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MODIFIED FORWARD-BACKWARD SPLITTING METHODS IN HILBERT SPACES

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Abstract

In this paper, for finding a zero of a monotone variational inclusion in Hilbert spaces, we introduce new modifications of the Halpern forward-backward splitting methods, strong convergence of which is proved under new condition on the resolvent parameter. We show that these methods are particular cases of two new methods, introduced for solving a monotone variational inequality problem over the set of zeros of the inclusion. Numerical experiments are given for illustration and comparison.

1. Introduction The problem, studied in this paper, is to find a zero p of the following variational inclusion

$$0 \in Tp, \; T = A + B, \tag{1.1}$$

where A and B are maximal monotone and A is single valued in a real Hilbert space H with inner product and norm denoted, respectively, by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Throughout this paper, we assume that $\Gamma := (A + B)^{-1}0 \neq \emptyset$.

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Note that there are two possibilities here: either T is also maximal monotone or T is not maximal monotone. A fundamental algorithm for finding a zero for a maximal monotone operator T in H is the proximal point algorithm: $x_1 \in H$ and either

$$x^{k+1} = J_k^T x^k + e^k, \ k \geq 1, \tag{1.2}$$

or

$$x^{k+1} = J_k^T (x^k + e^k), \ k \geq 1, \tag{1.3}$$

where $J_k^T = (I + r_k T)^{-1}$, I is the identity mapping in H, $r_k > 0$ is called a resolvent parameter and e^k is an error vector. This algorithm was firstly introduced by Martinet [23]. In [26], Rockafellar proved weak convergence of (1.2) or (1.3) to a point in Γ. In [15], G*ü*ler showed that, in general, it converges weakly in infinite dimensional Hilbert spaces. In order to obtain a strongly convergent sequence from the proximal point algorithm, several modifications of (1.2) or (1.3) has been proposed by Kakimura and Takahashi [18], Solodov and Svaiter [29], Lehdili and Moudafi [19], Xu [38], and then, they were modified and improved in [1, 2, 4, 9, 12-14, 16, 21 22, 27, 28, 30, 32-36, 40] and references therein.

In many cases, when T is not maximal monotone, even if T is maximal monotone, for a fixed $r_k > 0$, $I + r_k T$ is hard to invert, but $I + r_k A$ and $I + r_k B$ are easier to invert than $I + r_k T$, one of the popular iterative methods used in this case is the forward-backward splitting method introduced by Passty [25] which defines a sequence $\{x^k\}$ by

$$x^{k+1} = J_k (I - r_k A) x^k, \tag{1.4}$$

where $J_k = (I + r_k B)^{-1}$. Motivated by (1.4), Takahashi, Wong and Yao [31], for solving (1.1) when A is an α-inverse strongly monotone operator in H, introduced the Halpern-type method,

$$x^{k+1} = t_k u + (1 - t_k) J_k (I - r_k A) x^k \tag{1.5}$$

where u is a fixed point in H, and proved that the sequence $\{x^k\}$, generated by (1.5), as $k \to \infty$, converges strongly to a point $P_\Gamma u$, the projection of u onto Γ, under the following conditions:

(t) $t_k \in (0,1)$ for all $k \geq 1$, $\lim_{k\to\infty} t_k = 0$ and $\sum_{k=1}^{\infty} t_k = \infty$;
(t') $\sum_{k=1}^{\infty} |t_{k+1} - t_k| < \infty$; and
(r') $\{r_k\}$ satisfies

$$0 < \varepsilon \leq r_k \leq 2\alpha, \sum_{k=1}^{\infty} |r_{k+1} - r_k| < \infty,$$

where ε is some small constant. Several modified and improved methods for (1.1) were presented in [11, 17, 20, 31], strong convergence of which is guaranteed under some conditions one of which is (r'). Recently, combining (1.5) and the contraction proximal point algorithm [34, 40] with the viscosity approximation method [24] for nonexpansive operators, an iterative method,

$$x^{k+1} = t_k f(x^k) + (1 - t_k)J_k(I - r_k A)x^k \tag{1.6}$$

where f is a contraction on H, was investigated in [3], strong convergence of which is proved under the condition $0 < \varepsilon \le r_k \le \alpha$. In all the works, listed above, and references therein, it is easily to see that $\sum_{k=1}^{\infty} r_k = \infty$. Very recently, the last condition on r_k was replaced by

$(\tilde{r})$ $r_k \in (0, \alpha)$ for all $k \ge 1$ and $\sum_{k=1}^{\infty} r_k < +\infty$

for the method

$$x^{k+1} = T^k(t_k u + (1 - t_k)x^k + e^k) \tag{1.7}$$

and its equivalent form

$$z^{k+1} = t_k u + (1 - t_k)T^k z^k + e^k, \tag{1.8}$$

introduced by the authors [8], where $T^k = T_1 T_2 \cdots T_k$ and $T_i = J_i(I - r_i A)$ for each $i = 1, 2, \cdots, k$. They proved strongly convergent results under conditions (t), $(\tilde{r})$,

- (e) either $\sum_{k=1}^{\infty} \|e^k\| < \infty$ or $\lim_{k \to 0} \|e^k\|/t_k = 0$ and
- (d) $\|Ax\|$ and $|Bx| \le \varphi(\|x\|)$, where $|Bx| = \inf\{\|y\| : y \in Bx\}$ and $\varphi(t)$ is a non-negative and non-decreasing function for all $t \ge 0$.

