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ON RIGHT GENERALIZED (α, β)-DERIVATIONS IN PRIME RINGS

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Abstract

Brešar proved that the only semiderivations of prime rings are derivations and mappings of the form $D(x) = \lambda(x - \alpha(x))$, where α is an endomorphism and λ is an element in the extended centroid. De Filippis, Mamouni and Oukhtite extended this result to generalized semiderivations. The aim of this note is to extend their results to right generaliged (α, β)-derivations with reduction of some conditions, where α is an epimorphism and β is an endomorphism.

1 Introduction

Throughout this work R will be a ring with center $Z(R)$. We shall denote by C to the extended centroid of R and by U to the right Utumi quotient ring of R when R is a semiprime ring (see[1 and 6] for details). The symbol $[x, y]$ denotes the Lie structure or the commutator $xy - yx$. Let S be a nonempty subset of R. A map f: $S \to R$ is said to be commuting on S in case $[x, f(x)] = 0$ satisfies for all $x \in S$. Also, if S is a subring of R and α is an endomorphism of R, a map f: $S \to U$ is said to be α-commuting on S in case $[\alpha(x), f(x)] = 0$ holds for all $x \in S$ and is called additive modulo C on

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S in case $f(x+y) - f(x) - f(y) \in C$ is fulfilled for all $x, y \in S$. A ring R is said to be prime if for $a, b \in R$; $aRb = 0$ then either $a = 0$ or $b = 0$ and R is said to be semiprime if for $a \in R$; $aRa = 0$ then $a = 0$. Let α and β be mappings of R. A map D on R is called an (α, β)-derivation of R if it is additive and satisfying $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$, for all $x, y \in R$. Note that an (I, I)-derivation, where I is the identity map on R, is the usual definition of derivation. Let D be an (α, β)-derivation of R, a map F on R is called a right (left) generalized (α, β)-derivation of R associated with D if it is additive and satisfying $F(xy) = F(x)\alpha(y) + \beta(x)D(y)$ $(F(xy) = D(x)\alpha(y) + \beta(x)F(y))$, for all $x, y \in R$. Moreover, if F is both a right and left generalized (α, β)-derivation associated with D, then it is said to be a generalized (α, β)-derivation associated with D. Also, a generalized (I, I)-derivation on R is called a generalized derivation on R. (Note that this definition is different from that in [5]; generalized derivations in [5] are the right generalized derivations in our work.)

Let g be a mapping of R, a map d on R is said to be a semiderivation of R associated with g if it is additive and satisfying $d(xy) = d(x)y + g(x)d(y) = d(x)g(y) + xd(y)$, $d(g(x)) = g(d(x))$ for all $x, y \in R$. Let d be a semiderivation of R associated with a mapping g of R, a map G on R is called a generalized semiderivation of R associated with d and g if it is additive and satisfying $G(xy) = G(x)y + g(x)d(y) = d(x)g(y) + xG(y)$, $G(g(x)) = g(G(x))$ for all $x, y \in R$.

2 Preliminaries

We shall require throughout this paper to the following results.

Lemma 2.1 ([2], Lemma). *Let f and g be two mappings of a prime ring R. If $f(x)yg(z) = g(x)yf(z)$ for all $x, y, z \in R$ such that $f \neq 0$, then $g(x) = \lambda f(x)$ for all $x \in R$, where $\lambda \in C$.*

Lemma 2.2 ([3], Theorem 1). *If $d \neq 0$ is a semiderivation of a prime ring R associated with a mapping g of R, then g is an endomorphism.*

Lemma 2.3 ([7], Corollary 1). *Let α be an epimorphism of a semiprime ring R and $f\colon R \to U$ be an additive modulo C mapping. If f is α-commuting on R, then there exist $\lambda \in C$ and a mapping $\sigma\colon R \to C$ such that $f(x) = \lambda\alpha(x) + \sigma(x)$ for all $x \in R$.*

3 The results

To prove our result we need to generalize the result of Brešar [2], as in the following theorem.

Theorem 3.1. *Let α and β be two endomorphisms of a prime ring R and D be a mapping of R such that $D(xy) = D(x)\alpha(y) + \beta(x)D(y) = \alpha(x)D(y) + D(x)\beta(y)$ holds for all $x, y \in R$, where α or β is surjective. Then one of the following two cases is fulfilled.*

1. *$\alpha = \beta$ and so D satisfies $D(xy) = D(x)\alpha(y) + \alpha(x)D(y)$ for all $x, y \in R$.*
2. *$D(x) = \lambda(\alpha - \beta)(x)$ for all $x \in R$, where $\lambda \in C$.*

Proof Suppose that α is surjective. By the hypothesis, we have

$$D(xy) = D(x)\alpha(y) + \beta(x)D(y) = \alpha(x)D(y) + D(x)\beta(y) \quad \text{for all } x, y \in R. \tag{3.1}$$

If $\alpha = \beta$ in (3.1),

$$D(xy) = D(x)\alpha(y) + \alpha(x)D(y) \quad \text{for all } x, y \in R. \tag{3.2}$$

Therefore we may assume that $\alpha - \beta \neq 0$. Note that (3.1) can be written as

$$D(x)(\alpha - \beta)(y) = (\alpha - \beta)(x)D(y) \quad \text{for all } x, y \in R. \tag{3.3}$$

Putting in (3.1) yz for y, we get

$$D(x)(\alpha - \beta)(yz) = (\alpha - \beta)(x)D(yz) \quad \text{for all } x, y, z \in R. \tag{3.4}$$

