \tag{42}$$

$$\begin{aligned}&|f(t,U^2,V^2)-f(t,U^1,V^1)|\\ &\le c_0|u^2-u^1|+c_1|u_1^2-u_1^1|+c_2|u_2^2-u_2^1|+c_3|u_3^2-u_3^1|\\ &+c_4|v^2-v^1|+c_5|v_1^2-v_1^1|+c_6|v_2^2-v_2^1|+c_7|v_3^2-v_3^1|,\end{aligned} \tag{43}$$

$$\begin{aligned}
&|h(t,U^2,V^2)-h(t,U^1,V^1)| \\
&\le d_0|u^2-u^1|+d_1|u_1^2-u_1^1|+d_2|u_2^2-u_2^1|+d_3|u_3^2-u_3^1| \\
&+d_4|v^2-v^1|+d_5|v_1^2-v_1^1|+d_6|v_2^2-v_2^1|+d_7|v_3^2-v_3^1|,
\end{aligned} \tag{44}$$

for any $(t,U,V),(t,U^i,V^i)\in\mathcal{D}_M$ $(i=1,2)$, *and*

$$q:=\max\{q_1,q_2\}<1 \tag{45}$$

with

$$\begin{aligned}
q_1 &:= \frac{c_0+c_4}{24}+\frac{c_1+c_5}{12}+\frac{c_2+c_6}{8}+\frac{c_3+c_7}{2}, \\
q_2 &:= \frac{d_0+d_4}{24}+\frac{d_1+d_5}{12}+\frac{d_2+d_6}{8}+\frac{d_3+d_7}{2}.
\end{aligned}$$

Then, the problem (1)-(2) *has a unique solution* $(u(t),v(t))$ *such that*

$$\begin{aligned}
&|u(t)|\le\frac{M}{24}, \quad |u'(t)|\le\frac{M}{12}, \quad |u''(t)|\le\frac{M}{8}, \quad |u'''(t)|\le\frac{M}{2}, \\
&|v(t)|\le\frac{M}{24}, \quad |v'(t)|\le\frac{M}{12}, \quad |v''(t)|\le\frac{M}{8}, \quad |v'''(t)|\le\frac{M}{2}.
\end{aligned}$$

for any $0\le t\le 1$.

Proof. Under the assumption (42), as proven in Theorem 1, the operator T, defined by (19), maps the closed ball $B[0,M]$ into itself. The Lipschitz condition (43), (44) as shown in the proof of Theorem 3, implies that T is a contraction mapping. Thus, T is a contraction mapping from the closed ball $B[0,M]$ into itself. By the contraction principle the operator T has a unique fixed point in $B[0,M]$, which corresponds to a unique solution $(u(t),v(t)$ of the problem (1)-(2).

The estimations for $u(t),v(t)$ and their derivatives are obtained as in Theorem 1. Thus, the theorem is proved. □

Remark that in Theorem 3 the Lipschitz condition is required to be satisfied in $[0,1]\times\mathbb{R}^8$, while in Theorem 4, due to the condition (42) it is required only in $\mathcal{D}_M$.

3. Iterative method

Consider the following iterative process:

1. Given
$$w_0=(\varphi_0(t),\psi_0(t))\in B[0,M]. \tag{46}$$

2. Knowing $w_k=(\varphi_k,\psi_k)$ $(k=0,1,...)$ solve consecutively problems
$$\begin{cases} u''_{2k}=\varphi_k(t), \quad 0<t<1, \\ u_{2k}(0)=u_{2k}(1)=0, \end{cases} \tag{47}$$

$$\begin{cases} u_k'' = u_{2k}(t), \quad 0 < t < 1, \\ u_k(0) = u_k'(0) = 0, \end{cases} \tag{48}$$

$$\begin{cases} v_{2k}'' = \psi_k(t), \quad 0 < t < 1, \\ v_{2k}(0) = v_{2k}(1) = 0, \end{cases} \tag{49}$$

$$\begin{cases} v_k'' = v_{2k}(t), \quad 0 < t < 1, \\ v_k(0) = v_k'(0) = 0. \end{cases} \tag{50}$$