It is easily to see that methods (1.7) and (1.8) are quite complicated, when k is sufficiently large, because the number of forward-backward operators T_i is increased via each iteration step. Moreover, the second condition on r_k in $(\tilde{r})$ and condition (d) decrease the usage possibility of these methods. To overcome the drawback, in this paper, we introduce the new method

$$x^{k+1} = T_k T_c(t'_k u + (1 - t'_k)x^k + e^k) \tag{1.9}$$

and its equivalent form

$$x^{k+1} = t'_k u + (1 - t'_k)T_k T_c x^k + e^k, \tag{1.10}$$

that are simpler than (1.7) and (1.8), respectively, and two new methods,

$$x^{k+1} = t'_k u + \beta'_k T_c x^k + \gamma'_k T_k x^k + e^k, \tag{1.11}$$

and

$$x^{k+1} = t'_k f(T_c x^k) + \beta'_k T_c x^k + \gamma'_k J_k x^k + e^k, \tag{1.12}$$

with some conditions on positive parameters t'_k, β'_k and γ'_k, where, as for T_k, the operator $T_c = (I + cB)^{-1}(I - cA)$ with any sufficiently small positive number

c, i.e., $0 < c < \alpha$. Methods (1.9)-(1.12) contain only two forward-backward operators T_k and T_c at each iteration step k. As in [8], we will show that (1.9) with (1.10) and (1.11) with (1.12) are special cases of the methods

$$x^{k+1} = T_k T_c\big[(I - t_k F)x^k + e^k\big] \tag{1.13}$$

and

$$x^{k+1} = \beta_k (I - t_k F)T_c x^k + (1 - \beta_k)T_k x^k + e^k, \tag{1.14}$$

respectively, to solve the problem of finding a point $p_* \in \Gamma$ such that

$$\langle Fp_*, p_* - p\rangle \le 0 \ \forall p \in \Gamma, \tag{1.15}$$

where $F : H \to H$ is an η-strongly monotone and $\tilde{\gamma}$-strictly pseudocontractive operator with $\eta + \tilde{\gamma} > 1$. The last problem has been studied in [39], recently [7] in the case that $A \equiv 0$ and [8] (see, also references therein). We will show that the sequence $\{x^k\}$, generated by (1.13) or (1.14), converges strongly to the point p_* in (1.15), under conditions (t), (e),
(r) $c, r_k \in (0, \alpha)$ for all $k \ge 1$ and
(β) $\beta_k \in [a, b] \subset (0, 1)$ for all $k \ge 1$.

Clearly, the second requirement in $(\tilde{r})$ and condition (d) are removed for new simple methods (1.9)-(1.12).

The rest of the paper is organized as follows. In Section 2, we list some related facts, that will be used in the proof of our results. In Section 3, we prove strong convergent results for (1.13) with (1.14) and obtain their particular cases such as (1.9), (1.10), (1.11) and (1.12). A numerical example is given in Section 4 for illustration and comparison.

2. Preliminaries

The following facts will be used in the proof of our results in the next section.

Lemma 2.1 *Let H be a real Hilbert space. Then, the following inequality holds*

$$\|x + y\|^2 \le \|x\|^2 + \ 2\langle y, x + y\rangle, \ \forall x, y \in H.$$

Definition 2.1 Recall that an operator T in a real Hilbert space H, satisfying the conditions $\langle Tx - Ty, x - y\rangle \ge \eta\|x - y\|^2$ and

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \tilde{\gamma}\|(I - T)x - (I - T)y\|^2,$$

where $\eta > 0$ and $\tilde{\gamma} \in [0, 1)$ are some fixed numbers, is said to be η-strongly monotone and $\tilde{\gamma}$-strictly pseudocontractive, respectively.

Lemma 2.2 (*see,* [10]) *Let H be a real Hilbert space and let $F : H \to H$ be an η-strongly monotone and γ-strictly pseudocontractive operator with $\eta + \gamma > 1$. Then, for any $t \in (0, 1)$, $I - tF$ is contractive with constant $1 - t\tau$ where $\tau = 1 - \sqrt{(1 - \eta)/\gamma}$.*

Definitions 2.2 An operator T from a subset C of H into H is called:

(i) nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
(ii) α-inverse strongly monotone, if $\alpha\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in C$, where α is a positive real number.

We use $Fix(T) = \{x \in \mathcal{D}(T) : Tx = x\}$ to denote the set of fixed points of any operator T in H where $\mathcal{D}(T)$ is the domain of T.

