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THE HIT PROBLEM FOR THE POLYNOMIAL ALGEBRA OF FIVE VARIABLES IN DEGREE SEVENTEEN AND ITS APPLICATION

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Abstract

Let $P_k := \mathbb{F}_2[x_1, x_2, \ldots, x_k]$ be the polynomial algebra in k variables with the degree of each x_i being 1, regarded as a module over the mod-2 Steenrod algebra $\mathcal{A}$, and let GL_k be the general linear group over the prime field $\mathbb{F}_2$. We study the *Peterson hit problem* of finding a minimal set of generators for the polynomial algebra P_k as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The results are used to study the Singer algebraic transfer which is a homomorphism from the homology of the mod-2 Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2)$, to the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ consisting of all the GL_k-invariant classes of degree n.

In this paper, we explicitly determined the Peterson hit problem for $k = 5$ and the dgree 17. Using this result, we show that, Singer's conjecture for the fifth algebraic transfer is true in this degree.

1 Introduction and statement of results

Let $\mathbb{V}_k$ denote a k-dimensional $\mathbb{F}_2$-vector space and let $B\mathbb{V}_k$ denote the classifying space of $\mathbb{V}_k$. It may be thought as the product of k copies of the real projective space $\mathbb{R}\mathbb{P}^\infty$. As is well known,

$$P_k := H^*(B\mathbb{V}_k) \cong \mathbb{F}_2[x_1, x_2, \ldots, x_k],$$

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a polynomial algebra on k generators $x_1, x_2, \ldots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

Being the cohomology of a space, P_k is a module over the mod-2 Steenrod algebra $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is determined by the elementary properties of the Steenrod squares Sq^i and the Cartan formula $Sq^m(fg) = \sum_{j=0}^{m} Sq^j(f)Sq^{m-j}(g)$, for $f, g \in P_k$ (see Steenrod-Epstein [13]).

Let $GL_k := GL(\mathbb{V}_k)$ be the general linear group over the field $\mathbb{F}_2$. This group acts regularly on $\mathbb{V}_k$ and therefore on the cohomology of $B\mathbb{V}_k$. Since the two actions of $\mathcal{A}$ and GL_k upon $H^*(B\mathbb{V}_k)$ commute with each other, there is an inherited action of GL_k on QP_k.

A polynomial f in P_k is called *hit* if it can be written as a finite sum $f = \sum_{i\geqslant 0} Sq^{2^i}(f_i)$ for suitable polynomials f_i. That means f belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$. We study the *hit problem*, set up by Frank Peterson, of finding a minimal set of generators for the polynomial algebra P_k as a module over the Steenrod algebra. This means that we want to find a basis of the $\mathbb{F}_2$-vector space $QP_k := \mathcal{A}/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

The hit problem was first studied by Peterson [8], Wood [23], Singer [11], and Priddy [10], who showed its relationship to several classical problems respectively in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups. The vector space QP_k was explicitly calculated by Peterson [8] for $k = 1, 2$, by Kameko [6] for $k = 3$ and by Sum [15] for $k = 4$. However, for $k > 5$, the problem is still open.

Recently, many authors showed their interest in the study of the hit problem in conjunction with the transfer, which was defined by Singer [11]. This transfer is a homomorphism

$$\varphi_k : \operatorname{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_k)^{GL_k}_n,$$

where $\operatorname{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2)$ is isomorphic to $\operatorname{Ext}^{k,k+n}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$, the E_2 term of the Adams spectral sequence of spheres, $(QP_k)_n$ is the subspace of QP_k consisting of all the classes represented by the homogeneous polynomials of degree n in P_k and $(QP_k)^{GL_k}_n$ is the the subspace of $(QP_k)_n$ consisting of all the GL_k-invariant classes.

Singer showed in [11] that φ_k is an isomorphism for $k = 1, 2$. Boardman showed in [1] that φ_3 is also an isomorphism. Bruner-Ha-Hung [2], Hung [5], Ha [4], Sum [16] and Sum-Tin [19] have studied the transfer for $k = 4, 5$. However, for $k > 3$, the transfer is not a monomorphism. Singer made the following conjecture.

Conjecture 1.1 (Singer [11]). *The algebraic transfer φ_k is an epimorphism for any $k \geqslant 0$.*

The conjecture is true for $k \leqslant 3$. However, for $k > 3$, it is open.

In this paper, we explicitly determined the Peterson hit problem for $k = 5$ and the degree 17. This result is used to verify Singer's conjecture. One of our main results is the following.

Theorem 1.2. *$(QP_5)_{17}$ is the $\mathbb{F}_2$-vector space of dimension 566 with a basis consisting of all the classes represented by the monomials b_t, $1 \leqslant t \leqslant 566$, which are determined as in Subsection 4.2.*

The space $(QP_5)_{17}^{GL_5}$ is explicitly computed by using this theorem. We have

Theorem 1.3. *There exists uniquely a non-zero class in $(QP_5)_{17}^{GL_5}$.*

By combining Theorem 1.3 with the results of Singer [11] and Ha [4], one gets the following.

Theorem 1.4. *The homomorphism $\varphi_5 : \mathrm{Tor}^{\mathcal{A}}_{5,22}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_5)_{17}^{GL_5}$ is an isomorphism.*

The last theorem confirms that Singer's conjecture is true for $k = 5$ and the degree 17.

In Section 2, we recall some needed information on the admissible monomials in P_k, Singer's criterion on the hit monomials and Kameko's homomorphism. Our results will be proved in Section 3. Finally, in the appendix, we list all the admissible monomials of degrees 6, 17 in P_5.

2 Preliminaries

In this section, we recall some results in Kameko [6], Sum [15] and Singer [12] which will be used in the next section.

Notation 2.1. Let $\alpha_j(n)$ denote the j-th coefficients in dyadic expansion of a non-negative integer n. That means $n = \alpha_0(n)2^0 + \alpha_1(n)2^1 + \cdots + \alpha_j(n)2^j + \cdots$, for $\alpha_j(n) \in \{0, 1\}$ and $j \geqslant 0$.

Let $x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \in P_k$. Set $I_j(x) = \{i \in \mathbb{N}_k :\ \alpha_j(a_i) = 0\}$, for $j \geqslant 0$. Then we have

$$x = \prod_{j \geq 1} X_{I_{j-1}(x)}^{2^{j-1}}.$$

Definition 2.2. For a monomial $x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \in P_k$, we define two sequences associated with x by

$$\begin{aligned} \omega(x) &= (\omega_1(x), \omega_2(x), \ldots, \omega_j(x), \ldots) \\ \sigma(x) &= (a_1, a_2, \ldots, a_k), \end{aligned}$$

where $\omega_j(x) = \sum_{1 \leqslant i \leqslant k} \alpha_{j-1}(a_i) = \deg X_{I_{j-1}(x)}, j \geqslant 1$.

The sequence $\omega(x)$ is called the *weight vector* of the monomial x and $\sigma(x)$ called the *exponent vector* of the monomial x. The weight vectors and the exponent vectors can be ordered by the left lexicographical order.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a non-negative integer such that $\omega_i = 0$ for $i \gg 0$. Define $\deg\omega = \sum_{i \geqslant 1} 2^{i-1}\omega_i$. We denote

$$P_k(\omega) = \langle\{x \in P_k:\ \deg x = \deg\omega \text{ and } \omega(x) \leqslant \omega\}\rangle \subset P_k,$$
$$P_k^-(\omega) = \langle\{x \in P_k:\ \deg x = \deg\omega \text{ and } \omega(x) < \omega\}\rangle \subset P_k(\omega).$$

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k.

(i) $f \equiv g \bmod \mathcal{A}^+.P_k$ if and only if $f + g \in \mathcal{A}^+.P_k$. If $f \equiv 0$ then f is called hit.

(ii) $f \equiv_\omega g \bmod (\mathcal{A}^+.P_k + P_k^-(\omega))$ if and only if $f + g \in \mathcal{A}^+.P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$. Then we have

$$QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+.P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

For a polynomial $f \in P_k$, we denote by $[f]$ the classes in QP_k represented by f. If ω is a weight vector and $f \in P_k(\omega)$, then denote by $[f]_\omega$ the classes in $QP_k(\omega)$ represented by f. Denote by $|S|$ the cardinal of a set S.

It is easy to see that

$$QP_k(\omega) \cong QP_k^\omega := \langle\{[x] \in QP_k:\ x \text{ is admissible and } \omega(x) = \omega\}\rangle.$$

Then, we get

$$(QP_k)_n = \bigoplus_{\deg\omega = n} QP_k^\omega \cong \bigoplus_{\deg\omega = n} QP_k(\omega).$$

Hence, we can identify the vector space $QP_k(\omega)$ with $QP_k^\omega \subset QP_k$.

We note that the weight vector of a monomial is invariant under the permutation of the generators x_i, hence $QP_k(\omega)$ has an action of the symmetric group Σ_k.

For $1 \leqslant i \leqslant k$, define the $\mathcal{A}$-homomorphism $g_i : P_k \longrightarrow P_k$, which is determined by $g_i(x_i) = x_{i+1},\ g_i(x_{i+1}) = x_i,\ g_i(x_j) = x_j$ for $j \neq i, i+1,\ 1 \leqslant i < k$ and $g_k(x_1) = x_1 + x_2,\ g_k(x_j) = x_j$ for $j \geqslant 2$,. Observe that the general linear group $GL_k = GL(\mathbb{V}_k)$ is generated by $g_i,\ 0 \leqslant i \leqslant k$ and the symmetric group $\Sigma_k \subset GL_k$ is generated by $g_i,\ 1 \leqslant i \leqslant k-1$. Hence, a homogeneous polynomial $f \in P_k$ is an GL_k-invariant if and only if $g_i(f) \equiv f$ for $1 \leqslant i \leqslant k$. If $g_i(f) \equiv f$ for $1 \leqslant i \leqslant k-1$, then f is an Σ_k-invariant.

Lemma 2.4 (see Sum [17]). *If ω is a weight vector, then $QP_k(\omega)$ is the GL_k-module.*

Now, we recall some relations on the action of the Steenrod squares on P_k.