3. Update

$$\begin{cases} \varphi_{k+1} = f(t, U_k, V_k), \\ \psi_{k+1} = h(t, U_k, V_k). \end{cases} \tag{51}$$

Set $p_k = \dfrac{q^k}{1-q}\|w_1 - w_0\|_F$. We obtain the following result

Theorem 5. *Under the assumptions of Theorem 4 the above iterative method converges with the rate of geometric progression and there hold the estimates*

$$\begin{aligned} \|s_k - s\|_F \le \frac{p_k}{24}, \quad \|s_k' - s'\|_F \le \frac{p_k}{12}, \\ \|s_k'' - s''\|_F \le \frac{p_k}{8}, \quad \|s_k''' - s'''\|_F \le \frac{p_k}{2}, \end{aligned} \tag{52}$$

where $s = (u, v)$ is the exact solution of the problem (1)-(2).

Proof. Notice that the above iterative method is the successive iteration method for finding the fixed point of the operator T with the initial approximation (46) belonging to $B[O, M]$. Therefore, it converges with the rate of geometric progression and there is the estimate

$$\|w_k - w\|_F \le \frac{q^k}{1-q}\|w_1 - w_0\|_F. \tag{53}$$

Combining this with the estimates of the type (40) we obtain (52), and the theorem is proved. □

Below we illustrate the obtained theoretical results on some examples, where the exact solution of the problem is known or unknown.

To numerically realize the iterative method we use the difference schemes of fourth order accuracy for the problems (47)- (50) on uniform grids $\overline{\omega}_h = \{x_i = ih,\, i = 0, 1, ..., N;\, h = 1/N\}$. The iterations are performed until $e_k = \|s_k - s_{k-1}\| \le 10^{-16}$. In the tables of results of computation n is the number of grid points, $error = \|s_k - s_d\|$, where $s_d = (u_d, v_d)$ is the exact solution of the problem (1)- (2).

4. Examples

In this section we give some examples for demonstrating the applicability of the obtained theoretical results. First, we consider an example for the case of known exact solution.

Example 1. *Consider the boundary value problem*

$$\begin{cases} u^{(4)}(t)=\cos\left(-\dfrac{\sin\pi t}{\pi^2}-u''(t)\right)-\left(\dfrac{u'''(t)}{3}\right)^3-v^2(t)-\dfrac{v'(t)}{5}+\dfrac{\sin\pi t}{5\pi^5}+\dfrac{t^2}{5\pi^4}-\dfrac{t}{5\pi^4}\\ \quad+\left(-\dfrac{\cos\pi t}{\pi^6}+\dfrac{t^3}{3\pi^4}-\dfrac{t^2}{2\pi^4}+\dfrac{1}{\pi^6}\right)^2+\sin\pi t-\left(\dfrac{\cos\pi t}{3\pi}\right)^3-1,\quad 0<t<1\\ v^{(4)}(t)=-u^2-u'+\cos\left(\dfrac{\cos\pi t}{\pi^4}+\dfrac{2t}{\pi^4}-\dfrac{1}{\pi^4}-v''\right)-\dfrac{v'''(t)}{3}+\left(\dfrac{\sin\pi t}{\pi^4}-\dfrac{t}{\pi^3}\right)^2\\ \quad+\dfrac{1}{3}\left(-\dfrac{\sin\pi t}{\pi^3}+\dfrac{2}{\pi^4}\right)-\dfrac{\cos\pi t}{\pi^2}+\dfrac{\cos\pi t}{\pi^3}-\dfrac{1}{\pi^3}-1,\quad 0<t<1\\ u(0)=u'(0)=u''(0)=u''(1)=0,\\ v(0)=v'(0)=v''(0)=v''(1)=0. \end{cases}$$