Definitions 2.3 Let $B : H \to 2^H$ and $r > 0$.

(i) B is called a maximal monotone operator if B is monotone, i.e.,

$$\langle u - v, x - y\rangle \geq 0 \text{ for all } \ u \in Bx \text{ and } v \in By,$$

and the graph of B is not properly contained in the graph of any other monotone mapping;
(ii) $\mathcal{D}(B) := \{x \in H : Bx \neq \emptyset\}$ and $\mathcal{R}(B) = \{y \in Bx : x \in \mathcal{D}(B)\}$ are, respectively, the domain and range of B;
(iii) The resolvent of B with parameter r is denoted and defined by $J_r^B = (I + rB)^{-1}$.

It is well known that for $r > 0$,
i) B is monotone if and only if J_r^B is single-valued;
ii) B is maximal monotone if and only if J_r^B is single-valued and $\mathcal{D}(J_r^B)= H$.

Lemma 2.3 (*see,* [37]) *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following condition $a_{k+1} \leq (1-b_k)a_k+b_kc_k+d_k$, where $\{b_k\}, \{c_k\}$ and $\{d_k\}$ are sequences of real numbers such that*

(i) $b_k \in [0, 1]$ and $\sum_{k=1}^{\infty} b_k = \infty$;
(ii) $\limsup_{k\to\infty} c_k \leq 0$;
(iii) $\sum_{k=1}^{\infty} d_k < \infty$.
Then, $lim_{k\to\infty} a_k = 0$.

Lemma 2.4 (*see,* [3]) *Let H be a real Hilbert space, let B be a maximal monotone operator and let A be an α-inverse strongly monotone one in H with $\alpha > 0$ such that $\Gamma \neq \emptyset$. Then, for any $p \in \Gamma$, $z \in \mathcal{D}(A)$ and $r \in (0, \alpha)$, we have*

$$\|T_rz - p\|^2 \leq \|z - p\|^2 - \|T_rz - z\|^2/2,$$

where $T_r = J_r^B(I - rA)$.

Proposition 2.1 (*see,* [5, 6]) *Let H be a real Hilbert space, let F be as in Lemma 2.2 and let T be a nonexpansive operator on H such that $Fix(T) \neq \emptyset$. Then, for any bounded sequence $\{z^k\}$ in H such that $\lim_{k\to\infty} \|Tz^k - z^k\| = 0$, we have*

$$\limsup_{k\to\infty}\langle Fp_*, p_* - z^k\rangle \leq 0, \tag{2.1}$$

where p_* is the unique solution of (1.15) *with Γ replaced by $Fix(T)$.*

3. Main Results

First, we prove the following result.

Theorem 3.1 *Let H, B and A be as in Lemma 2.4 with $\mathcal{D}(A) = H$ and let F be an η-strongly monotone and $\tilde{\gamma}$-strictly pseudocontractive operator on H such that $\eta + \tilde{\gamma} > 1$. Then, as $k \to \infty$, the sequence $\{z^k\}$, defined by*

$$z^{k+1} = T_k T_c (I - t_k F) z^k \tag{3.1}$$

with conditions (r) and (t), converges strongly to p_*, solving (1.15) with $\Gamma = (A+B)^{-1}0$.

Proof. First, we prove that $\{z^k\}$ is bounded. We know that $p \in \Gamma$ if and only if $p \in Fix(T_r)$, that is defined in Lemma 2.4 for any $r \in (0, \alpha)$. It means that $\Gamma = Fix(T_r)$ for any $r \in (0, \alpha)$. Thus, for any point $p \in \Gamma$, from the nonexpansivity of T_k and T_c (see, [3]), condition (r), (3.1) and Lemma 2.2, we have that

$$\begin{aligned} \|z^{k+1} - p\| &= \|T_k T_c (I - t_k F) z^k - T_k T_c p\| \\ &\leq \|(I - t_k F) z^k - p\| \\ &\leq (1 - t_k \tau) \|z^k - p\| + t_k \|Fp\| \\ &\leq \max \{\|z^1 - p\|, \|Fp\|/\tau\}, \end{aligned}$$

by mathematical induction. Therefore, $\{z^k\}$ is bounded. So, is the sequence $\{Fz^k\}$. Without any loss of generality, we assume that they are bounded by a positive constant M_1. Put $y^k = (I - t_k F) z^k$. By using again the nonexpansivity of T_k and T_c, Lemmas 2.4 and 2.2, we obtain the following inequalities,

$$\begin{aligned} \|z^{k+1} - p\|^2 &= \|T_k T_c y^k - T_k p\|^2 \leq \|T_c y^k - p\|^2 \\ &\leq \|y^k - p\|^2 - \|T_c y^k - y^k\|^2/2 \\ &= \|(I - t_k F) z^k - p\|^2 - \|T_c y^k - y^k\|^2/2 \\ &\leq (1 - t_k \tau) \|z^k - p\|^2 + 2t_k \langle Fp, p - z^k + t_k F z^k \rangle - \|T_c y^k - y^k\|^2/2 \\ &\leq \|z^k - p\|^2 + 2t_k \|Fp\| (\|p\| + 2M_1) - \|T_c y^k - y^k\|^2/2. \end{aligned}$$

