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AN UPPER LENGTH ESTIMATE FOR CURVES IN CAT(K) SPACES

Mantana Chudtong and **Chaiwat Maneesawarng***

Department of Mathematics, Faculty of Science,
Mahidol University, Bangkok 10400, Thailand
e-mail: mchudtong@gmail.com

†*Dept. of Mathematics, Faculty of Science, Mahidol University*
Centre of Excellence in Mathematics, Commission on Higher Education,
Si Ayutthaya Rd., Bangkok 10400, Thailand.
e-mail: chaiwat.man@mahidol.ac.th

Abstract

In Euclidean space, upper estimates for curvelength have been studied mostly in the previous century. Many of these have been extended over time, either to a larger class of spaces or to a larger class of curves. Due to limited tools, extentions to a larger class of spaces often end up with a restricted class of curves. With an appropriate variation of Reshetnyak's fan construction technique in comparison geometry, the obstacle is overcome and a sharp upper length estimate for curves in terms of total curvature and the radius of a circumball are presented in this paper for CAT(K) spaces. The configurations of maximizers, which exist in standard spaces of constant curvature, are also completely determined. An interesting part is that in spaces of negative constant curvature, the maximizing configurations are totally different from the case of nonnegative curvature.

1 Introduction

In this paper we search for a sharp upper estimate of the length of a curve with known total curvature that is confined to a closed ball of a given radius in a metric space in which total curvature (an amount measuring deviation

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from being "straight") of curves is meaningful. We are also interested in the configurations of the curves rendering the extremal length. This problem was discussed by A. D. Alexandrov and Yu. G. Reshetnyak in [6] and its solution was given there for the Euclidean plane. Our attempt is to extend Alexandrov and Reshetnyak's setting to a more general one that covers at least the standard spaces of constant curvature, namely, the Euclidean spaces, the spheres and the hyperbolic spaces, and seek for solutions there. Earlier (partial) extensions were made by B. V. Dekster, who proved an analog of the original result for piecewise smooth curves in [13, 14] in Riemannian manifolds of curvature bounded below with boundaries and in Riemannian manifolds of curvature bounded above, respectively. These extensions were partial because only certain ranges of total curvature (at most $\frac{\pi}{2}$, depending on the radius of the circumball and the spatial curvature bound) were considered but in a more general class of spaces.

Here we work in the class of CAT(K) spaces, which turns out to be an effective way to complete Dekster's unfinished part. The theory on these spaces was developed in early 1950s ([3, 4, 5]; see also [1, 7, 8, 9] and [10]). It was built on the ground of comparison of a basic element of the space, namely the triangles, with those in the standard 2-dimensional spaces of constant curvature K (denoted S_K throughout). Indeed, a metric space is said to be CAT(K) if and only if each pair of its points can be joined by a curve realizing the distance between them (a so-called *minimizing geodesic*) and if the distance between any pair of points on any minimizing geodesic triangle in it with perimeter less than $\frac{2\pi}{\sqrt{K}}$ ($=\infty$ if $K \leq 0$) is dominated by the distance between corresponding points on the *comparison triangle*—the triangle with the same sidelengths—in S_K. (Such a space is also called an R_K domain.) The key feature is that in such a space angles between geodesics (and also between curves that behave well enough) issuing from a common point can be intrinsically measured, which makes it possible to define total curvature of curves.

In a smooth case the total curvature of a curve is the integral with respect to arclength of its (unsigned) pointwise scalar curvature. An extensive development of the theory on this subject for general curves in the Euclidean space and on the spheres was presented in [6] while total curvature for closed curves was introduced independently in [15]. The concept was then extended to CAT(0) spaces (and hence CAT(K) spaces for all $K \geq 0$) in [2]. The hyperbolic case as well as the CAT(K) case for negative K, however, needed special care and were treated later in [12]. The unified definition in CAT(K) spaces as found in [12] was given by first considering *polysegments*, which are curves formed by concatenation of finitely many minimizing geodesics called their (*geodesic*) *segments*. The term *bisegments*, *trisegments* and *n-segments* are customarily adopted with the obvious meanings. For a curve of this special type its total curvature, initially called *total rotation*, is naturally defined as the sum of the supplementary angles of the angles at its interior vertices. The *total curvature* of an arbitrary curve is then defined to be the limit supremum of the total rotations of its inscribed polysegments that are arbitrarily closed to it in a certain sense. Needless to say, for polysegments this coincides with the total rotation formerly defined. For more details, readers are advised to consult [12].