The exact solution of the problem is

$$\begin{cases} u(t)=\dfrac{\sin(\pi t)}{\pi^4}-\dfrac{t}{\pi^3},\\ v(t)=-\dfrac{\cos(\pi t)}{\pi^6}+\dfrac{t^3}{3\pi^4}-\dfrac{t^2}{2\pi^4}+\dfrac{1}{\pi^6}. \end{cases}$$

In this example

$$\begin{aligned} f(t,U,V)=&\cos\left(-\frac{\sin\pi t}{\pi^2}-u_2\right)-\left(\frac{u_3}{3}\right)^3-v^2-\frac{v_1}{5}+\frac{\sin\pi t}{5\pi^5}+\frac{t^2}{5\pi^4}-\frac{t}{5\pi^4}\\ &+\left(-\frac{\cos\pi t}{\pi^6}+\frac{t^3}{3\pi^4}-\frac{t^2}{2\pi^4}+\frac{1}{\pi^6}\right)^2+\sin\pi t-\left(\frac{\cos\pi t}{3\pi}\right)^3-1,\end{aligned}$$

$$\begin{aligned} h(t,U,V)=&-u^2-u_1+\cos\left(\frac{\cos\pi t}{\pi^4}+\frac{2t}{\pi^4}-\frac{1}{\pi^4}-v_2\right)-\frac{v_3}{3}+\left(\frac{\sin\pi t}{\pi^4}-\frac{t}{\pi^3}\right)^2\\ &+\frac{1}{3}\left(-\frac{\sin\pi t}{\pi^3}+\frac{2}{\pi^4}\right)-\frac{\cos\pi t}{\pi^2}+\frac{\cos\pi t}{\pi^3}-\frac{1}{\pi^3}-1.\end{aligned}$$

It is easy to see that the function $f(t,u,u_1,u_2,u_3,v,v_1,v_2,v_3)$ does not satisfy the Nagumo-type condition with respect to the variable u_3, therefore, [7, Theorem 6] cannot guarantee the existence of a solution of the problem. Below, using the obtained theoretical results in Section 2 we show that the problem has a unique solution and the iterative method is very efficient for finding the solution.

First, choose M such that $\max\{|f|,|h|\} \le M$. We have

$$
\begin{aligned}
|f| &\le 1 + \left(\frac{M}{6}\right)^3 + \left(\frac{M}{24}\right)^2 + \frac{M}{60} + \frac{1}{5\pi^5} + \frac{1}{5\pi^4} + \frac{1}{5\pi^4} \\
&+ \left(\frac{1}{\pi^6} + \frac{1}{3\pi^4} + \frac{1}{2\pi^4} + \frac{1}{\pi^6}\right)^2 + 1 + \frac{1}{27\pi^3} + 1 \\
&\approx \left(\frac{M}{6}\right)^3 + \left(\frac{M}{24}\right)^2 + \frac{M}{60} + 3.006, \\
|h| &\le \left(\frac{M}{24}\right)^2 + \frac{M}{12} + 1 + \frac{M}{6} + \left(\frac{1}{\pi^4} + \frac{1}{\pi^3}\right)^2 \\
&+ \frac{1}{3}\left(\frac{1}{\pi^3} + \frac{2}{\pi^4}\right) + 1 + 1 + \frac{1}{\pi^3} + 1 \\
&\approx \left(\frac{M}{24}\right)^2 + \frac{M}{4} + 2.1852
\end{aligned}
$$