Thus,

$$(\|T_c y^k - y^k\|^2/2) - 2t_k \|Fp\| (\|p\| + 2M_1) \leq \|z^k - p\|^2 - \|z^{k+1} - p\|^2. \tag{3.2}$$

Only two cases need to be discussed. When $(\|T_c y^k - y^k\|^2/2) \leq 2t_k \|Fp\| (\|p\| + 2M_1)$ for all $k \geq 1$, from condition (t), it follows that

$$\lim_{k \to \infty} \|T_c y^k - y^k\|^2 = 0. \tag{3.3}$$

When $(\|T_c y^k - y^k\|^2/2) > t_k\|Fp\|(\|p\| + 2M_1)$, considering analogue of (3.2) from $k = 1$ to M, summing them side-by-side, we get that

$$\sum_{k=1}^{M}\big[(\|T_c y^k - y^k\|^2/2) - 2t_k\|Fp\|(\|p\| + 2M_1)\big] \le \|z^1 - p\|^2 - \|z^{M+1} - p\|^2 \le \|z^1 - p\|^2.$$

Then,

$$\sum_{k=1}^{\infty}\big[(\|T_c y^k - y^k\|^2/2) - 2t_k\|Fp\|(\|p\| + 2M_1)\big] < +\infty.$$

Consequently,

$$\lim_{k\to\infty}\big[(\|T_c y^k - y^k\|^2/2) - 2t_k\|Fp\|(\|p\| + 2M_1)\big] = 0,$$

that together with condition (t) implies (3.3). Next, from the definition of y^k, we have that $\|y^k - z^k\| = t_k\|Fz^k\| \le t_k M_1 \to 0$ as $k \to \infty$. Thus, $\lim_{k\to\infty}\|T_c z^k - z^k\| = 0$. Consequently, $\{z^k\}$ satisfies (2.1) with $T = T_c$.

Now, we estimate the value $\|z^{k+1} - p_*\|^2$ as follows.

$$\begin{aligned}\|z^{k+1} - p_*\|^2 &= \|T_k T_c(I - t_k F)z^k - T_k T_c p_*\|^2\\ &\le \|(I - t_k F)z^k - p_*\|^2\\ &\le (1 - t_k\tau)\|z^k - p_*\|^2 + 2t_k\langle Fp_*, p_* - z^k + t_k Fz^k\rangle\\ &= (1 - b_k)\|x^k - p_*\|^2 + b_k c_k,\end{aligned} \tag{3.4}$$

where $b_k = t_k\tau$ and

$$c_k = (2/\tau)\big[\langle Fp_*, p_* - z^k\rangle + t_k\langle Fp_*, Fz^k\rangle\big].$$

Since $\sum_{k=1}^{\infty} t_k = \infty$, $\sum_{k=1}^{\infty} b_k = \infty$. So, from (3.4), (2.1), the condition (t) and Lemma 2.3, it follows that $\lim_{k\to\infty}\|z^k - p_*\|^2 = 0$. This completes the proof. □

Remarks 1

1.1. Since $y^k = (I - t_k F)z^k$, from (3.1) with re-denoting $t_k := t_{k+1}$, we get the method

$$y^{k+1} = (I - t_k F)T_k T_c y^k. \tag{3.5}$$

Moreover, if $t_k \to 0$ then $\{z^k\}$ is convergent if and only if $\{y^k\}$ is so and their limits coincide. Indeed, from the definition of y^k, it follows that $\|y^k - z^k\| \le t_k\|Fz^k\|$. Therefore, when $\{z^k\}$ is convergent, $\{z^k\}$ is bounded, and hence, $\{Fz^k\}$ is also bounded. Since $t_k \to 0$ as $k \to \infty$, from the last inequality and the convergence of $\{z^k\}$ it follows the convergence of $\{y^k\}$ and that their limits coincide. The case, when $\{y^k\}$ converges, is similar.

It is well known (see, [6]) that the operator $F = I - f$, where $f = \overline{a}I + (1-\overline{a})u$ for a fixed number $\overline{a} \in (0,1)$ and a fixed point $u \in H$, is η-strongly monotone with $\eta = 1 - \overline{a}$ and $\tilde{\gamma}$-strictly pseudocontractive with a fixed $\tilde{\gamma} \in (\overline{a}, 1)$, and hence, $\eta + \tilde{\gamma} > 1$. Replacing F in (3.1) and (3.5) by $I - f$ and denoting $t'_k := (1 - \overline{a})t_k$, we get, respectively, the following methods,

$$\begin{aligned} z^{k+1} &= T_k T_c(t'_k u + (1 - t'_k)z^k), \\ y^{k+1} &= t'_k u + (1 - t'_k)T_k T_c y^k. \end{aligned} \tag{3.6}$$