The major results that answer the main question covering all possible values of total curvature and all types of curves are proved in sections 2 and 3 (Theorems 2.4 and 3.13, respectively). Contrary to what one would naturally expect (an analogy between the classical case and the nonzero spatial curvature cases), a maximizing curve can take a totally different shape in the hyperbolic case. Indeed, only in the case of a negative spatial curvature bound K with sufficiently large radius do the maximizers rendering the sharp upper curvelength bound that exist in the corresponding hyperbolic space of constant curvature K deform through polysegments as the radius grows, and never contains a nontrivial circular arc of the boundary of the circumdisk as do the maximizers in the other cases. A reason that makes this happen is that, unlike the case of nonnegative curvature bound, the (pointwise) curvature of the boundary of the circumdisk converges to a positive number as the radius approaches infinity. Thus it costs more (curvature) for a curve to follow the boundary of a large disk than it would in the case of nonnegative curvature bound. As a consequence, a totally different technique is needed in optimizing curvelength in this case.

The main results may be summarized in one theorem, stated in uniformity below. Here and throughout, $\kappa(\gamma)$ and $\ell(\gamma)$ denote the total curvature and the length of γ, respectively. For convenience, rational multiples of $\frac{\pi}{\sqrt{K}}$ assume the value ∞ if $K \leq 0$, as did in an earlier paragraph. R_0 is the cut-point of the radius beyond which a circumball is considered "large." $S(R,\kappa)$ is the sharp upper length bound. The precise definitions of R_0 and $S(R,\kappa)$, together with those of the symbols $\Lambda_{R,\kappa}$, $\Gamma_{R,\kappa}$, $\Pi^n_{R,\kappa}$ and $\mathit{\Pi}^n_{R,\kappa}$, which stand for certain families of curves, are given in sections 2 and 3. Roughly speaking, these are curves in a disk of radius R in S_K with total curvature κ, where $\Lambda_{R,\kappa}$ is an isosceles bisegment, $\Gamma_{R,\kappa}$ is an isosceles bisegment with an arc of the boundary of the disk inserted in the middle (also called a round-tip bisegment), $\Pi^n_{R,\kappa}$ is an $(n+2)$-segment with all the n body segments (i.e., all those except the two ends) of equal length, while $\mathit{\Pi}^n_{R,\kappa}$ is an $(n+2)$-segment with one body segment exceptionally shorter than the others.

Theorem 1.1. *Let γ be a curve contained in a closed ball of radius $R < \frac{\pi}{2\sqrt{K}}$ in an R_K space. Then $\ell(\gamma) \leq S(R,\kappa(\gamma))$. Moreover, for any $R < \frac{\pi}{2\sqrt{K}}$ and any $\kappa \geq 0$, curves in a closed disk of radius R that realize the upper length bound $S(R,\kappa)$ exist in S_K and are as described below:*

(i) If $R \leq R_0$ then there exists a real number $\overline{\kappa}_0$ such that maximizers are $\Lambda_{R,\kappa}$ if $0 \leq \kappa \leq \overline{\kappa}_0$ and $\Gamma_{R,\kappa}$ if $\overline{\kappa}_0 < \kappa < \infty$.

(ii) If $R > R_0$ then there exist a positive integer N and strictly increasing sequences κ_n and $\overline{\kappa}_n$ of positive reals with $\overline{\kappa}_{n-1} \leq \kappa_n < \overline{\kappa}_n$ for all n and

$\overline{\kappa}_{n-1} = \kappa_n$ for and only for $n \geq N$ such that maximizers are $\Lambda_{R,\kappa}$ if $0 \leq \kappa \leq \overline{\kappa}_0$ and, for $\kappa > \overline{\kappa}_0$,

- *$\mathit{\Pi}^n_{R,\kappa}$ if $\overline{\kappa}_{n-1} < \kappa < \kappa_n$,*
- *$\Pi^n_{R,\kappa}$ if $\kappa_n < \kappa < \overline{\kappa}_n$, or if $\kappa = \kappa_n$ with $n = 1$ or 2, or if $\kappa = \overline{\kappa}_n$ with $n \leq N-2$,*
- *both $\mathit{\Pi}^n_{R,\kappa}$ and $\Pi^n_{R,\kappa}$ if $\kappa = \kappa_n$ with $2 < n < N$, and*
- *both $\Pi^n_{R,\kappa}$ and $\Pi^{n-1}_{R,\kappa}$ if $\kappa = \kappa_n$ with $n \geq N$,*

where n is a unique integer (necessarily exists) such that one and only one of the above exhaustive conditions hold.