This number M may be defined from the inequality

$$\max(|f|,|h|) \le M$$

Clearly, $M = 4$ is a suitable choice. Then in the domain $\mathcal{D}_4$, since

$$
\begin{aligned}
&f'_u = 0, \quad f'_{u_1} = 0, \quad f'_{u_2} = \sin\left(-\frac{\sin \pi t}{\pi^2} - u_2\right), \quad f'_{u_3} = -\left(\frac{u_3}{3}\right)^2, \\
&f'_v = -2v, \quad f'_{v_1} = -\frac{1}{5}, \quad f'_{v_2} = 0, \quad f'_{v_3} = 0, \\
&h'_u = -2u, \quad h'_{u_1} = -1, \quad h'_{u_2} = 0, \quad h'_{u_3} = 0, \\
&h'_v = 0, \quad h'_{v_1} = 0, \quad h'_{v_2} = \sin\left(\frac{\cos \pi t}{\pi^4} + \frac{2t}{\pi^4} - \frac{1}{\pi^4} - v_2\right), \quad h'_{v_3} = -\frac{1}{3}.
\end{aligned}
$$

we can take

$$
\begin{aligned}
c_0 = c_1 = 0, c_2 = 1, c_3 = \frac{4}{9}, c_4 = \frac{1}{3}, c_5 = \frac{1}{5}, c_6 = c_7 = 0, \\
d_0 = \frac{1}{3}, d_1 = 1, d_2 = d_3 = d_4 = d_5 = 0, d_6 = 1, d_7 = \frac{1}{3}.
\end{aligned}
$$

Then $q = \max\{(\frac{c_0+c_4}{24} + \frac{c_1+c_5}{12} + \frac{c_2+c_6}{8} + \frac{c_3+c_7}{2}), (\frac{d_0+d_4}{24} + \frac{d_1+d_5}{12} + \frac{d_2+d_6}{8} + \frac{d_3+d_7}{2})\} \approx 0.389 < 1$. All the conditions of Theorem 4 are satisfied. Hence, the problem has a unique solution, and the iterative method converges.

The convergence of the iterative method for Example 1 is given in Table 1 and Fig. 1.

Table 1: The convergence in Example 1.

n	k	$error$
30	12	$3.7122e-08$
50	12	$4.8556e-09$
100	12	$3.0467e-10$
500	12	$4.9041e-13$
900	12	$3.8684e-14$

From Table 1 we observe that the convergence of the iterative method does not depend on the grid size.

In the next examples, the exact solution of problem (1)-(2) is not known.

Example 2. *Consider the problem*

$$\begin{cases} u^{(4)}(t) = \dfrac{uv'}{12} - u'u'' + \dfrac{(u''')^3}{10} + e^{-v^2}, \quad 0 < t < 1 \\ v^{(4)}(t) = \dfrac{uv}{20} + (u')^2u''' + \dfrac{v'}{2} + (v'')^2 + \dfrac{3}{2}, \quad 0 < t < 1 \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \\ v(0) = v'(0) = v''(0) = v''(1) = 0. \end{cases}$$

In this example

$$\begin{aligned} f(t,U,V) &= \frac{uv_1}{12} - u_1u_2 + \frac{u_3^3}{10} + e^{-v^2}, \\ h(t,U,V) &= \frac{uv}{20} + (u_1)^2u_3 + \frac{v_1}{2} + (v_2)^2 + \frac{3}{2}. \end{aligned}$$

As in the previous example, obviously, that the function f does not satisfy the Nagumo-type condition with respect to the variable u_3, therefore, [7, Theorem 6] cannot guarantee the existence of a solution of the problem.

Analogously as in Example 1 we can choose $M = 2$, and therefore, the Lipschitz coefficients in Theorem 4 are

$$\begin{aligned} c_0 &= \frac{1}{72}, c_1 = \frac{1}{4}, c_2 = \frac{1}{6}, c_3 = \frac{3}{10}, c_4 = \frac{1}{6}, c_5 = \frac{1}{144}, c_6 = c_7 = 0, \\ d_0 &= \frac{1}{240}, d_1 = \frac{1}{3}, d_2 = 0, d_3 = \frac{1}{36}, d_4 = \frac{1}{240}, d_5 = \frac{1}{2}, d_6 = \frac{1}{2}, d_7 = 0. \end{aligned}$$

Then, $q \approx 0.199 < 1$. All the conditions of Theorem 4 are satisfied. Hence, the problem has a unique solution (u, v), and the iterative method converges.