Then, from Theorem 3.1, we obtain that the sequences $\{z^k\}$ and $\{y^k\}$, defined by (3.6), as $k \to \infty$, under conditions (t) and (r), converge strongly to a point p_* in Γ, solving the variational inequality $\langle p_* - u, p_* - p\rangle \leq 0$ for all $p \in \Gamma$, i.e., $p_* = P_\Gamma u$. Beside, we have still that

$$\begin{aligned} \|x^{k+1} - z^{k+1}\| &= \|T_k T_c(I - t_k F)x^k + e^k) - T_k T_c(I - t_k F)z^k\| \\ &\leq (1 - t_k\tau)\|x^k - z^k\| + \|e^k\|, \end{aligned}$$

where x^k and z^k are defined, respectively, by (1.13) and (3.1). Thus, by Lemma 2.3, under conditions (t), (r) and (e), $\|x^k - z^k\| \to 0$ as $k \to \infty$, and hence, the sequence $\{x^k\}$ converges strongly to the point p_*. By the same argument as the above, we obtain that the sequence $\{x^k\}$ defined by either (1.9) or (1.10), under conditions (t), (r) and (e), converges strongly to the point $p_* = P_\Gamma u$, as $k \to \infty$.

1.2. Now, we consider the case, when A maps a closed and convex subset C of H into H and $\mathcal{D}(B) \subseteq C$. Then, algorithms in (3.6) work well when u and x^1 are chosen such that $u, x^1 \in C$.

1.3. $t_k = 1/\ln(1+k)$ does not satisfy conditions in (r'). But, it can be used in our methods.

Further, we have the following result.

Theorem 3.2 *Let H, B, A, Γ and F be as in Theorem* 3.1. *Then, as $k \to \infty$, the sequence $\{x^k\}$, generated by* (1.14) *with conditions* (β), (t), (r) *and* (e), *converges strongly to p_*, solving* (1.15).

Proof. Obviously, for $\{z^k\}$, generated by

$$z^{k+1} = \beta_k(I - t_k F)T_c z^k + (1 - \beta_k)T_k z^k, \tag{3.7}$$

from (1.14), we get that

$$\begin{aligned} \|x^{k+1} - z^{k+1}\| &= \|\big[\beta_k(I - t_k F)T_c x^k - (I - t_k F)T_c z^k\big] + (1 - \beta_k)(T_k x^k - T_k z^k) + e^k\| \\ &\leq \beta_k(1 - t_k\tau)\|x^k - z^k\| + (1 - \beta_k)\|x^k - z^k\| + \|e^k\| \\ &= (1 - \beta_k t_k\tau)\|x^k - z^k\| + \|e^k\|. \end{aligned}$$

By Lemma 2.3 with condition (t), (β) and (e), $\|x^k - z^k\| \to 0$ as $k \to \infty$. So, it is sufficient to prove that $\{z^k\}$, defined by (3.7), converges to the point p_*. For this purpose, first, we prove that $\{z^k\}$ is bounded. Since $T_k p = p$ for any point $p \in \Gamma$, from the nonexpansivity of T_k, (3.7) and Lemma 2.2, we have that

$$\begin{aligned}\|z^{k+1} - p\| &= \|\beta_k((I - t_k F)T_c z^k - p) + (1 - \beta_k)(T_k z^k - p)\| \\ &\le \beta_k \|(I - t_k F)T_c z^k - p\| + (1 - \beta_k)\|T_k z^k - p\| \\ &\le (1 - \beta_k t_k \tau)\|z^k - p\| + \beta_k t_k \|Fp\| \\ &\le \max\,\{\|z^1 - p\|, \|Fp\|/\tau\},\end{aligned}$$

by mathematical induction. Therefore, $\{z^k\}$ is bounded. So, are the sequences $\{T_c z^k\}$ and $\{FT_c z^k\}$. Without any loss of generality, we assume that they are bounded by a positive constant M_2. By using Lemmas 2.4 and 2.2, we obtain the following inequalities,

$$\begin{aligned}\|z^{k+1} - p\|^2 &\le \beta_k \|(I - t_k F)T_c z^k - p\|^2 + (1 - \beta_k)\|T_k z^k - p\|^2 \\ &\le \beta_k \big[(1 - t_k\tau)\|T_c z^k - p\|^2 + 2t_k\langle Fp, p - T_c z^k + t_k FT_c z^k\rangle\big] \\ &\quad + (1 - \beta_k)\|z^k - p\|^2 \\ &\le (1 - \beta_k t_k \tau)\|z^k - p\|^2 + 2\beta_k t_k \langle Fp, p - T_c z^k + t_k FT_c z^k\rangle \\ &\quad - c_2\|T_c z^k - z^k\|^2/2 \\ &\le \|z^k - p\|^2 + 2\beta_k t_k \|Fp\|(\|p\| + 2M_1) - c_2\|T_c z^k - z^k\|^2/2,\end{aligned} \tag{3.8}$$

where c_2 is a positive constant such that $c_2 \le \beta_k(1 - t_k\tau)$ for all $k \ge 1$. The existence of the constant is due to conditions (β) and (t). Thus, as in the proof of Theorem 3.1, we can obtain (3.3) with $y^k = z^k$. So, $\{z^k\}$ satisfies (2.1) with $T = T_c$.