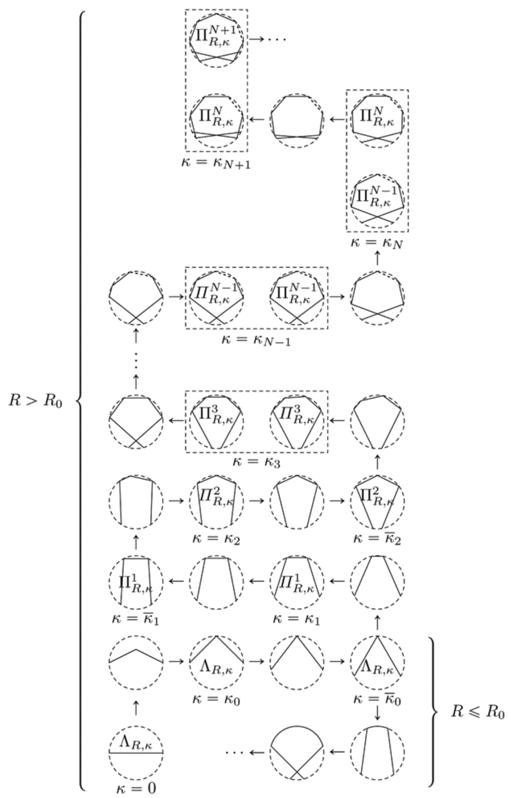

Figure 1: Configurations of maximizing curves. Dashed rectangles indicate co-maximizers.

According to the above result, the configurations of maximizers change as total curvature grows. Figure 1 summarizes the deformation scheme of the maximizers as total curvatures increase. In the case of "small" circumdisks, the deformation starts from merely a geodesic, advances through bisegments and ends in a series of round-tip bisegments as shown in the lower two rows of Figure 1. This is exactly the same result as discussed in [6] in the classical case. For "large" circumdisks in the negative spatial curvature case, on the other hand, the maximizers deform through a series of polysegments with gradually increasing numbers of segments. The upward-series part of Figure 1 illustrates this.

2 The Case of Small Circumballs: the Classical Configurations Revisited

As is noted earlier, pointwise curvatures of the boundaries of large disks in the model spaces behave differently when spatial curvatures change. For small disks, on the contrary, the response of curvelength to the change in its total curvature for curves near the boundary can be controlled in some manner no matter what the spatial curvature is. We therefore consider the case of small and large circumballs separately. This section will be devoted to the case of small circumballs. However, we first discuss common tools that apply to both cases.

Despite being more general and having less structure, CAT(K) spaces accommodate quite several interesting common properties, mostly obtained through the use of one type of comparison theorems or another. The following comparison theorem, also given in [12], makes use of Reshetnyak's fan construction technique. A version of it appears in [16] as part of the proof of Reshetnyak's majorization theorem, hence the accreditation. As is described in [12], a *supporting half space* of an n-segment γ in S_K corresponding to any of its minimizing geodesic segment σ is a closed half space of S_K containing all segments of γ adjacent to σ, with boundary containing σ. Two such half spaces corresponding to adjacent segments are *compatible* if a deformation by rotation about their common vertex from one half space to the other exists in such a way that the two segments always lie in the intermediate half spaces. The polysegment γ is *weakly convex* with respect to a point $O \in S_K$ if every segment of γ furnishes a supporting half space containing O such that every pair of these half spaces corresponding to adjacent segments are compatible.