The numerical experiment for $N = 100$ shows that with the above stopping criterion after $k = 8$ iterations the iterative process stops and $e_8 = 2.0817e{-}17$. The graph of the approximate solution for Example 2 is depicted in Figure 2.

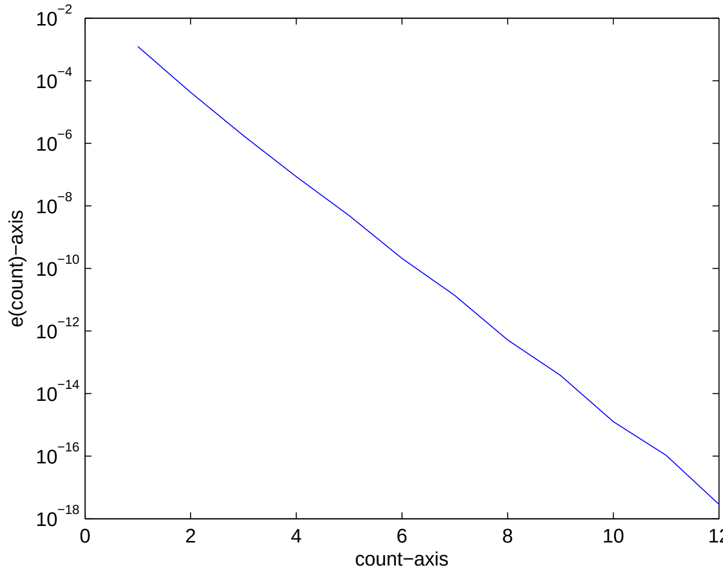

Figure 1: The graph of e_k in Example 1 for $n = 100$.

Moreover, below we show theoretically that this solution satifies $u \le 0,\ v \le 0$. Indeed, consider the domain

$$\mathcal{D}_2^{--} = \{(t, u, u_1, u_2, u_3, v, v_1, v_2, v_3)\}$$

where

$$\begin{aligned} 0 \le t \le 1,\ -\frac{1}{12} \le u \le 0; -\frac{1}{6} \le u_1 \le 0,\ -\frac{1}{4} \le u_2 \le 0,\ |u_3| \le 1, \\ -\frac{1}{12} \le v \le 0; -\frac{1}{6} \le v_1 \le 0,\ -\frac{1}{4} \le v_2 \le 0,\ |v_3| \le 1,, \end{aligned}$$

and the strip S_2^{++}

$$S_2^{++} = \left\{ w \in (C[0,1])^2 \ |0 \le \varphi(t) \le 2,\ 0 \le \psi(t) \le 2 \right\}.$$

Therefore, it is easy to see that in $\mathcal{D}_2^{--}$ we have

$$0 \le f(t, U, V) \le 2, \quad 0 \le h(t, U, V) \le 2$$

and all the conditions of Theorem 2 (i) are satisfied. Hence, the problem has a unique solution ($u \le 0,\ v \le 0$).

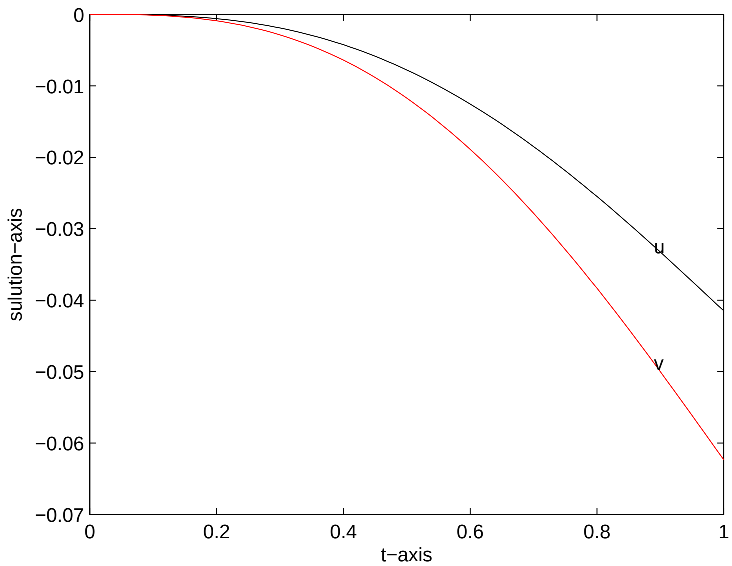

Figure 2: The graph of the approximate solution in Example 2.