Now, from (3.8), we estimate the value $\|z^{k+1} - p_*\|^2$ as follows.

$$\begin{aligned}\|z^{k+1} - p_*\|^2 &= \|T_k T_c(I - t_k F)z^k - T^k T_c p_*\|^2 \\ &\le \|(I - t_k F)z^k - p_*\|^2 \\ &\le (1 - t_k\tau)\|z^k - p_*\|^2 + 2t_k\langle Fp_*, p_* - T_c z^k + t_k FT_c z^k\rangle \\ &= (1 - b_k)\|z^k - p_*\|^2 + b_k c_k,\end{aligned} \tag{3.9}$$

where $b_k = \beta_k t_k \tau$ and

$$c_k = (2/\tau)\big[\langle Fp_*, p_* - z^k\rangle + \langle Fp_*, z^k - T_c z^k\rangle + t_k\langle Fp_*, FT_c z^k\rangle\big].$$

Since $\sum_{k=1}^{\infty} t_k = \infty$, $\sum_{k=1}^{\infty} b_k = \infty$. So, from (3.3) with $y^k = z^k$, (3.9) and Lemma 2.3, it follows that $\lim_{k\to\infty} \|z^k - p_*\|^2 = 0$. The proof is completed. □

Remarks 2

2.1. Replacing F in (1.14) by $I - f$, that is defined in remark 1.1, we obtain method (1.11) with $t'_k = \beta_k t_k(1-\overline{a})$, $\beta'_k = \beta_k - t'_k$ and $\gamma'_k = 1 - \beta_k$.
2.2. Let $\tilde{a} > 1$ and let f be an $\tilde{a}$-inverse strongly monotone operator on H. It is easily seen that f is a contraction with constant $1/\tilde{a} \in (0,1)$, and hence, $F := I - f$ is an η-strongly monotone operator with $\eta = 1 - (1/\tilde{a})$. Moreover,

$$\begin{aligned}\langle Fx - Fy, x - y\rangle &= \|x-y\|^2 - \langle f(x) - f(y), j(x-y)\rangle \\ &\le \|x-y\|^2 - \tilde{a}\|f(x) - f(y)\|^2 \\ &\le \|x-y\|^2 - \gamma\|(I-F)x - (I-F)y\|^2,\end{aligned}$$

for any $\gamma \in (0, \tilde{a}]$. Taking any fixed $\gamma \in ((1/\tilde{a}), \tilde{a}]$, we get that F is a γ–strictly pseudocontractive operator with $\eta + \gamma > 1$. Next, by replacing F by $I - f$ in (1.14), we obtain method (1.12) with the same t'_k, β'_k and γ'_k.
2.3. Further, take $f = \overline{a}I$ with a fixed number $\overline{a} \in (0,1)$. Then,

$$\langle f(x) - f(y), j(x-y)\rangle = \overline{a}\|x-y\|^2 = (1/\overline{a})\|f(x) - f(y)\|^2,$$

and hence, f is $\tilde{a}$-inverse strongly monotone operator on H with $\tilde{a} = (1/\overline{a}) > 1$. By the similar argument, we get a new method,

$$x^{k+1} = \beta_k(1 - t'_k)T_c x^k + (1-\beta_k)T_k x^k + e^k.$$

2.4. For a given α-inverse strongly monotone operator f on H, we can obtain an $\tilde{\alpha}$-inverse strongly monotone operator $\tilde{f}$ with $\tilde{\alpha} > 1$ by considering $\tilde{f} := \beta f$ with a positive real number $\beta < \alpha$. Indeed,

$$\begin{aligned}\langle \tilde{f}(x) - \tilde{f}(y), x - y\rangle &= \langle \beta f(x) - \beta f(y), x - y\rangle \\ &\ge \beta\alpha\|f(x) - f(y)\|^2 = \tilde{\alpha}\|\tilde{f}(x) - \tilde{f}(y)\|^2,\end{aligned}$$

where $\tilde{\alpha} = \alpha/\beta > 1$.