Then we have

$$D(x)(\alpha - \beta)(yz) = D(x)\alpha(y)(\alpha - \beta)(z) + D(x)(\alpha - \beta)(y)\beta(z), \tag{3.5}$$

and

$$(\alpha - \beta)(x)D(yz) = (\alpha - \beta)(x)\alpha(y)D(z) + (\alpha - \beta)(x)D(y)\beta(z). \tag{3.6}$$

Comparing the last two relations and using (3.3), we obtain

$$D(x)\alpha(y)(\alpha - \beta)(z) = (\alpha - \beta)(x)\alpha(y)D(z) \quad \text{for all } x, y, z \in R. \tag{3.7}$$

Since α is surjective, we get

$$D(x)y(\alpha - \beta)(z) = (\alpha - \beta)(x)yD(z) \quad \text{for all } x, y, z \in R. \tag{3.8}$$

Lemma 2.1 now gives us $D(x) = \lambda(\alpha - \beta)(x)$ for all $x \in R$, where $\lambda \in C$. $\square$

Corollary 3.2 ([2], Theorem). *If D is a semiderivation of a prime ring R associated with the mapping α of R, then either D is a derivation or $D(x) = \lambda(I - \alpha)(x)$, where $\lambda \in C$, I is the identity map on R and α is an endomorphism.*

Proof If $D = 0$ we have nothing to prove. Assume that $D \neq 0$, then by Lemma 2.2, α is an endomorphism; and so by Theorem 3.1 we get the desired result. $\square$

Now, we will mention and prove the main result in this note.

Theorem 3.3. *Let α be an epimorphism of a prime ring R and β be an endomorphism of R. If F is a right generalized (α, β)-derivation associated with an (α, β)-derivation D of R such that $F(xy) = D(x)\beta(y) + \alpha(x)F(y)$ for all $x, y \in R$ and $D(xy) = D(x)\beta(y) + \alpha(x)D(y)$ for all $x, y \in R$, then one of the following two cases is fulfilled.*

1. *$\alpha = \beta$ and so F is a generalized (α, α)-derivation.*
2. *$F(x) = \lambda\alpha(x) + \mu\beta(x) = (\lambda + \mu)\alpha(x) + D(x)$ for all $x \in R$, where $\lambda, \mu \in C$.*

Proof Given that

$$F(xy) = F(x)\alpha(y) + \beta(x)D(y) = D(x)\beta(y) + \alpha(x)F(y) \quad \text{for all } x, y \in R. \tag{3.9}$$

If $\alpha = \beta$ in (3.9), then F is a generalized (α, α)-derivation. Now, assume that $\alpha - \beta \neq 0$, then by Theorem 3.1 there exists $\lambda_1 \in C$ such that $D(x) = \lambda_1(\alpha - \beta)(x)$ for all $x \in R$. Then from (3.9),

$$F(x)\alpha(y) + \lambda_1\beta(x)\alpha(y) - \lambda_1\beta(x)\beta(y) = \lambda_1\alpha(x)\beta(y) - \lambda_1\beta(x)\beta(y) + \alpha(x)F(y); \tag{3.10}$$

that is,

$$F(x)\alpha(y) + \lambda_1\beta(x)\alpha(y) - \lambda_1\alpha(x)\beta(y) - \alpha(x)F(y) = 0. \tag{3.11}$$

Denote $G = F + \lambda_1\beta$. If $G = 0$, then $F(x) = -\lambda_1\beta(x)$ for all $x \in R$ and we get the desired result. Therefore suppose that $G \neq 0$, and by (3.11), we obtain

$$G(x)\alpha(y) - \alpha(x)G(y) = 0 \quad \text{for all } x, y \in R. \tag{3.12}$$

Putting $x = y$ in (3.12), then G is an additive α-commuting map on R. By Lemma 2.3, we get $\lambda_2 \in C$ and additive mapping σ: $R \to C$ such that

$$G(x) = \lambda_2\alpha(x) + \sigma(x) \quad \text{for all } x \in R. \tag{3.13}$$

Thus (3.12) implies that

$$\sigma(x)\alpha(y) = \alpha(x)\sigma(y) \quad \text{for all } x, y \in R. \tag{3.14}$$

Since $\sigma(R) \subseteq C$ and by (3.14), we have

$$\sigma(R)R[\alpha(x), \alpha(y)] = 0 \quad \text{for all } x, y \in R. \tag{3.15}$$

If $\sigma(R) \neq 0$ and since R is prime, we get

$$[\alpha(x), \alpha(y)] = 0 \quad \text{for all } x, y \in R; \tag{3.16}$$

and since α is surjective, R is commutative and then $R \subseteq C$. Moreover, since $G \neq 0$, we get $x_0 \in R$ such that $G(x_0) \neq 0$ and denote $G(x_0) = \lambda_3 \in C$, and $(\alpha(x_0))^{-1} = \lambda_4 \in C$. Thus by (3.12), we have $G(y) = \lambda_4\lambda_3\alpha(y)$ for all $y \in R$, therefore, $F(y) = \lambda_4\lambda_3\alpha(y) - \lambda_1\beta(y)$ for all $y \in R$, as required.

Hence we may assume that $\sigma(R) = 0$ and so $G(x) = \lambda_2\alpha(x)$, that is, $F(x) = \lambda_2\alpha(x) - \lambda_1\beta(x)$ for all $x \in R$ and we are done again. □

From Theorem 3.3 we can get

Corollary 3.4 ([4], Theorem 17). *Let α be an endomorphism of a prime ring R and D be a semiderivation of R associated with α. If F is a generalized semiderivation of R associated with D and α, then one of the following two cases is satisfied*

1. *$\alpha = I$ and so F is a right generalized derivation.*
2. *$F(x) = \lambda x + \mu\alpha(x) = (\lambda + \mu)x + D(x)$ for all $x \in R$, where $\lambda, \mu \in C$.*

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