Example 3. *Consider the problem*

$$\begin{cases} u^{(4)}(t) = -\dfrac{u'''}{24} - uu'' - \dfrac{e^{-v}}{2} - \dfrac{e^{-v'}}{3}, \quad 0 < t < 1 \\ v^{(4)}(t) = -e^{-u} - uv' + \dfrac{(v''')^3}{12}, \quad 0 < t < 1 \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \\ v(0) = v'(0) = v''(0) = v''(1) = 0. \end{cases}$$

In this example

$$\begin{aligned} f(t, U, V) &= -\frac{u_3}{24} - uu_2 - \frac{e^{-v}}{2} - \frac{e^{-v_1}}{3}, \\ h(t, U, V) &= -e^{-u} - uv_1 + \frac{(v_3)^3}{12}. \end{aligned}$$

As in the previous example, obviously, that the function h does not satisfy the Nagumo-type condition with respect to the variable v_3, therefore, [7, Theorem 6] cannot guarantee the existence of a solution of the problem.

Analogously as in Example 1 we can choose $M = 2$, and therefore, the

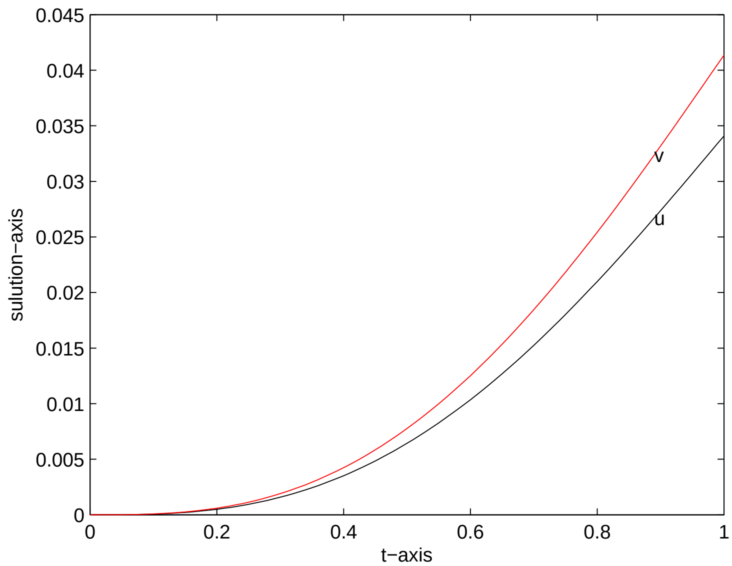

Figure 3: The graph of the approximate solution in Example 3.

Lipschitz coefficients in Theorem 4 are

$$c_0 = \frac{1}{4}, c_1 = 0, c_2 = \frac{1}{2}, c_3 = \frac{1}{24}, c_4 \approx 0.5435, c_5 \approx 0.3938, c_6 = c_7 = 0,$$
$$d_0 \approx 1.2536, d_1 = d_2 = d_3 = d_4 = 0, d_5 = \frac{1}{12}, d_6 = 0, d_7 = \frac{1}{4}.$$

Then, $q \approx 0.1842 < 1$. All the conditions of Theorem 4 are satisfied. Hence, the problem has a unique solution (u, v), and the iterative method converges.

The numerical experiment for $N = 100$ shows that with the above stopping criterion after $k = 8$ iterations the iterative process stops and $e_8 = 2.0817e{-}17$. The graph of the approximate solution for Example 3 is depicted in Figure 3.