4. Numerical experiments

We can apply our methods to the following variational inequality problem: find a point

$$p \in C \text{ such that } \langle Ap, p - x\rangle \le 0 \text{ for all } x \in C, \tag{4.1}$$

where C is a closed convex subset in a Hilbert space H and A is an α-inverse strongly monotone operator on H. We know that p is a solution of (4.1) if and only if it is a zero for inclusion (1.1), where B is the normal cone to C, defined by

$$N_C x = \{w \in H : \langle w, v - x\rangle \le 0,\ \forall v \in C\}.$$

Let φ be a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. Then, the subdifferential $\partial\varphi$ of φ is defined as follows:

$$\partial\varphi(x) = \{z \in H : \varphi(x) + \langle z, y - x\rangle \leq \varphi(y), y \in H\}$$

for all $x \in H$; see, for instance, [11]. We know that $\partial\varphi$ is maximal monotone. Let χ_C be the indicator function of C, i.e.,

$$\chi_C = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, χ_C is a proper semicontinuous convex function of H into $(-\infty, \infty]$ and then the subdifferential $\partial\chi_C$ is a maximal monotone operator. Next, we can define the resolvent $J_{r_k}^{\partial\chi_C}$ for $r_k > 0$, i.e., $J_{r_k}^{\partial\chi_C} y = (I + r_k\partial\chi_C)^{-1}y$, for all $y \in H$. We have (see, [30]) that $x = J_{r_k}^{\partial\chi_C} y \Longleftrightarrow x = P_C y$ for any $y \in H$ and $x \in C$.

For computation, we consider the example in [8], when

$$C = \{x \in \mathbb{E}^n : \sum_{j=1}^{n} (x_j - a_j)^2 \leq r^2\}, \tag{4.2}$$

where $a_j, r \in (-\infty; +\infty)$, for all $1 \leq j \leq n$.

Numerical computations are implemented with $n = 3$, $a_1 = a_2 = a_3 = 2$, $r = 1$ and $Ax = \varphi'(x)$ where $\varphi(x) = [(x_1 - 1.5)^2 + (x_2 - 1.3)^2]/2$ for all $x \in \mathbb{E}^3$. Clearly, A is an 1-inverse strongly monotone operator on Euclidean space $\mathbb{E}^3$. With taking $u = (2.0; 1.0; 1.5)$, we get that $p_* = P_\Gamma u = (1.5; 1.3; 1.5)$ is a solution of (4.1)-(4.2) where $\Gamma = \{(1.5; 1.3; (-\infty, \infty))\} \cap C$ is the solution set of the stated problem. The computational results, using each method from (1.9), (1.11) and (1.12) with a starting point $x^1 = (2.7; 2.5; 2.3)$, $t'_k = 1/(k+1)$, $\gamma'_k = 0.1 + 1/(k + 1)$, $\beta'_k = 1 - t'_k - \gamma'_k$, $c = 0.5$, $r_k = 1/(k + 1)$ and either $e^k = 0$ or $e^k = (1.0; 1.0; 1.0)/k^2$, are presented in numerical tables 1-6. Note that, using $f = 0.9I$ in method (1.14), we have p_* is the point in Γ with minimal norm, where $p_* = (1.5; 1.3; 2 - \sqrt{0.74}) \approx (1.5; 1.3; 1.13397674733)$. We do not calculate by method (1.10), because it is equivalent to (1.9). Analyzing the numerical results, we can conclude that the calculation by methods (1.9) and (1.11) is better than that by (1.12). Moreover, the calculation without errors, i.e., $e^k = (0; 0; 0)$ for all $k \geq 1$, is also better than that with errors $e^k = (1; 1; 1)/k^2$. The numerical results above show that our methods work good and they are simpler than that in [8].

Further, for comparison, we give numerical results by methods (1.5) and (1.10) with the same t'_k and $r_k = 0.1+1/(3k)$, satisfying conditions (r') and (r), where $c = 0.4$ and $e^k = (0; 0; 0)$ in the tables 7 and 8, respectively. Numerical results computed by (1.10) and (1.8) with new $r_k = 1/(k(k + 1))$, that has

Table 1: Computational results by (1.9) with $e^k = (0; 0; 0)$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
10	1.5372038029	1.2776932141	1.7272727273
20	1.5215419538	1.2870748319	1.5380952381
30	1.5150884495	1.2909469303	1.5258064516
40	1.5116002380	1.2930398572	1.5195212195
50	1.5094194541	1.2943483276	1.5156862745
100	1.5048024654	1.2970885207	1.5079207921
200	1.5024628903	1.2985223138	1.5039800995
300	1.5016500922	1.2990999447	1.5039867110
400	1.5012406639	1.2991556016	1.5019950125
500	1.5009404019	1.2994035880	1.5015968064

properties (r) and $(\tilde{r})$, are given in tables 9 and 10, respectively. Tables of numerical results 7, 8, 9 and 10 show that method (1.8) gives the best result than the others. Perhaps, the quantity of information $(T_i, i = 1, 2, \cdots, k)$, using at kth iteration step, for method (1.8) is more than that for the rest methods.

5. Conclusion

We have presented several iterative methods of Halpern or viscosity approximation types for finding a zero of a monotone inclusion in Hilbert spaces. We have showed that they are particular cases of our new two methods, designed for solving a monotone variational inequality problem over the set of zeros for the inclusion problem. Both these two methods are two combinations of the steepest-descent method with the forward-backward splitting one, strong convergence results of which have been proved under a new condition on resolvent parameter and weaker conditions on iterative parameter than that for other methods in literature. A numerical example was given for illustrating our methods and the comparisons of our new methods with others in literature have been done by computations with the same values of iterative parameters.