Proposition 2.5. *Let f be a homogeneous polynomial in P_k and the Steenrod squares $Sq^i : (P_k)_n \longrightarrow (P_k)_{n+i},\ i \geqslant 0$.*

(i) If $i > \deg f$ then $Sq^i(f) = 0$. If $i = \deg f$ then $Sq^i(f) = f^2$.

(ii) If i is not divisible by 2^s then $Sq^i(f^{2^s}) = 0$ while $Sq^{r2^s}(f^{2^s}) = (Sq^r(f))^{2^s}$.

Definition 2.6. Let x, y be monomials in P_k. We say that $x < y$ if and only if one of the following holds

(i) $\omega(x) < \omega(y)$;

(ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.7. A monomial x is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_l$ such that $y_t < x$, for $1 \leqslant t \leqslant l$ and $x = \sum_{t=1}^{l} y_t \bmod (\mathcal{A}^+.P_k)$.

A monomial x is said to be admissible if is not inadmissible.

Obviously the set of all the admissible monomials of degree n in P_k is a miniaml set of $\mathcal{A}$-generators for P_k in degree n.

Definition 2.8. A monomial x in P_k is said to be strictly inadmissible if and only if there exist monomials $y_1, y_2, \ldots, y_t$ such that $y_j < x$, for $j = 1, 2, \ldots, t$ and

$$x = \sum_{j=1}^{t} y_j + \sum_{u=1}^{2^s-1} Sq^u(h_u)$$

with $s = \max\{i : \omega_i(x) > 0\}$ and suitable polynomials $h_u \in P_k$.

It is easy to see that if x is strictly inadmissible, then it is inadmissible. We recall the following.

Theorem 2.9 (Kameko [6], Sum [15]). *Let x, y, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*

(i) If w is inadmissible, then xw^{2^r} is also inadmissible.

(ii) If w is strictly inadmissible, then wy^{2^s} is also strictly inadmissible.

Now, we recall a result of Singer [12].

Definition 2.10. A monomial $z = x_1^{b_1} x_2^{b_2} \ldots x_k^{b_k}$ is called a spike if $b_i = 2^{s_i} - 1$ for s_i a non-negative integer and $1 \leqslant i \leqslant k$. If z is a spike with $s_1 > s_2 > \ldots > s_{l-1} \geqslant s_l$ and $s_j = 0$ for $j \geqslant l+1$, then it is called a minimal spike.

For a positive integer n, by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1\leqslant i\leqslant r}(2^{d_i}-1)$, where $d_i > 0$. In [12], Singer showed that if $\mu(n) \leqslant k$, then there exists uniquely a minimal spike of degree n in P_k.

The following is a criterion for the hit monomials in P_k.

Theorem 2.11 (Singer [12]). *Suppose $x \in P_k$ is a monomial of degree n, where $\mu(n) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$ then x is hit.*

From this theorem, we see that if z is a minimal spike, then $P_k^-(\omega(z)) \subset \mathcal{A}^+P_k$. We set

$$P_k^0 = \langle\{x = x_1^{a_1}x_2^{a_2}\dots x_k^{a_k} \mid a_1a_2\dots a_k = 0\}\rangle,$$
$$P_k^+ = \langle\{x = x_1^{a_1}x_2^{a_2}\dots x_k^{a_k} \mid a_1a_2\dots a_k > 0\}\rangle.$$

It is easy to see that P_k^0 and P_k^+ are the $\mathcal{A}$-submodules of P_k. Furthermore, we have the following.

Proposition 2.12. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces*

$$QP_k = QP_k^0 \oplus QP_k^+.$$

Here $QP_k^0 = P_k^0/\mathcal{A}^+.P_k^0$ and $QP_k^+ = P_k^+/\mathcal{A}^+.P_k^+$.

Definition 2.13. For $1 \leqslant i \leqslant k$, define the homomorphism $f_i = f_{k;\,i} : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j & \text{if } 1 \leqslant i \leqslant j-1, \\ x_{j+1} & \text{if } j \leqslant i \leqslant k-1. \end{cases}$$

For a subset $\mathcal{B} \subset P_k$, we denote $[\mathcal{B}] = \{[f] : f \in \mathcal{B}\}$. If $\mathcal{B} \subset P_k(\omega)$, then we set $[\mathcal{B}]_\omega = \{[f]_\omega : f \in \mathcal{B}\}$. From Theorem 2.11, we see that if ω is the weight vector of a minimal spike in P_k, then $[\mathcal{B}]_\omega = [\mathcal{B}]$.

Clearly, we have

Proposition 2.14 (Sum [15]). *If $\mathcal{B}$ is a minimal set of generators for $(P_{k-1})_n$, then $f(\mathcal{B}) = \bigcup_{i=1}^k f_i(\mathcal{B})$ is the minimal set of generators for $(P_k^0)_n$.*

From now on, we denote by $\mathcal{B}_k(n)$ the set of all admissible monomials in $(P_k)_n$, $\mathcal{B}_k^0(n) = \mathcal{B}_k(n) \cap (P_k^0)_n$, $\mathcal{B}_k^+(n) = \mathcal{B}_k(n) \cap (P_k^+)_n$. For a weight ω of degree n, we set $\mathcal{B}_k(\omega) = \mathcal{B}_k(n) \cap P_k(\omega)$, $\mathcal{B}_k^+(\omega) = \mathcal{B}_k(n) \cap P_k^+(\omega)$.

Then, $[\mathcal{B}_k(\omega)]_\omega$ and $[\mathcal{B}_k^+(\omega)]_\omega$ are respectively the basis of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap (QP_k^+)_n$.

For any monomials $z, z_1, \dots, z_t \in P_k(\omega)$ with $t \geqslant 1$, we denote

$$\Sigma_k(z_1,\dots,z_t) = \{\sigma z_i : \sigma \in \Sigma_k,\ 1 \leqslant i \leqslant t\} \subset P_k(\omega),$$
$$[\mathcal{B}(z_1,\dots,z_t)]_\omega = [\mathcal{B}_k(\omega)]_\omega \cap \langle[\Sigma_k(z_1,\dots,z_t)]_\omega\rangle,$$
$$p(z) = \sum_{x\in\mathcal{B}_k(n)\cap\Sigma_k(z)} x.$$

We denote

$\mathcal{N}_k = \{(i;I) \mid I = (i_1, i_2, \ldots, i_r), 1 \leqslant i < i_1 < i_2 < \ldots < i_r \leqslant k, 0 \leqslant r \leqslant k-1\}.$

Definition 2.15. For any $(i;I) \in \mathcal{N}_k$, we define the homomorphism $p_{(i;I)}: P_k \to P_{k-1}$ of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j & \text{if } 1 \leqslant j \leqslant i-1, \\ \sum\limits_{s\in I} x_{s-1} & \text{if } j = i, \\ x_{j-1} & \text{if } i+1 \leqslant j \leqslant k. \end{cases}$$

Then $p_{(i,I)}$ is a homomorphism of $\mathcal{A}$-modules. In particular, we have $p_{(i;\emptyset)}(x_i) = 0$ for $1 \leqslant i \leqslant k$.

Lemma 2.16 (see [9]). *If x is a monomial in P_k, then $p_{(i,I)}(x) \in P_{k-1}(\omega(x))$.*

Lemma 2.16 implies that if ω is a weight vector and $x \in P_k(\omega(x))$, then $p_{(i,I)}(x) \in P_{k-1}(\omega)$. Moreover, $p_{(i,I)}$ passes to a homomorphism from $QP_k(\omega)$ to $QP_{k-1}(\omega)$.

We end this section by recalling the definition of Kameko's homomorphism $\widetilde{Sq}^0_*: QP_k \to QP_k$. This homomorphism is an GL_k-homomorphism induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_*: P_k \to P_k$, given by

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if } x = x_1x_2\ldots x_k y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However, $\widetilde{Sq}^0_* Sq^{2t} = Sq^t \widetilde{Sq}^0_*$ and $\widetilde{Sq}^0_* Sq^{2t+1} = 0$ for any non-negative integer t. Denote by $(\widetilde{Sq}^0_*)_{(k,d)}: (QP_k)_{2d+k} \to (QP_k)_d$ Kameko's homomomorphism in degree $2d+k$.

3 Proofs of the results

In this section, we prove our results which are stated in the introduction.

3.1 Proof of Theorem 1.2

Consider Kameko's homomorphism $(\widetilde{Sq}^0_*)_{(5,6)}: (QP_5)_{17} \to (QP_5)_6$. Since this homomorphism is an epimorphism, we get

$$\begin{aligned}(QP_5)_{17} &\cong \mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)} \oplus (QP_5)_6 \\ &\cong (QP_5^0)_{17} \oplus (\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)} \cap (QP_5^+)_{17}) \oplus (QP_5)_6.\end{aligned}$$

The computation of $(QP_5)_6$ is easy. We have the following.

Proposition 3.1.1. *$(QP_5)_6$ is the $\mathbb{F}_2$-vector space of dimension 74 with a basis consisting of all the classes represented by the monomials $a_t,\ 1 \leqslant t \leqslant 75$, which are determined as in Subsection 4.1.*

From a result in [15] and Proposition 2.14, we easily obtain

Proposition 3.1.2. *$(QP_5)^0_{17}$ is the $\mathbb{F}_2$-vector space of dimension 335 with a basis consisting of all the classes represented by the monomials $b_t,\ 1 \leqslant t \leqslant 335$, which are determined as in Subsection 4.2.*

Now, we compute $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)} \cap (QP_5^+)_{17}$.

Proposition 3.1.3. *The set $\{[b_t] : 336 \leqslant t \leqslant 492\}$ is the basis of the $\mathbb{F}_2$-vector space $Ker(\widetilde{Sq}^0_*)_{(5,6)} \cap (QP_5^+)_{17}$. Here the monomials $b_t,\ 336 \leqslant t \leqslant 492$, which are determined as in Subsection 4.2.*

By combining Proposition 3.1.1-3.1.3, we get $\dim(QP_5)_{17} = 566$.
We prove the proposition by proving some lemmas.