Moreover, below we show theoretically that this solution satifies $u \geq 0,\ v \geq 0$. Indeed, consider the domain

$$\mathcal{D}_2^{++} = \{(t, u, u_1, u_2, u_3, v, v_1, v_2, v_3)\}$$

where

$$0 \leq t \leq 1,\, 0 \leq u \leq \frac{1}{12}; 0 \leq u_1 \leq \frac{1}{6},\ 0 \leq u_2 \leq \frac{1}{4},\ |u_3| \leq 1,$$
$$0 \leq v \leq \frac{1}{12},\ 0 \leq v_1 \leq \frac{1}{6},\ 0 \leq v_2 \leq \frac{1}{4},\ |v_3| \leq 1,$$

and the strip S_2^{--}

$$S_2^{--} = \left\{ w \in (C[0,1])^2 \mid -2 \le \varphi(t) \le 0,\ -2 \le \psi(t) \le 0 \right\}.$$

Therefore, it is easy to see that in $\mathcal{D}_2^{++}$ we have

$$-2 \le f(t,U,V) \le 0, \quad -2 \le h(t,U,V) \le 0$$

and all the conditions of Theorem 2 (i) are satisfied. Hence, the problem has a unique solution $(u \ge 0,\ v \ge 0)$.

Example 4. *Consider the problem*

$$\begin{cases} u^{(4)}(t) = -uv - e^{-u/2} - |v'''|^{1/2}, \quad 0 < t < 1 \\ v^{(4)}(t) = -u'^3 v' - v'' - \left(\dfrac{u'''}{12}\right)^3 - 1, \quad 0 < t < 1 \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \\ v(0) = v'(0) = v''(0) = v''(1) = 0. \end{cases}$$

In this example

$$\begin{aligned} f(t,U,V) &= -uv - e^{-u/2} - |v_3|^{1/2}, \\ h(t,U,V) &= -u_1^3 v_1 - v_2 - \left(\frac{u_3}{12}\right)^3 - 1, \end{aligned}$$

As in the previous example, obviously, that the function h does not satisfy the Nagumo-type condition with respect to the variable u_3. Therefore, [7, Theorem 6] cannot guarantee the existence of a solution of the problem if this theorem is valid.

Analogously as in Example 1 we can choose $M = 3$ such that $\max\{|f|, |h|\} \le M$. In this example, the function f does not satisfy the Lipshitz condition, but in D_3 all the conditions of Theorem 1 are satisfied. Hence, the problem has a solution.

The numerical experiment for $N = 100$ shows that with the above stopping criterion after $k = 16$ iterations the iterative process stops and $e_{16} = 6.2450e-17$. The graph of the approximate solution for Example 4 is depicted in Figure 4.

Moreover, below we show theoretically that this solution satifies $u \ge 0,\ v \ge 0$. Indeed, consider the domain

$$\mathcal{D}_3^{++} = \{(t, u, u_1, u_2, u_3, v, v_1, v_2, v_3)\}$$

where

$$\begin{aligned} 0 \le t \le 1,\ 0 \le u \le \frac{1}{8}; 0 \le u_1 \le \frac{1}{4},\ 0 \le u_2 \le \frac{3}{8},\ |u_3| \le \frac{3}{2}, \\ 0 \le v \le \frac{1}{8},\ 0 \le v_1 \le \frac{1}{4},\ 0 \le v_2 \le \frac{3}{8},\ |v_3| \le \frac{3}{2}, \end{aligned}$$