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Table 2: Computational results by (1.9) with $e^k = (1;1;1)/k^2$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
10	1.5464178232	1.2869072344	1.9798850896
20	1.5239296975	1.2894625776	1.7854239477
30	1.5161650112	1.2920234920	1.7066898467
40	1.5122103973	1.2936500166	1.6633850488
50	1.5098111775	1.2947406484	1.6357713347
100	1.5049014855	1.2971877541	1.5754689249
200	1.5024876866	1.2985471901	1.5413829726
300	1.5016611665	1.2990210190	1.5289842863
400	1.5012468984	1.2992618361	1.5224747304
500	1.5009980120	1.2994075801	1.5184346497

Table 3: Computational results by (1.11) with $e^k = (0;0;0)$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
10	1.6109762181	1.2343058229	1.5727272727
20	1.5555754564	1.2666565954	1.5380952381
30	1.5370406269	1.2777756281	1.5258064516
40	1.5277788571	1.2833326857	1.5195121951
50	1.5222226527	1.2866664084	1.5156862745
100	1.5111111368	1.2933333179	1.5079207921
200	1.5055555571	1.2966666657	1.5039800995
300	1.5037037040	1.2977777776	1.5026078073
400	1.5027777779	1.2983333333	1.5019950125
500	1.5022222223	1.2986666667	1.5015968064

Table 4: Computational results by (1.11) with $e^k = (1;1;1)/k^2$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
10	1.6440673591	1.2674396185	1.8810373796
20	1.6519480053	1.2730292336	1.7336465758
30	1.5397242253	1.2804592266	1.6716068529
40	1.5295244149	1.2848079776	1.6368649315
50	1.5231546812	1.2875984368	1.6144512404
100	1.5113385081	1.2935606892	1.5647033328
200	1.5056117427	1.2967228513	1.5359733964
300	1.5037285808	1.2978026543	1.5253719115
400	1.5027917447	1.2983473001	1.5197633197
500	1.5022311510	1.2986755954	1.5015968064

Table 5: Computational results by (1.12) with $e^k = (0;0;0)$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
10	1.6838845367	1.4316385735	0.9460475410
20	1.6607901923	1.4360056970	1.0249545450
30	1.6517732101	1.4357200727	1.0551478617
40	1.6470038387	1.4351359901	1.0711414366
50	1.440569072	1.4346259790	1.0810516556
100	1.6379679014	1.4332109466	1.1016324519
200	1.6348245088	1.4322892090	1.1123282170
300	1.6337619066	1.4319480339	1.1159568595
400	1.6332278173	1.4317708904	1.1177833634
500	1.6329064700	1.4316624810	1.1188832032

Table 6: Computational results by (1.12) with $e^k = (1; 1; 1)/k^2$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
10	1.6997343568	1.4394420335	0.9555823001
20	1.6639790863	1.4379547688	1.0275707923
30	1.6531272890	1.4365743834	1.0563427660
40	1.6477489034	1.4356122668	1.0563427660
50	1.6445275278	1.4349289576	1.0814923391
100	1.6380825145	1.4332856541	1.1017448445
200	1.6348527888	1.4323077406	1.1123561663
300	1.6337744205	1.4319562476	1.1159695279
400	1.6332348410	1.4317755041	1.1177905026
500	1.6329109593	1.4316654313	1.1188877774

Table 7: Computational results by (1.5) with $e^k = (0; 0; 0)$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
100	1.5477401998	1.2713559444	1.5079207920
200	1.5244482200	1.2853310680	1.5039800995
300	1.5164230401	1.2901461759	1.5026578073
400	1.5123633874	1.2925819676	1.5019950125
500	1.5099127274	1.2940523636	1.5015968064

Table 8: Computational results by (1.10) with $e^k = (0; 0; 0)$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
100	1.5107147983	1.2935711210	1.5079207920
200	1.5053959438	1.2967624337	1.5039800995
300	1.5036059047	1.2978364572	1.5026578073
400	1.5027076630	1.2983754022	1.5019950125
500	1.5021676845	1.2986993893	1.5015968064

Table 9: Computational results by (1.5) with $e^k = (0; 0; 0)$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
100	1.5123743414	1.2925753951	1.5079207920
200	1.5062186699	1.2962687981	1.5039800995
300	1.5041527542	1.2975083475	1.5026578073
400	1.5031771776	1.2981296934	1.5019950125
500	1.5024949949	1.2985030030	1.5015968064

Table 10: Computational results by (1.8) with $e^k = (0; 0; 0)$.

k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}
100	1.5070685001	1.2957589000	1.5079207920
200	1.5035443335	1.2978733999	1.5039800995
300	1.5023651487	1.2985809107	1.5026578073
400	1.5017747116	1.2989351730	1.5019950125
500	1.5014201780	1.2991478932	1.5015968064

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