Theorem 2.1. *[16] For any n-segment γ in a closed ball of radius $R < \frac{\pi}{2\sqrt{K}}$ centered at a point O in a CAT(K) space, there exist a closed disk D of radius R centered at some point O' in S_K, and an n-segment η in D that is weakly convex with respect to O' with geodesic segments of the same sequence of lengths and with an angle at each interior vertex no smaller than the corresponding angle of γ.*

Because the total curvature of a polysegment is defined in terms of the sum of supplementary angles of the angles at interior vertices, the above theorem equips us with a way to produce a polysegment in a disk of the same radius in the model space that is no shorter and with no greater total curvature than the original polysegment. The next proposition implies further that, as far as only polysegments are concerned, in order to obtain the upper estimate we are aiming at we only need to consider the class of bisegments and "admissible" polysegments. A polysegment in a closed disk D of radius $R < \frac{\pi}{2\sqrt{K}}$ in S_K is D-*admissible* if it is weakly convex with respect to the center of D and if all of its vertices lie on the boundary of D.

Proposition 2.2. *Let η be a polysegment in a closed disk D of radius less than $\frac{\pi}{2\sqrt{K}}$ in S_K that is weakly convex with respect to the center of D. Then there exists a polysegment σ in D, which is either a D-admissible polysegment or a bisegment with endpoints on the boundary of D, such that $\ell(\eta) \le \ell(\sigma)$ and $\kappa(\eta) \ge \kappa(\sigma)$.*

Proof. Let us first extend the segments at the two ends of η until they meet the boundary of D and call the newly obtained polysegment η as well. The claim is then obvious if the number n of geodesic segments of η is 1 or 2. Suppose $n \ge 3$. We note that weak convexity of η with respect to the center of D implies the following property:

($*$) For any consecutive vertices A, B, C and E of η the segments AB and CE both lie on the same closed halfspace whose boundary contains the segment BC.

Let A, B, C and E be any four consecutive vertices of η. We then prove the assertion by breaking it into two cases:

Case I. $K \ge 0$. We extend the geodesic segments AB through B and CE through C. If the extensions meet at P before they exit the disk we replace the segment BC by BPC, thereby reducing the number of vertices of η by 1; otherwise we let P_1 be the point at which the extension of AB meets the boundary of the disk, P_2 the analogous point for the extension of CE, and then replace the segment BC by BP_1P_2C, thereby increasing the number of vertices on the boundary. In either case the length of the resulting polysegment is no smaller than that of the original one, while the total curvature is no greater. The latter is true by the Gauss-Bonnet formula. Since the weak convexity of all polysegments and the property ($*$) are both preserved, this implies the existence of σ with the required properties.

Case II. $K < 0$. We note that the Gauss-Bonnet formula does not apply here. Let us consider the deformation of η by fixing all of its vertices except B, which moves in such a way that the length of segment AB increases but the total length of the polysegment remains unchanged, until either the segments BC and CE form a geodesic or the resulting polysegment hits the boundary of D. By property $(*)$, this deformation can be shown not to increase total curvature. On the other hand, it either increases the number of vertices on the boundary or reduces the total number of vertices without destroying the weak convexity. Since the resulting polysegment still satisfies $(*)$, the process can be repeated until every interior vertex not on the boundary is either eliminated or brought to the boundary without total curvature increase or length decrease. We therefore have the existence of σ with the required properties, completing the proof of the proposition. $\square$

The following fact makes it possible to shift from the case of polysegments to the general case. The proofs are found in [6] and [12]. Here, the *modulus* $\mu_\gamma(\sigma)$ of a polysegment σ inscribed in a curve γ is the maximum diameter of the subcurves which the vertices of σ cut γ into.

Theorem 2.3. *[6, 12] Let σ_n be any sequence of polysegments inscribed in a curve γ in an R_K space such that $\mu_\gamma(\sigma_n) \to 0$. Then $\sigma_n \to \gamma$ pointwise, $\ell(\sigma_n) \to \ell(\gamma)$ and $\kappa(\sigma_n) \to \kappa(\gamma)$.*

Let us now make some notation agreement so that our main theorem in this section as well as further development can be concisely presented. For a metric curvature bound K and a positive real number $R < \frac{\pi}{2\sqrt{K}}$, we put $\lambda = \sqrt{|K|}$ and define constants and functions as shown in the following

Table 1.