Lemma 3.1.4. *If x is an admissible monomial of degree 17 in P_5 and $[x]$ belong to $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)}$, then $\omega(x)$ is one of the sequences: $(3,1,1,1)$, $(3,1,3)$, $(3,3,2)$.*

Proof. Observe $z = x_1^{15}x_2x_3$ is the minimal spike of degree 17 in P_5 and $\omega(z) = (3,1,1,1)$. Since $[x] \neq [0]$, by Theorem 2.11, we get either $\omega_1(x) = 3$ or $\omega_1(x) = 5$. If $\omega_1(x) = 5$ then $x = x_1x_2x_3x_4x_5y^2$ with y a monomial of degree 6 in P_5. Since x is admissible, by Theorem 2.9, y is admissible. Hence, $(\widetilde{Sq}^0_*)_{(5,6)}([x]) = [y] \neq [0]$. This contradicts the fact that $[x] \in \mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)}$. Hence, $\omega_1(x) = 3$. Then, we have $x = x_ix_jx_\ell y_1^2$ with $1 \leqslant i < j < \ell \leqslant 5$ and y_1 is an admissible monomial of degree 7 in P_5. Since y_1 is admissible, according to a result in Tin [20], we have either $\omega(y_1) = (1,1,1)$ or $\omega(y_1) = (1,3)$ or $\omega(y_1) = (3,2)$. The lemma is proved. $\square$

Lemma 3.1.5. *The following monomials are stricly inadmissible:*

$x_2^2x_3x_4x_5$	$x_1^2x_3x_4x_5$	$x_1^2x_2x_4x_5$	$x_1^2x_2x_3x_5$	$x_1^2x_2x_3x_4$
$x_2^3x_3^{12}x_4x_5$	$x_1^3x_3^{12}x_4x_5$	$x_1^3x_2^{12}x_4x_5$	$x_1^3x_2^{12}x_3x_5$	$x_1^3x_2^{12}x_3x_4$
$x_2^3x_3^4x_4x_5^9$	$x_2^3x_3^4x_4^9x_5$	$x_1^3x_3^4x_4x_5^9$	$x_1^3x_3^4x_4^9x_5$	$x_1^3x_2^4x_4x_5^9$
$x_1^3x_2^4x_4^9x_5$	$x_1^3x_2^4x_3x_5^9$	$x_1^3x_2^4x_3x_4^9$	$x_1^3x_2^4x_3^9x_5$	$x_1^3x_2^4x_3^9x_4$
$x_1^3x_2^4x_3x_4x_5^8$	$x_1^3x_2^4x_3x_4^8x_5$	$x_1^3x_2^4x_3^8x_4x_5.$		

Proof. We prove the lemma for the monomial $x = x_1^3x_2^4x_3^8x_4x_5$. The others can be proved by a similar computation. By a direct computation, we have

$$\begin{aligned}x &= x_1^3x_2x_3^8x_4^4x_5 + x_1^3x_2x_3^8x_4x_5^4 + x_1^2x_2x_3^{12}x_4x_5 + Sq^1(x_1^3x_2x_3^{10}x_4x_5)\\ &\quad + Sq^2(x_1^5x_2^2x_3^6x_4x_5 + x_1^5x_2x_3^6x_4^2x_5 + x_1^5x_2x_3^6x_4x_5^2 + x_1^2x_2x_3^{10}x_4x_5)\\ &\quad + Sq^4(x_1^3x_2^2x_3^6x_4x_5 + x_1^3x_2x_3^6x_4^2x_5 + x_1^3x_2x_3^6x_4x_5^2) \bmod\big(P_5^-(3,1,1,1)\big).\end{aligned}$$

Hence, x is strictly inadmissible. □

Lemma 3.1.6. *The $\mathbb{F}_2$-vector space $QP_5^+(3,1,1,1)$ is spanned by the set*

$$\{[b_t]:\ 336 \leqslant t \leqslant 356\}.$$

Proof. Let x be an admissible monomial in P_5 such that $\omega(x) = (3,1,1,1)$. Then, $\omega_1(x) = 3$, $x = x_ix_jx_\ell y^2$ with $1 \leqslant i < j < \ell << 5$ and y a monomial of degree 7 in P_5. Since x is admissible, according to Theorem 2.9, $y \in B_5(1,1,1)$ (see [20]).

Let $z \in B_5(1,1,1)$ and $1 \leqslant i < j < \ell \leqslant 5$. By a direct computation, we see that if $x_ix_jx_\ell z^2 \neq b_t, \forall t, 336 \leqslant t \leqslant 356$, then there is a monomial w which is given in Lemma 3.1.5 such that $x_ix_jx_\ell z^2 = wz_1^{2^u}$ with suitable monomial $z_1 \in P_5$, and $u = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.9, $x_ix_jx_\ell z^2$ is inadmissible. Since $x = x_ix_jx_\ell y^2$ with $y \in B_5(1,1,1)$ and x is admissible, one gets $x = b_t$ for some t. The lemma is proved. □

Lemma 3.1.7. *The following monomials are stricly inadmissible:*

$x_1^3x_2^4x_3x_4^4x_5^5$	$x_1^3x_2^4x_3x_4^5x_5^4$	$x_1^3x_2^4x_3^4x_4x_5^5$	$x_1^3x_2^4x_3^4x_4^5x_5$
$x_1^3x_2^4x_3^5x_4x_5^4$	$x_1^3x_2^4x_3^5x_4^4x_5$	$x_1^3x_2^5x_3^4x_4x_5^4$	$x_1^3x_2^5x_3^4x_4^4x_5$

Proof. We only prove the lemma for the monomial $x = x_1^3x_2^4x_3x_4^5x_5^5$. We have

$$\begin{aligned}x &= x_1^3x_2x_3^4x_4^4x_5^5 + Sq^1(x_1^3x_2x_3^1x_2x_5^9)\\ &\quad + Sq^2(x_1^5x_2^2x_3x_4^2x_5^5 + x_1^5x_2x_3^2x_4^2x_5^5 + x_1^5x_2x_3x_4^2x_5^6)\\ &\quad + Sq^4(x_1^3x_2^2x_3x_4^2x_5^5 + x_1^3x_2x_3^2x_4^2x_5^5 + x_1^3x_2x_3x_4^2x_5^6) \bmod\big(P_5^-(3,1,3)\big).\end{aligned}$$

This equality implies x is strictly inadmissible. □

Lemma 3.1.8. *The $\mathbb{F}_2$-vector space $QP_5^+(3,1,3)$ is spanned by the set*

$$\{[b_t]_{(3,1,3)}:\ 357 \leqslant t \leqslant 366\}.$$

Proof. Let x be an admissible monomial in P_5 such that $\omega(x) = (3,1,3)$. Then, $\omega_1(x) = 3$, $x = x_ix_jx_\ell y^2$ with $1 \leqslant i < j < \ell <\leqslant 5$ and y a monomial of degree 7 in P_5. Since x is admissible, according to Theorem 2.9, $y \in B_5(1,3)$.

Let $z \in B_5(1,3)$ and $1 \leqslant i < j < \ell \leqslant 5$. A direct computation shows that if $x_ix_jx_\ell z^2 \neq b_t, \forall t, 357 \leqslant t \leqslant 366$, then there is a monomial w which is given in one of Lemmas 3.1.5, 3.1.7 such that $x_ix_jx_\ell z^2 = wz_1^{2^r}$ with suitable monomial $z_1 \in P_5$, and $r = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.9, $x_ix_jx_\ell z^2$ is inadmissible. Since $x = x_ix_jx_\ell y^2$ with $y \in B_5(1,3)$ and x is admissible, one gets $x = b_t$ for some t, $357 \leqslant t \leqslant 366$, completing the proof. $\square$

By an easy computation, we get the following.

Lemma 3.1.9. *If (i,j,ℓ,m,n) is a permutation of $(1,2,3,4,5)$ such that $i < j$, then the monomials $x_i^2x_jx_\ell^3x_m^3$ is stricly inadmissible.*

Lemma 3.1.10. *The following monomials are stricly inadmissible:*

$x_1x_2^2x_3^6x_4x_5^7$	$x_1x_2^2x_3^6x_4^7x_5$	$x_1x_2^2x_3^7x_4^6x_5$	$x_1x_2^6x_3^2x_4x_5^7$	$x_1x_2^6x_3^2x_4^7x_5$
$x_1x_2^6x_3^7x_4^2x_5$	$x_1x_2^7x_3^2x_4^6x_5$	$x_1x_2^7x_3^6x_4^2x_5$	$x_1^7x_2x_3^2x_4^6x_5$	$x_1^7x_2x_3^6x_4^2x_5$
$x_1x_2^6x_3^3x_4^6x_5$	$x_1x_2^6x_3^6x_4x_5^3$	$x_1x_2^6x_3^6x_4^3x_5$	$x_1x_2^2x_3^2x_4^5x_5^7$	$x_1x_2^2x_3^2x_4^7x_5^5$
$x_1x_2^2x_3^7x_4^2x_5^5$	$x_1x_2^7x_3^2x_4^2x_5^5$	$x_1^7x_2x_3^2x_4^2x_5^5$	$x_1x_2^2x_3^6x_4^3x_5^5$	$x_1x_2^2x_3^6x_4^5x_5^3$
$x_1x_2^6x_3^2x_4^3x_5^5$	$x_1x_2^6x_3^2x_4^5x_5^3$	$x_1x_2^6x_3^3x_4^2x_5^5$	$x_1x_2^6x_3^3x_4^4x_5^3$	$x_1^3x_2^4x_3x_4^6x_5^3$
$x_1^3x_2^4x_3^3x_4^3x_5^4$	$x_1^3x_2^4x_3^3x_4^4x_5^3$			