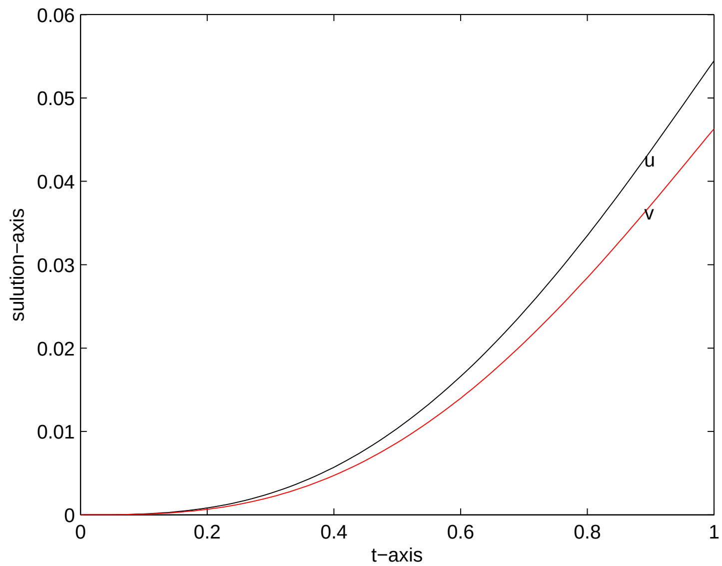

Figure 4: The graph of a approximate solution in Example 4.

and the strip S_3^{--}

$$S_3^{--} = \left\{ w \in (C[0,1])^2 \mid -3 \le \varphi(t) \le 0,\ -3 \le \psi(t) \le 0 \right\}.$$

It is easy to see that in $\mathcal{D}_3^{++}$ we have

$$-3 \le f(t,U,V) \le 0, \quad -3 \le h(t,U,V) \le 0$$

and all the conditions of Theorem 2 (i) are satisfied. Hence, the problem has a solution $(u \ge 0,\ v \ge 0)$.

Example 5. *Consider the problem*

$$\begin{cases} u^{(4)}(t) = u(t)v(t) + u'(t)^3 v'(t) + (u''(t))^{1/3} + v''(t) \\ \qquad\qquad + u'''(t) + (v'''(t))^{1/3}, \quad 0 < t < 1 \\ v^{(4)}(t) = u(t)^2 v(t) + u'(t)v'(t)^2 + e^{u''(t)} \sin(v''(t)) \\ \qquad\qquad + \dfrac{1}{4}(u'''(t))^{1/5} + v'''(t) + 1, \quad 0 < t < 1 \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \\ v(0) = v'(0) = v''(0) = v''(1) = 0. \end{cases}$$

In this example

$$f(t, U, V) = uv + u_1^3 v_1 + u_2^{1/3} + v_2 + u_3 + v_3^{1/3},$$
$$h(t, U, V) = u^2 v + u_1 v_1^2 + e^{u_2} \sin(v_2) + \frac{1}{4}(u_3)^{1/5} + v_3 + 1,$$

Analogously as in Example 1 we can choose $M = 8$ such that $\max\{|f|, |h|\} \leq M$. In D_8 all the conditions of Theorem 1 are satisfied. Hence, the problem has a solution.

Remark 1. In Example 4, the problem has a sulution. Since the function f does not satisfy the Lipshitz condition, Theorem 3 cannot guarantee the uniqueness of a solution. In spite of that the convergence of the iterative method to a solution is confirmed by the numerical experiment.

Remark 2. In Examples 1-4, the right-hand side functions do not satisfy the Nagumo-type condition, therefore, [7, Theorem 6] cannot guarantee the existence of a solution. But as seen above, using the theory in Sections 2 and 3 we have established the existence and uniqueness (or the existence) of a solution and the convergence of the iterative method. This convergence is also confirmed by numerical experiments.

5. Conclusion

In this paper we have proposed a method for investigating the solvability and iterative solution of coupled beams equations with fully nonlinear terms. In this method, by the reduction of the problem to an operator equation for the right-hand side functions, we have established the existence and uniqueness of a solution and the convergence of an iterative process. We have also studied the sign of the solution. The proposed method differs from the methods of other authors, where the problem is reduced to an operator equation for the pair of functions to be sought or is studied by the method of lower and upper solutions.

The proposed approach can be used for some other systems of nonlinear boundary value problems for ordinary and partial differential equations. This is the direction of our future research.

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