Proof. We only prove the lemma for the monomials $x = x_1x_2^2x_3^6x_4x_5^7$ and $y = x_1x_2^6x_3^3x_4^4x_5^3$. The others can be proved by a similar computation. A direct computation shows

$$\begin{aligned}
x &= x_1x_2^2x_3^5x_4^2x_5^7 + x_1x_2x_3^6x_4^2x_5^7 + Sq^1(x_1^2x_2x_3^5x_4x_5^7)\\
&\quad + Sq^2(x_1x_2x_3^5x_4x_5^7 + x_1x_2x_3^3x_4x_5^9) \bmod\big(P_5^-(3,3,2)\big),\\
y &= x_1x_2^3x_3^3x_4^4x_5^6 + x_1x_2^3x_3^3x_4^6x_5^4 + x_1x_2^3x_3^4x_4^3x_5^6 + x_1x_2^3x_3^4x_4^6x_5^3\\
&\quad + x_1x_2^3x_3^6x_4^3x_5^4 + x_1x_2^3x_3^6x_4^4x_5^3 + x_1x_2^4x_3^3x_4^3x_5^6 + x_1x_2^4x_3^3x_4^6x_5^3\\
&\quad + x_1x_2^4x_3^6x_4^3x_5^3 + x_1x_2^6x_3^2x_4^3x_5^5 + x_1x_2^6x_3^2x_4^5x_5^3 + x_1x_2^6x_3^3x_4^3x_5^4\\
&\quad + Sq^1(x_1^2x_2^5x_3^3x_4^3x_5^3 + x_1^2x_2^3x_3^5x_4^3x_5^3 + x_1^2x_2^3x_3^3x_4^5x_5^3 + x_1^2x_2^3x_3^3x_4^3x_5^5)\\
&\quad + Sq^2(x_1x_2^5x_3^3x_4^3x_5^3 + x_1x_2^3x_3^5x_4^3x_5^3 + x_1x_2^3x_3^3x_4^5x_5^3 + x_1x_2^3x_3^3x_4^3x_5^5\\
&\quad + x_1x_2^6x_3^2x_4^3x_5^3) \bmod\big(P_5^-(3,3,2)\big).
\end{aligned}$$

Hence, x, y are strictly inadmissible. $\square$

Lemma 3.1.11. *The $\mathbb{F}_2$-vector space $QP_5^+(\omega)$ is spanned by the set*

$$\{[b_t]_{\bar\omega},\ 367 \leqslant t \leqslant 492\},$$

where $\omega = (3,3,2)$.

Proof. Let x be an admissible monomial in P_5 such that $\omega(x) = (3,3,2)$. Then, $\omega_1(x) = 3$, $x = x_ix_jx_\ell y^2$ with $1 \leqslant i < j < \ell <\leqslant 5$ and y a monomial of degree 7 in P_5. Since x is admissible, according to Theorem 2.9, $y \in B_5(3,2)$.

Let $z \in B_5(3,2)$ and $1 \leqslant i < j < \ell \leqslant 5$. A routine computation shows that if $x_ix_jx_\ell z^2 \neq b_t, \forall t, 367 \leqslant t \leqslant 492$, then there is a monomial w which is given in one of Lemmas 3.1.9, 3.1.10 such that $x_ix_jx_\ell z^2 = wz_1^{2^r}$ with suitable monomial $z_1 \in P_5$, and $r = \max\{s \in \mathbb{Z} : \omega_s(w) > 0\}$. By Theorem 2.9, $x_ix_jx_\ell z^2$ is inadmissible. Since $x = x_ix_jx_\ell y^2$ with $y \in B_5(3,2)$ and x is admissible, one gets $x = b_t$ for some $t,\ 367 \leqslant t \leqslant 492$, completing the proof. □

Now, we are ready to prove Proposition 3.1.3.

Proof. [Proof of Proposition 3.1.3] Lemmas 3.1.6, 3.1.8, 3.1.11 and 3.1.4 imply that the space $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)} \cap (QP_5^+)_{17}$ is spanned by the set $\{[b_t] : 336 \leqslant t \leqslant 492\}$. Now, we prove this set is linearly independent in QP_5.

Suppose there is a linear relation $\mathcal{S} = \sum_{t=336}^{492} \gamma_t b_t \equiv 0$, where $\gamma_t \in \mathbb{F}_2$, for all $t,\ 336 \leqslant t \leqslant 492$. For $(i;I) \in \mathcal{N}_5$, we explicitly compute $p_{(i;I)}(\mathcal{S})$ in terms ot the admissible monomials of degree 17 in P_4. By a direct computation from the relations $p_{(i;I)}(\mathcal{S}) \equiv 0$ with either $I = (j), 1 \leqslant i < j \leqslant 5$ or $i = 1, I = (2,3), (2,4), (3,4)$ we will obtain $\gamma_t = 0$ for all $t,\ 336 \leqslant t \leqslant 492$. The proposition is proved. □

3.2 Proof of Theorem 1.3

To prove the theorem, we need to compute $(QP_5)_6^{GL_5}$.

Proposition 3.2.1. $(QP_5)_6^{GL_5} = 0$.

Proof. By Proposition 3.1.1, if $[h] \in (QP_5)_6^{GL_5}$, then $h \equiv \sum_{t=1}^{74} \gamma_t a_t$ with $\gamma_t \in \mathbb{F}_2$ and $g_i(h) \equiv h$ for $i = 1,2,3,4,5$. By a direct computation from the relations $g_i(h) \equiv h$ for $i = 1,2,3,4$, we obtain

$$\begin{cases} \gamma_t = \gamma_1, \forall t,\ 1 \leqslant t \leqslant 10, \\ \gamma_t = \gamma_{11}, \forall t,\ 11 \leqslant t \leqslant 40, \\ \gamma_t = 0, \forall t,\ 41 \leqslant t \leqslant 50, \text{ and } \ 71 \leqslant t \leqslant 74, \\ \gamma_t = \gamma_{51}, \forall t,\ 51 \leqslant t \leqslant 70. \end{cases}$$

Now, computing directly from the relation $g_5(h) \equiv h$, one gets $\gamma_1 = \gamma_{11} = \gamma_{51} = 0$. Hence, $\gamma_t = 0,\ \forall t,\ 1 \leqslant t \leqslant 74$. The proposition is proved. □

Recall that Kameko's homomorphism $(\widetilde{Sq}^0_*)_{(5,6)} : (QP_5)_{17} \longrightarrow (QP_5)_6$ is a homomorphism of GL_5-modules. Hence, using Proposition 3.2.1, we need to compute $(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)}))^{GL_5}$. We have a direct summand decomposition of the Σ_5-modules:

$$(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)}))^{GL_5} = (QP_5^0)_{17}\bigoplus\Big((\mathrm{Ker}(\widetilde{Sq}^0_*)_{(5,6)}))^{GL_5}\bigcap(QP_5^+)_{17}\Big).$$

By Theorem 1.2, we have

$$(QP_5^0)_{17} = QP_5^0(3,1,1,1)\bigoplus QP_5^0(3,1,3)\bigoplus QP_5^0(3,3,2).$$

Lemma 3.2.2. *We have*

$$QP_5^0(3,1,1,1)^{\Sigma_5} = \langle[p(b_1)],[p(b_{31})],[p(b_{61})],[p_1]\rangle,$$

where $p_1 = \sum_{t=71}^{110} b_t$.

Proof. [Outline of the proof] By a direct computation using the results of Theorem 1.2, we see that see that there is a direct summand decomposition of the Σ_5-modules:

$$QP_5^0(3,1,1,1) = \langle[\Sigma_5(b_1)]\rangle\bigoplus\langle[\Sigma_5(b_{31})]\rangle\bigoplus\langle[\Sigma_5(b_{61})]\rangle\bigoplus\langle[\Sigma_5(b_{71},b_{136})],$$

where

$$B_5(b_1) = \{b_t : 1\leqslant t\leqslant 30\},\quad B_5(b_{31}) = \{b_t : 31\leqslant t\leqslant 60\},$$
$$B_5(b_{61}) = \{b_t : 61\leqslant t\leqslant 70\},\quad B_5(b_{71},b_{136}) = \{b_t : 71\leqslant t\leqslant 160\}.$$

Let $[f]\in\langle[\Sigma_5(b_j)]\rangle^{\Sigma_5}$, $j = 1,31,61$. Then, $f\equiv_\omega \sum_{z\in B(b_i)}\gamma_z.z$. By a direct computation, we can see that the action of Σ_5 on QP_5 induces the one of it on the set $[B(b_i)]$. Furthermore, this action is transitive. Hence, we get $\gamma_z = \gamma_{z'} = \gamma\in\mathbb{F}_2$, for all $z,z'\in B(b_i)$. This implies $f\equiv\gamma p(b_i)$.

If $[f]\in\langle[\Sigma_5(b_{71},b_{136})]\rangle^{\Sigma_5}$. Then $f\equiv\sum_{t=71}^{160}\gamma_t b_t$. Computing directly from the relation $g_i(f)\equiv f$ for $i = 1,2,3,4$, gives $\gamma_t = \gamma_{71}, 71\leqslant t\leqslant 110$ and $\gamma_t = 0$ for $t > 110$. Hence $f\equiv\gamma_{71}p_1$. The lemma is proved. □

Proposition 3.2.3. *Let* $\omega = (3,3,2)$. *Then,* $QP_5(\omega)^{GL_5} = \langle[p_2]_\omega\rangle$, *where*

$$\begin{aligned}p_2 &= b_{428}+b_{429}+b_{439}+b_{440}+b_{490}+b_{491}+b_{492}\\ &= x_1x_2x_3^6x_4^3x_5^6 + x_1x_2x_3^6x_4^6x_5^3 + x_1x_2^2x_3^3x_4^5x_5^6 + x_1x_2^2x_3^3x_4^6x_5^5\\ &\quad + x_1^3x_2^3x_3^3x_4^4x_5^4 + x_1^3x_2^3x_3^4x_4^3x_5^4 + x_1^3x_2^3x_3^4x_4^4x_5^3.\end{aligned}$$

We prepare some lemmas for the proof of the proposition.

From Theorem 1.2, there is a direct summand decomposition of the Σ_5-modules:

$$\begin{aligned}QP_5^0(\omega) = \langle[\Sigma_5(b_{161})]_\omega\rangle&\bigoplus\langle[\Sigma_5(b_{191})]_\omega\rangle\bigoplus\langle[\Sigma_5(b_{221})]_\omega\rangle\\ &\bigoplus\langle[\Sigma_5(b_{321})]_\omega\rangle\bigoplus\langle[\Sigma_5(b_{392})]_\omega\rangle\bigoplus\langle[\Sigma_5(b_{439})]_\omega\rangle.\end{aligned}$$

where

$$B_5(b_{161}) = \{b_t : 161 \leqslant t \leqslant 190\},\quad B_5(b_{191}) = \{b_t : 191 \leqslant t \leqslant 220\},$$
$$B_5(b_{221}) = \{b_t : 221 \leqslant t \leqslant 320\},\quad B_5(b_{321}) = \{b_t : 321 \leqslant t \leqslant 335\}$$
$$B_5(b_{392}) = \{b_t : 367 \leqslant t \leqslant 426\},\quad B_5(b_{439}) = \{b_t : 427 \leqslant t \leqslant 492\}.$$

Lemma 3.2.4. *We have*

i) $\langle[\Sigma_5(b_i)]_\omega\rangle^{\Sigma_5} = \langle[p(b_i)]_\omega\rangle$ *for* $i = 161, 191, 321$.
ii) $\langle[\Sigma_5(b_{221})]_\omega\rangle^{\Sigma_5} = \langle[p_3]_\omega\rangle$ *with* $p_3 = \sum_{t=221}^{300} b_t$.
iii) $\langle[\Sigma_5(b_{392})]_\omega\rangle^{\Sigma_5} = \langle[p_4]_\omega\rangle$ *with* $p_3 = \sum_{t=267}^{426} b_t$.
iv) $\langle[\Sigma_5(b_{439})]_\omega\rangle^{\Sigma_5} = \langle[p_2]_\omega\rangle$.

Proof. [Outline of the proof] Let $[f] \in \langle[\Sigma_5(b_u)]_\omega\rangle^{\Sigma_5}$, $u = 161, 191, 221, 321, 392$. Then, $f \equiv_\omega \sum_{z\in B(b_u)} \gamma_z.z$. The lemma is proved by a direct computation from the relations $g_i(f) \equiv_\omega f$ for $i = 1, 2, 3, 4$. □

Proof. [Proof of Proposition 3.2.3] Using Lemma 3.2.4, we have

$$QP_5(\omega)^{\Sigma_5} = \langle[p(b_{161})]_\omega, [p(b_{191})]_\omega, [p(b_{221})]_\omega, [p(b_{321})]_\omega, [p_2]_\omega, [p_3]_\omega, [p_4]_\omega\rangle.$$

Let $f \in P_5(\omega)$ such that $[f]_\omega \in QP_5(\omega)^{GL_5}$. Then,

$$f \equiv_\omega \gamma_1 p(a_{161}) + \gamma_2 p(a_{191}) + \gamma_3 p(a_{321}) + \gamma_4 p_2 + \gamma_5 p_3 + \gamma_6 p_4$$

with $\gamma_j \in \mathbb{F}_2$ for $j = 1, 2, 3, 4, 5, 6$. By a direct computation, we have

$$g_5(f) + f \equiv_\omega \gamma_1 b_{176} + \gamma_2 b_{191} + \gamma_3 b_{224} + \gamma_5 b_{314} + \gamma_6 b_{301} + \text{other terms} \equiv_\omega 0.$$

This equality implies $\gamma_j = 0$ for $j = 1, 2, 3, 5, 6$. So, $f \equiv_\omega \gamma_4 p_2$. The proposition is proved. □

We now prove Theorem 1.3.

Proof. [Proof of Theorem 1.3] Let $f \in P_5(\omega)$ such that $[f] \in (QP_5)_{17}^{GL_5}$. Using Theorem 1.2, we have $[f]_\omega \in QP_5(\omega)^{GL_5}$. By Proposition 3.2.3, $f \equiv_\omega \gamma p_2$. Hence, using Theorem 1.2, one gets

$$f \equiv \gamma p_2 + \sum_{t=336}^{366} \gamma_t b_t + f^*,$$

where $f* \in (QP_5^0)_{17}$. Since $\gamma p_2 + \sum_{t=336}^{366} \gamma_t b_t \in (QP_5^+)_{17}$, by computing from the relations $g_i(f) \equiv f$, we $g_i(f^*) \equiv f^*$ for $i = 1, 2, 3, 4$. Hence $[f^*] \in (QP_5^0)_{17}^{\Sigma_5}$. Using Lemma 3.2.2, we have $f^* \equiv \gamma_1 p(b_1) + \gamma_2 p(b_{31}) + \gamma_3 p(b_{61}) + \gamma_4 p_1$ with

$\gamma_j \in \mathbb{F}_2$ for $j = 1, 2, 3, 4$. Now, by a direct computation using the relations $g_j(f) \equiv f$ for $j = 1, 2, 3, 4, 5$, we obtain

$$\begin{cases} \gamma_1 = \gamma_2 = \gamma_3 = \gamma_{357} = 0, \\ \gamma_t = 0, \ \forall t, \ 326 \leqslant t \leqslant 356, \text{ and } t \neq 336, 350, 351, 354, \\ \gamma_t = \gamma, \ t = 4, 336, 350, 351, 354, 358, 359, \ldots, 366. \end{cases}$$

The last equality implies $f \equiv \gamma(p_1 + p_2 + b_{336} + b_{350} + b_{351} + b_{354} + \sum_{t=358}^{366} b_t)$. The theorem is completely proved. □

3.3 Proof of Theorem 1.4

From the results of Tangora [21], Lin [7] and Chen [3], we have

$$\mathrm{Tor}^{\mathcal{A}}_{5,22}(\mathbb{F}_2, \mathbb{F}_2) = \langle (h_2 d_0)^* \rangle,$$

and $h_2 d_0 \neq 0$, where h_2 denotes the Adams element in $\mathrm{Ext}^{1,4}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ and $d_0 \in \mathrm{Ext}^{4,18}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$.

In [11], Singer showed that the Adams elements h_2 is in the image of φ_1^*. Ha showed in [4] that the element d_0 is in the image of φ_4^*. Since $\varphi^* = \bigoplus_{k \geqslant 0} \varphi_k^*$ is the homomorphism of algebras, we see that the element $h_2 d_0$ is in the image of φ_5^*. This fact implies that $\varphi_5((h_2 d_0)^*) \neq 0$. Hence, from Theorem 1.3, the homomorphism

$$\varphi_5 : \mathrm{Tor}^{\mathcal{A}}_{5,22}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_5)^{GL_5}_{17}$$

is also an isomorphism. Therefore, Singer's conjecture is true in the case $k = 5$ and the degree 17. Theorem 1.4 is proved.

4 Appendix

In the appendix, we list all the admissible monomials of degrees 6 and 17 in P_5.

4.1 The admissible monomials of degree 6 in $\mathbf{P_5}$.

$B_5(6) = B_5^0(6) \cup B_5^+(6)$, where $B_5^0(6)$ is the set of 70 monomials a_t, $1 \leqslant t \leqslant 70$:

1. $x_4^3x_5^3$	2. $x_3^3x_5^3$	3. $x_3^3x_4^3$	4. $x_2^3x_5^3$
5. $x_2^3x_4^3$	6. $x_2^3x_3^3$	7. $x_1^3x_5^3$	8. $x_1^3x_4^3$
9. $x_1^3x_3^3$	10. $x_1^3x_2^3$	11. $x_3x_4^2x_5^3$	12. $x_3x_4^3x_5^2$
13. $x_3^3x_4x_5^2$	14. $x_2x_4^2x_5^3$	15. $x_2x_4^3x_5^2$	16. $x_2x_3^2x_5^3$
17. $x_2x_3^2x_4^3$	18. $x_2x_3^3x_5^2$	19. $x_2x_3^3x_4^2$	20. $x_2^3x_4x_5^2$
21. $x_2^3x_3x_5^2$	22. $x_2^3x_3x_4^2$	23. $x_1x_4^2x_5^3$	24. $x_1x_4^3x_5^2$
25. $x_1x_3^2x_5^3$	26. $x_1x_3^2x_4^3$	27. $x_1x_3^3x_5^2$	28. $x_1x_3^3x_4^2$
29. $x_1x_2^2x_5^3$	30. $x_1x_2^2x_4^3$	31. $x_1x_2^2x_3^3$	32. $x_1x_2^3x_5^2$
33. $x_1x_2^3x_4^2$	34. $x_1x_2^3x_3^2$	35. $x_1^3x_4x_5^2$	36. $x_1^3x_3x_5^2$
37. $x_1^3x_3x_4^2$	38. $x_1^3x_2x_5^2$	39. $x_1^3x_2x_4^2$	40. $x_1^3x_2x_3^2$
41. $x_2x_3x_4^2x_5^2$	42. $x_2x_3^2x_4x_5^2$	43. $x_1x_3x_4^2x_5^2$	44. $x_1x_3^2x_4x_5^2$
45. $x_1x_2x_4^2x_5^2$	46. $x_1x_2x_3^2x_5^2$	47. $x_1x_2x_3^2x_4^2$	48. $x_1x_2^2x_4x_5^2$
49. $x_1x_2^2x_3x_5^2$	50. $x_1x_2^2x_3x_4^2$	51. $x_2x_3x_4x_5^3$	52. $x_2x_3x_4^3x_5$
53. $x_2x_3^3x_4x_5$	54. $x_2^3x_3x_4x_5$	55. $x_1x_3x_4x_5^3$	56. $x_1x_3x_4^3x_5$
57. $x_1x_3^3x_4x_5$	58. $x_1x_2x_4x_5^3$	59. $x_1x_2x_4^3x_5$	60. $x_1x_2x_3x_5^3$
61. $x_1x_2x_3x_4^3$	62. $x_1x_2x_3^3x_5$	63. $x_1x_2x_3^3x_4$	64. $x_1x_2^3x_4x_5$
65. $x_1x_2^3x_3x_5$	66. $x_1x_2^3x_3x_4$	67. $x_1^3x_3x_4x_5$	68. $x_1^3x_2x_4x_5$
69. $x_1^3x_2x_3x_5$	70. $x_1^3x_2x_3x_4$		

$B_5^+(6)$ is the set of 4 monomials $a_t,\ 71 \leqslant t \leqslant 74$:

71. $x_1x_2x_3x_4x_5^2$ 72. $x_1x_2x_3x_4^2x_5$ 73. $x_1x_2x_3^2x_4x_5$ 74. $x_1x_2^2x_3x_4x_5$.

4.2 The admissible monomials of degree 17 in P_5.

We have $B_5(17) = B_5^0(17) \cup B_5^+(3,1,1,1) \cup B_5^+(3,1,3) \cup B_5^+(3,3,2) \cup \psi(B_5(6))$, where $\psi : P_5 \to P_5$ is the $\mathbb{F}_2$-linear map determined by $\psi(x) = c_1x_2x_3x_4x_5x^2$ for any monomials x in P_5.

$B_5^0(17)$ is the set of 335 monomials: $b_t,\ 1 \leqslant t \leqslant 335$:

1. $x_3x_4x_5^{15}$	2. $x_3x_4^{15}x_5$	3. $x_3^{15}x_4x_5$	4. $x_2x_4x_5^{15}$
5. $x_2x_4^{15}x_5$	6. $x_2x_3x_5^{15}$	7. $x_2x_3x_4^{15}$	8. $x_2x_3^{15}x_5$
9. $x_2x_3^{15}x_4$	10. $x_2^{15}x_4x_5$	11. $x_2^{15}x_3x_5$	12. $x_2^{15}x_3x_4$
13. $x_1x_4x_5^{15}$	14. $x_1x_4^{15}x_5$	15. $x_1x_3x_5^{15}$	16. $x_1x_3x_4^{15}$
17. $x_1x_3^{15}x_5$	18. $x_1x_3^{15}x_4$	19. $x_1x_2x_5^{15}$	20. $x_1x_2x_4^{15}$
21. $x_1x_2x_3^{15}$	22. $x_1x_2^{15}x_5$	23. $x_1x_2^{15}x_4$	24. $x_1x_2^{15}x_3$
25. $x_1^{15}x_4x_5$	26. $x_1^{15}x_3x_5$	27. $x_1^{15}x_3x_4$	28. $x_1^{15}x_2x_5$
29. $x_1^{15}x_2x_4$	30. $x_1^{15}x_2x_3$	31. $x_3x_4^3x_5^{13}$	32. $x_3^3x_4x_5^{13}$
33. $x_3^3x_4^{13}x_5$	34. $x_2x_4^3x_5^{13}$	35. $x_2x_3^3x_5^{13}$	36. $x_2x_3^3x_4^{13}$
37. $x_2^3x_4x_5^{13}$	38. $x_2^3x_4^{13}x_5$	39. $x_2^3x_3x_5^{13}$	40. $x_2^3x_3x_4^{13}$
41. $x_2^3x_3^{13}x_5$	42. $x_2^3x_3^{13}x_4$	43. $x_1x_4^3x_5^{13}$	44. $x_1x_3^3x_5^{13}$

45. $x_1x_3^3x_4^{13}$	46. $x_1x_2^3x_5^{13}$	47. $x_1x_2^3x_4^{13}$	48. $x_1x_2^3x_3^{13}$
49. $x_1^3x_4x_5^{13}$	50. $x_1^3x_4^{13}x_5$	51. $x_1^3x_3x_5^{13}$	52. $x_1^3x_3x_4^{13}$
53. $x_1^3x_3^{13}x_5$	54. $x_1^3x_3^{13}x_4$	55. $x_1^3x_2x_5^{13}$	56. $x_1^3x_2x_4^{13}$
57. $x_1^3x_2x_3^{13}$	58. $x_1^3x_2^{13}x_5$	59. $x_1^3x_2^{13}x_4$	60. $x_1^3x_2^{13}x_3$
61. $x_3^3x_4^5x_5^9$	62. $x_2^3x_4^5x_5^9$	63. $x_2^3x_3^5x_5^9$	64. $x_2^3x_3^5x_4^9$
65. $x_1^3x_4^5x_5^9$	66. $x_1^3x_3^5x_5^9$	67. $x_1^3x_3^5x_4^9$	68. $x_1^3x_2^5x_5^9$
69. $x_1^3x_2^5x_4^9$	70. $x_1^3x_2^5x_3^9$	71. $x_2x_3x_4x_5^{14}$	72. $x_2x_3x_4^{14}x_5$
73. $x_1x_3x_4x_5^{14}$	74. $x_1x_3x_4^{14}x_5$	75. $x_1x_2x_4x_5^{14}$	76. $x_1x_2x_4^{14}x_5$
77. $x_1x_2x_3x_5^{14}$	78. $x_1x_2x_3x_4^{14}$	79. $x_1x_2x_3^{14}x_5$	80. $x_1x_2x_3^{14}x_4$
81. $x_2x_3^3x_4x_5^{12}$	82. $x_2x_3^3x_4^{12}x_5$	83. $x_2^3x_3x_4x_5^{12}$	84. $x_2^3x_3x_4^{12}x_5$
85. $x_1x_3^3x_4x_5^{12}$	86. $x_1x_3^3x_4^{12}x_5$	87. $x_1x_2^3x_4x_5^{12}$	88. $x_1x_2^3x_4^{12}x_5$
89. $x_1x_2^3x_3x_5^{12}$	90. $x_1x_2^3x_3x_4^{12}$	91. $x_1x_2^3x_3^{12}x_5$	92. $x_1x_2^3x_3^{12}x_4$
93. $x_1^3x_3x_4x_5^{12}$	94. $x_1^3x_3x_4^{12}x_5$	95. $x_1^3x_2x_4x_5^{12}$	96. $x_1^3x_2x_4^{12}x_5$
97. $x_1^3x_2x_3x_5^{12}$	98. $x_1^3x_2x_3x_4^{12}$	99. $x_1^3x_2x_3^{12}x_5$	100. $x_1^3x_2x_3^{12}x_4$
101. $x_2^3x_3^5x_4x_5^8$	102. $x_2^3x_3^5x_4^8x_5$	103. $x_1^3x_3^5x_4x_5^8$	104. $x_1^3x_3^5x_4^8x_5$
105. $x_1^3x_2^5x_4x_5^8$	106. $x_1^3x_2^5x_4^8x_5$	107. $x_1^3x_2^5x_3x_5^8$	108. $x_1^3x_2^5x_3x_4^8$
109. $x_1^3x_2^5x_3^8x_5$	110. $x_1^3x_2^5x_3^8x_4$	111. $x_2x_3^{14}x_4x_5$	112. $x_1x_3^{14}x_4x_5$
113. $x_1x_2^{14}x_4x_5$	114. $x_1x_2^{14}x_3x_5$	115. $x_1x_2^{14}x_3x_4$	116. $x_2x_3x_4^2x_5^{13}$
117. $x_2x_3^2x_4x_5^{13}$	118. $x_2x_3^2x_4^{13}x_5$	119. $x_1x_3x_4^2x_5^{13}$	120. $x_1x_3^2x_4x_5^{13}$
121. $x_1x_3^2x_4^{13}x_5$	122. $x_1x_2x_4^2x_5^{13}$	123. $x_1x_2x_3^2x_5^{13}$	124. $x_1x_2x_3^2x_4^{13}$
125. $x_1x_2^2x_4x_5^{13}$	126. $x_1x_2^2x_4^{13}x_5$	127. $x_1x_2^2x_3x_5^{13}$	128. $x_1x_2^2x_3x_4^{13}$
129. $x_1x_2^2x_3^{13}x_5$	130. $x_1x_2^2x_3^{13}x_4$	131. $x_2x_3x_4^3x_5^{12}$	132. $x_1x_3x_4^3x_5^{12}$
133. $x_1x_2x_4^3x_5^{12}$	134. $x_1x_2x_3^3x_5^{12}$	135. $x_1x_2x_3^3x_4^{12}$	136. $x_2x_3^2x_4^5x_5^9$
137. $x_1x_3^2x_4^5x_5^9$	138. $x_1x_2^2x_4^5x_5^9$	139. $x_1x_2^2x_3^5x_5^9$	140. $x_1x_2^2x_3^5x_4^9$
141. $x_2x_3^3x_4^4x_5^9$	142. $x_2^3x_3x_4^4x_5^9$	143. $x_1x_3^3x_4^4x_5^9$	144. $x_1x_2^3x_4^4x_5^9$
145. $x_1x_2^3x_3^4x_5^9$	146. $x_1x_2^3x_3^4x_4^9$	147. $x_1^3x_3x_4^4x_5^9$	148. $x_1^3x_2x_4^4x_5^9$
149. $x_1^3x_2x_3^4x_5^9$	150. $x_1^3x_2x_3^4x_4^9$	151. $x_2x_3^3x_4^5x_5^8$	152. $x_2^3x_3x_4^5x_5^8$
153. $x_1x_3^3x_4^5x_5^8$	154. $x_1x_2^3x_4^5x_5^8$	155. $x_1x_2^3x_3^5x_5^8$	156. $x_1x_2^3x_3^5x_4^8$
157. $x_1^3x_3x_4^5x_5^8$	158. $x_1^3x_2x_4^5x_5^8$	159. $x_1^3x_2x_3^5x_5^8$	160. $x_1^3x_2x_3^5x_4^8$
161. $x_3^3x_4^7x_5^7$	162. $x_3^7x_4^3x_5^7$	163. $x_3^7x_4^7x_5^3$	164. $x_2^3x_4^7x_5^7$
165. $x_2^3x_3^7x_5^7$	166. $x_2^3x_3^7x_4^7$	167. $x_2^7x_4^3x_5^7$	168. $x_2^7x_4^7x_5^3$
169. $x_2^7x_3^3x_5^7$	170. $x_2^7x_3^3x_4^7$	171. $x_2^7x_3^7x_5^3$	172. $x_2^7x_3^7x_4^3$
173. $x_1^3x_4^7x_5^7$	174. $x_1^3x_3^7x_5^7$	175. $x_1^3x_3^7x_4^7$	176. $x_1^3x_2^7x_5^7$
177. $x_1^3x_2^7x_4^7$	178. $x_1^3x_2^7x_3^7$	179. $x_1^7x_4^3x_5^7$	180. $x_1^7x_4^7x_5^3$
181. $x_1^7x_3^3x_5^7$	182. $x_1^7x_3^3x_4^7$	183. $x_1^7x_3^7x_5^3$	184. $x_1^7x_3^7x_4^3$
185. $x_1^7x_2^3x_5^7$	186. $x_1^7x_2^3x_4^7$	187. $x_1^7x_2^3x_3^7$	188. $x_1^7x_2^7x_5^3$
189. $x_1^7x_2^7x_4^3$	190. $x_1^7x_2^7x_3^3$	191. $x_2x_3^2x_4^7x_5^7$	192. $x_2x_3^7x_4^2x_5^7$
193. $x_2x_3^7x_4^7x_5^2$	194. $x_2^7x_3x_4^2x_5^7$	195. $x_2^7x_3x_4^7x_5^2$	196. $x_2^7x_3^7x_4x_5^2$
197. $x_1x_3^2x_4^7x_5^7$	198. $x_1x_3^7x_4^2x_5^7$	199. $x_1x_3^7x_4^7x_5^2$	200. $x_1x_2^2x_4^7x_5^7$
201. $x_1x_2^2x_3^7x_5^7$	202. $x_1x_2^2x_3^7x_4^7$	203. $x_1x_2^7x_4^2x_5^7$	204. $x_1x_2^7x_4^7x_5^2$

205.	$x_1x_2^7x_3^2x_5^7$	206.	$x_1x_2^7x_3^2x_4^7$	207.	$x_1x_2^7x_3^7x_5^2$	208.	$x_1x_2^7x_3^7x_4^2$
209.	$x_1^7x_3x_4^2x_5^7$	210.	$x_1^7x_3x_4^7x_5^2$	211.	$x_1^7x_3^7x_4x_5^2$	212.	$x_1^7x_2x_4^2x_5^7$
213.	$x_1^7x_2x_4^7x_5^2$	214.	$x_1^7x_2x_3^2x_5^7$	215.	$x_1^7x_2x_3^2x_4^7$	216.	$x_1^7x_2x_3^7x_5^2$
217.	$x_1^7x_2x_3^7x_4^2$	218.	$x_1^7x_2^7x_4x_5^2$	219.	$x_1^7x_2^7x_3x_5^2$	220.	$x_1^7x_2^7x_3x_4^2$
221.	$x_2x_3^6x_4^3x_5^7$	222.	$x_2x_3^6x_4^7x_5^3$	223.	$x_2x_3^7x_4^6x_5^3$	224.	$x_2^3x_3x_4^6x_5^7$
225.	$x_2^3x_3x_4^7x_5^6$	226.	$x_2^3x_3^7x_4x_5^6$	227.	$x_2^7x_3x_4^6x_5^3$	228.	$x_2^7x_3^3x_4x_5^6$
229.	$x_1x_3^6x_4^3x_5^7$	230.	$x_1x_3^6x_4^7x_5^3$	231.	$x_1x_3^7x_4^6x_5^3$	232.	$x_1x_2^6x_4^3x_5^7$
233.	$x_1x_2^6x_4^7x_5^3$	234.	$x_1x_2^6x_3^3x_5^7$	235.	$x_1x_2^6x_3^3x_4^7$	236.	$x_1x_2^6x_3^7x_5^3$
237.	$x_1x_2^6x_3^7x_4^3$	238.	$x_1x_2^7x_4^6x_5^3$	239.	$x_1x_2^7x_3^6x_5^3$	240.	$x_1x_2^7x_3^6x_4^3$
241.	$x_1^3x_3x_4^6x_5^7$	242.	$x_1^3x_3x_4^7x_5^6$	243.	$x_1^3x_3^7x_4x_5^6$	244.	$x_1^3x_2x_4^6x_5^7$
245.	$x_1^3x_2x_4^7x_5^6$	246.	$x_1^3x_2x_3^6x_5^7$	247.	$x_1^3x_2x_3^6x_4^7$	248.	$x_1^3x_2x_3^7x_5^6$
249.	$x_1^3x_2x_3^7x_4^6$	250.	$x_1^3x_2^7x_4x_5^6$	251.	$x_1^3x_2^7x_3x_5^6$	252.	$x_1^3x_2^7x_3x_4^6$
253.	$x_1^7x_3x_4^6x_5^3$	254.	$x_1^7x_3^3x_4x_5^6$	255.	$x_1^7x_2x_4^6x_5^3$	256.	$x_1^7x_2x_3^6x_5^3$
257.	$x_1^7x_2x_3^6x_4^3$	258.	$x_1^7x_2^3x_4x_5^6$	259.	$x_1^7x_2^3x_3x_5^6$	260.	$x_1^7x_2^3x_3x_4^6$
261.	$x_2^3x_3^5x_4^2x_5^7$	262.	$x_2^3x_3^5x_4^7x_5^2$	263.	$x_2^3x_3^7x_4^5x_5^2$	264.	$x_2^7x_3^3x_4^5x_5^2$
265.	$x_1^3x_3^5x_4^2x_5^7$	266.	$x_1^3x_3^5x_4^7x_5^2$	267.	$x_1^3x_3^7x_4^5x_5^2$	268.	$x_1^3x_2^5x_4^2x_5^7$
269.	$x_1^3x_2^5x_4^7x_5^2$	270.	$x_1^3x_2^5x_3^2x_5^7$	271.	$x_1^3x_2^5x_3^2x_4^7$	272.	$x_1^3x_2^5x_3^7x_5^2$
273.	$x_1^3x_2^5x_3^7x_4^2$	274.	$x_1^3x_2^7x_4^5x_5^2$	275.	$x_1^3x_2^7x_3^5x_5^2$	276.	$x_1^3x_2^7x_3^5x_4^2$
277.	$x_1^7x_3^3x_4^5x_5^2$	278.	$x_1^7x_2^3x_4^5x_5^2$	279.	$x_1^7x_2^3x_3^5x_5^2$	280.	$x_1^7x_2^3x_3^5x_4^2$
281.	$x_2^3x_3^3x_4^4x_5^7$	282.	$x_2^3x_3^3x_4^7x_5^4$	283.	$x_2^3x_3^7x_4^3x_5^4$	284.	$x_2^7x_3^3x_4^3x_5^4$
285.	$x_1^3x_3^3x_4^4x_5^7$	286.	$x_1^3x_3^3x_4^7x_5^4$	287.	$x_1^3x_3^7x_4^3x_5^4$	288.	$x_1^3x_2^3x_4^4x_5^7$
289.	$x_1^3x_2^3x_4^7x_5^4$	290.	$x_1^3x_2^3x_3^4x_5^7$	291.	$x_1^3x_2^3x_3^4x_4^7$	292.	$x_1^3x_2^3x_3^7x_5^4$
293.	$x_1^3x_2^3x_3^7x_4^4$	294.	$x_1^3x_2^7x_4^3x_5^4$	295.	$x_1^3x_2^7x_3^3x_5^4$	296.	$x_1^3x_2^7x_3^3x_4^4$
297.	$x_1^7x_3^3x_4^3x_5^4$	298.	$x_1^7x_2^3x_4^3x_5^4$	299.	$x_1^7x_2^3x_3^3x_5^4$	300.	$x_1^7x_2^3x_3^3x_4^4$
301.	$x_2x_3^3x_4^6x_5^7$	302.	$x_2x_3^3x_4^7x_5^6$	303.	$x_2x_3^7x_4^3x_5^6$	304.	$x_2^7x_3x_4^3x_5^6$
305.	$x_1x_3^3x_4^6x_5^7$	306.	$x_1x_3^3x_4^7x_5^6$	307.	$x_1x_3^7x_4^3x_5^6$	308.	$x_1x_2^3x_4^6x_5^7$
309.	$x_1x_2^3x_4^7x_5^6$	310.	$x_1x_2^3x_3^6x_5^7$	311.	$x_1x_2^3x_3^6x_4^7$	312.	$x_1x_2^3x_3^7x_5^6$
313.	$x_1x_2^3x_3^7x_4^6$	314.	$x_1x_2^7x_4^3x_5^6$	315.	$x_1x_2^7x_3^3x_5^6$	316.	$x_1x_2^7x_3^3x_4^6$
317.	$x_1^7x_3x_4^3x_5^6$	318.	$x_1^7x_2x_4^3x_5^6$	319.	$x_1^7x_2x_3^3x_5^6$	320.	$x_1^7x_2x_3^3x_4^6$
321.	$x_2^3x_3^3x_4^5x_5^6$	322.	$x_2^3x_3^5x_4^3x_5^6$	323.	$x_2^3x_3^5x_4^6x_5^3$	324.	$x_1^3x_3^3x_4^5x_5^6$
325.	$x_1^3x_3^5x_4^3x_5^6$	326.	$x_1^3x_3^5x_4^6x_5^3$	327.	$x_1^3x_2^3x_4^5x_5^6$	328.	$x_1^3x_2^3x_3^5x_5^6$
329.	$x_1^3x_2^3x_3^5x_4^6$	330.	$x_1^3x_2^5x_4^3x_5^6$	331.	$x_1^3x_2^5x_4^6x_5^3$	332.	$x_1^3x_2^5x_3^3x_5^6$
333.	$x_1^3x_2^5x_3^3x_4^6$	334.	$x_1^3x_2^5x_3^6x_5^3$	335.	$x_1^3x_2^5x_3^6x_4^3$		

$B_5^+(3,1,1,1)$ is the set of 21 monomials: $b_t,\ 336 \leqslant t \leqslant 356$:

336.	$x_1x_2x_3x_4^2x_5^{12}$	337.	$x_1x_2x_3^2x_4x_5^{12}$	338.	$x_1x_2x_3^2x_4^{12}x_5$	339.	$x_1x_2^2x_3x_4x_5^{12}$
340.	$x_1x_2^2x_3x_4^{12}x_5$	341.	$x_1x_2^2x_3^{12}x_4x_5$	342.	$x_1x_2x_3^2x_4^4x_5^9$	343.	$x_1x_2^2x_3x_4^4x_5^9$
344.	$x_1x_2^2x_3^4x_4x_5^9$	345.	$x_1x_2^2x_3^4x_4^9x_5$	346.	$x_1x_2x_3^2x_4^5x_5^8$	347.	$x_1x_2^2x_3x_4^5x_5^8$
348.	$x_1x_2^2x_3^5x_4x_5^8$	349.	$x_1x_2^2x_3^5x_4^8x_5$	350.	$x_1x_2x_3^3x_4^4x_5^8$	351.	$x_1x_2^3x_3x_4^4x_5^8$
352.	$x_1x_2^3x_3^4x_4x_5^8$	353.	$x_1x_2^3x_3^4x_4^8x_5$	354.	$x_1^3x_2x_3x_4^4x_5^8$	355.	$x_1^3x_2x_3^4x_4x_5^8$
356.	$x_1^3x_2x_3^4x_4^8x_5$						

$B_5^+(3,1,3)$ is the set of 10 monomials: $b_t,\ 357 \leqslant t \leqslant 366$:

357.	$x_1x_2^2x_3^4x_4^5x_5^5$	358.	$x_1x_2^2x_3^5x_4^4x_5^5$	359.	$x_1x_2^2x_3^5x_4^5x_5^4$	360.	$x_1x_2^3x_3^4x_4^4x_5^5$
361.	$x_1x_2^3x_3^4x_4^5x_5^4$	362.	$x_1x_2^3x_3^5x_4^4x_5^4$	363.	$x_1^3x_2x_3^4x_4^4x_5^5$	364.	$x_1^3x_2x_3^4x_4^5x_5^4$
365.	$x_1^3x_2x_3^5x_4^4x_5^4$	366.	$x_1^3x_2^5x_3x_4^4x_5^4$				

$B_5^+(3,3,2)$ is the set of 126 monomials: $b_t,\ 367 \leqslant t \leqslant 492$:

367.	$x_1x_2x_3^2x_4^6x_5^7$	368.	$x_1x_2x_3^2x_4^7x_5^6$	369.	$x_1x_2x_3^6x_4^2x_5^7$	370.	$x_1x_2x_3^6x_4^7x_5^2$
371.	$x_1x_2x_3^7x_4^2x_5^6$	372.	$x_1x_2x_3^7x_4^6x_5^2$	373.	$x_1x_2^6x_3x_4^2x_5^7$	374.	$x_1x_2^6x_3x_4^7x_5^2$
375.	$x_1x_2^6x_3^7x_4x_5^2$	376.	$x_1x_2^7x_3x_4^2x_5^6$	377.	$x_1x_2^7x_3x_4^6x_5^2$	378.	$x_1x_2^7x_3^6x_4x_5^2$
379.	$x_1^7x_2x_3x_4^2x_5^6$	380.	$x_1^7x_2x_3x_4^6x_5^2$	381.	$x_1^7x_2x_3^6x_4x_5^2$	382.	$x_1x_2^2x_3^3x_4^4x_5^7$
383.	$x_1x_2^2x_3^3x_4^7x_5^4$	384.	$x_1x_2^2x_3^4x_4^3x_5^7$	385.	$x_1x_2^2x_3^4x_4^7x_5^3$	386.	$x_1x_2^2x_3^7x_4^3x_5^4$
387.	$x_1x_2^2x_3^7x_4^4x_5^3$	388.	$x_1x_2^3x_3^2x_4^4x_5^7$	389.	$x_1x_2^3x_3^2x_4^7x_5^4$	390.	$x_1x_2^3x_3^7x_4^2x_5^4$
391.	$x_1x_2^7x_3^2x_4^3x_5^4$	392.	$x_1x_2^7x_3^2x_4^4x_5^3$	393.	$x_1x_2^7x_3^3x_4^2x_5^4$	394.	$x_1^3x_2x_3^2x_4^4x_5^7$
395.	$x_1^3x_2x_3^2x_4^7x_5^4$	396.	$x_1^3x_2x_3^4x_4^2x_5^7$	397.	$x_1^3x_2x_3^4x_4^7x_5^2$	398.	$x_1^3x_2x_3^7x_4^2x_5^4$
399.	$x_1^3x_2x_3^7x_4^4x_5^2$	400.	$x_1^3x_2^4x_3x_4^2x_5^7$	401.	$x_1^3x_2^4x_3x_4^7x_5^2$	402.	$x_1^3x_2^4x_3^7x_4x_5^2$
403.	$x_1^3x_2^7x_3x_4^2x_5^4$	404.	$x_1^3x_2^7x_3x_4^4x_5^2$	405.	$x_1^3x_2^7x_3^4x_4x_5^2$	406.	$x_1^7x_2x_3^2x_4^3x_5^4$
407.	$x_1^7x_2x_3^2x_4^4x_5^3$	408.	$x_1^7x_2x_3^3x_4^2x_5^4$	409.	$x_1^7x_2^3x_3x_4^2x_5^4$	410.	$x_1^7x_2^3x_3x_4^4x_5^2$
411.	$x_1^7x_2^3x_3^4x_4x_5^2$	412.	$x_1x_2^2x_3x_4^6x_5^7$	413.	$x_1x_2^2x_3x_4^7x_5^6$	414.	$x_1x_2^2x_3^7x_4x_5^6$
415.	$x_1x_2^7x_3^2x_4x_5^6$	416.	$x_1^7x_2x_3^2x_4x_5^6$	417.	$x_1x_2^2x_3^5x_4^2x_5^7$	418.	$x_1x_2^2x_3^5x_4^7x_5^2$
419.	$x_1x_2^2x_3^7x_4^5x_5^2$	420.	$x_1x_2^7x_3^2x_4^5x_5^2$	421.	$x_1^7x_2x_3^2x_4^5x_5^2$	422.	$x_1x_2^3x_3^4x_4^2x_5^7$
423.	$x_1x_2^3x_3^4x_4^7x_5^2$	424.	$x_1x_2^3x_3^7x_4^4x_5^2$	425.	$x_1x_2^7x_3^3x_4^4x_5^2$	426.	$x_1^7x_2x_3^3x_4^4x_5^2$
427.	$x_1x_2x_3^3x_4^6x_5^6$	428.	$x_1x_2x_3^6x_4^3x_5^6$	429.	$x_1x_2x_3^6x_4^6x_5^3$	430.	$x_1x_2^3x_3x_4^6x_5^6$
431.	$x_1x_2^3x_3^6x_4x_5^6$	432.	$x_1x_2^3x_3^6x_4^6x_5$	433.	$x_1x_2^6x_3x_4^3x_5^6$	434.	$x_1x_2^6x_3x_4^6x_5^3$
435.	$x_1x_2^6x_3^3x_4x_5^6$	436.	$x_1^3x_2x_3x_4^6x_5^6$	437.	$x_1^3x_2x_3^6x_4x_5^6$	438.	$x_1^3x_2x_3^6x_4^6x_5$
439.	$x_1x_2^2x_3^3x_4^5x_5^6$	440.	$x_1x_2^2x_3^3x_4^6x_5^5$	441.	$x_1x_2^2x_3^5x_4^3x_5^6$	442.	$x_1x_2^2x_3^5x_4^6x_5^3$
443.	$x_1x_2^3x_3^2x_4^5x_5^6$	444.	$x_1x_2^3x_3^2x_4^6x_5^5$	445.	$x_1x_2^3x_3^5x_4^2x_5^6$	446.	$x_1x_2^3x_3^5x_4^6x_5^2$
447.	$x_1x_2^3x_3^6x_4^2x_5^5$	448.	$x_1x_2^3x_3^6x_4^5x_5^2$	449.	$x_1x_2^6x_3^3x_4^5x_5^2$	450.	$x_1^3x_2x_3^2x_4^5x_5^6$
451.	$x_1^3x_2x_3^2x_4^6x_5^5$	452.	$x_1^3x_2x_3^5x_4^2x_5^6$	453.	$x_1^3x_2x_3^5x_4^6x_5^2$	454.	$x_1^3x_2x_3^6x_4^2x_5^5$
455.	$x_1^3x_2x_3^6x_4^5x_5^2$	456.	$x_1^3x_2^5x_3x_4^2x_5^6$	457.	$x_1^3x_2^5x_3x_4^6x_5^2$	458.	$x_1^3x_2^5x_3^2x_4x_5^6$
459.	$x_1^3x_2^5x_3^2x_4^6x_5$	460.	$x_1^3x_2^5x_3^6x_4x_5^2$	461.	$x_1^3x_2^5x_3^6x_4^2x_5$	462.	$x_1x_2^3x_3^3x_4^4x_5^6$
463.	$x_1x_2^3x_3^3x_4^6x_5^4$	464.	$x_1x_2^3x_3^4x_4^3x_5^6$	465.	$x_1x_2^3x_3^4x_4^6x_5^3$	466.	$x_1x_2^3x_3^6x_4^3x_5^4$
467.	$x_1x_2^3x_3^6x_4^4x_5^3$	468.	$x_1x_2^6x_3^3x_4^3x_5^4$	469.	$x_1^3x_2x_3^3x_4^4x_5^6$	470.	$x_1^3x_2x_3^3x_4^6x_5^4$
471.	$x_1^3x_2x_3^4x_4^3x_5^6$	472.	$x_1^3x_2x_3^4x_4^6x_5^3$	473.	$x_1^3x_2x_3^6x_4^3x_5^4$	474.	$x_1^3x_2x_3^6x_4^4x_5^3$
475.	$x_1^3x_2^3x_3x_4^4x_5^6$	476.	$x_1^3x_2^3x_3x_4^6x_5^4$	477.	$x_1^3x_2^3x_3^4x_4x_5^6$	478.	$x_1^3x_2^4x_3x_4^3x_5^6$
479.	$x_1^3x_2^4x_3^3x_4x_5^6$	480.	$x_1^3x_2^5x_3^2x_4^2x_5^5$	481.	$x_1^3x_2^5x_3^2x_4^5x_5^2$	482.	$x_1^3x_2^3x_3^4x_4^5x_5^2$
483.	$x_1^3x_2^3x_3^5x_4^2x_5^4$	484.	$x_1^3x_2^3x_3^5x_4^4x_5^2$	485.	$x_1^3x_2^4x_3^3x_4^5x_5^2$	486.	$x_1^3x_2^5x_3^2x_4^3x_5^4$
487.	$x_1^3x_2^5x_3^2x_4^4x_5^3$	488.	$x_1^3x_2^5x_3^3x_4^2x_5^4$	489.	$x_1^3x_2^5x_3^3x_4^4x_5^2$	490.	$x_1^3x_2^3x_3^3x_4^4x_5^4$
491.	$x_1^3x_2^3x_3^4x_4^3x_5^4$	492.	$x_1^3x_2^3x_3^4x_4^4x_5^3$				

We denote by $b_t = \psi(a_{t-492}),\ 493 \leqslant t \leqslant 566$.

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