Constant/ Function	$K<0$	$K=0$	$K>0$
R_0	$\frac{1}{\lambda}\sinh^{-1} 1$	∞	$\frac{1}{\lambda}\sin^{-1} 1$
c_R	$\tanh \lambda R$	R	$\tan \lambda R$
$\kappa_0 = \kappa_{0,R}$	$2\tan^{-1}(\cosh \lambda R)$	$\frac{\pi}{2}$	$2\tan^{-1}(\cos \lambda R)$
$\overline{\kappa}_0 = \overline{\kappa}_{0,R}$	$2\cos^{-1}\frac{1-\sqrt{c_R^4-c_R^2+1}}{c_R^2}$	$\frac{2\pi}{3}$	$2\cos^{-1}\frac{-1+\sqrt{c_R^4+c_R^2+1}}{c_R^2}$
$r_R(\varphi)$	$\frac{2}{\lambda}\tanh^{-1}(c_R\sin\varphi)$	$2c_R\sin\varphi$	$\frac{2}{\lambda}\tan^{-1}(c_R\sin\varphi)$
$s_R(r,\kappa)$	$\frac{2}{\lambda}\sinh^{-1}(\frac{\sinh\frac{\lambda r}{2}}{\cos\frac{\kappa}{2}})$	$\frac{r}{\cos\frac{\kappa}{2}}$	$\frac{2}{\lambda}\sin^{-1}(\frac{\sin\frac{\lambda r}{2}}{\cos\frac{\kappa}{2}})$

Note that, since the number K will always be fixed, it will be omitted every time a symbol depending on K is introduced. The formula for $r_R(\varphi)$ indeed gives the length of a chord of a circle of radius R that makes an angle φ, $0 < \varphi < \frac{\pi}{2}$, with the tangent at one of its endpoints. If the length of an isosceles bisegment with minimizing segments in S_K does not exceed $\frac{\pi}{\sqrt{K}}$, then it is given by $s_R(r,\kappa)$ defined above, where r is the chordlength, i.e., the distance between the two endpoints.

A curve γ in S_K is called a *round-tip bisegment* if γ consists of a chord of a circle of radius R followed by an arc of that circle and then another chord of the circle. The *rotation* at an interior point p corresponding to a parameter value t of a curve is the supplementary angle of the angle between the two subcurves which t cuts the curve into. For each $\kappa \geq \overline{\kappa}_0$, let $\Gamma_{R,\kappa}$ be a round-tip bisegment of total curvature κ with rotation $\frac{\overline{\kappa}_0}{2}$ at each end of its circular arc. For each $\kappa \in [0,\pi]$, let $\Lambda_{R,\kappa}$ be any isosceles bisegment of total curvature κ in a closed disk of radius R, with endpoints at the ends of a diameter of the disk if $0 \leq \kappa \leq \kappa_0$, and with the three vertices on the boundary of the disk if $\kappa_0 \leq \kappa \leq \pi$. Indeed, Λ_{R,κ_0} is an isosceles bisegment with all three vertices on the boundary and both endpoints at the end of a diameter of the disk.

For $R > 0$ such that $R < \frac{\pi}{2\sqrt{K}}$ and $R \leq R_0$, let $\widetilde{S}(R,\kappa)$ (also depending on K) be the length of a $\Lambda_{R,\kappa}$ when $0 \leq \kappa \leq \overline{\kappa}_0$ and that of a $\Gamma_{R,\kappa}$ when $\overline{\kappa}_0 \leq \kappa < \infty$. It is easily verified that $\widetilde{S}(R,\kappa)$ is nondecreasing and continuous in κ. We are now ready to prove the main theorem of this section:

Theorem 2.4. *(Length estimate for curves in small circumballs.) Let γ be a curve contained in a closed ball of radius $R < \frac{\pi}{2\sqrt{K}}$ in an R_K space, with total curvature κ. If $0 < R \leq R_0$ then $\ell(\gamma) \leq \widetilde{S}(R,\kappa)$. Moreover, the upper bound $\widetilde{S}(R,\kappa)$ is achieved by $\Lambda_{R,\kappa}$ in a closed disk of radius R in S_K if $\kappa \leq \overline{\kappa}_0$ and by $\Gamma_{R,\kappa}$ if $\kappa > \overline{\kappa}_0$.*

Proof. We first prove the theorem for the case of bisegments and D-admissible polysegments in a closed disk D of radius R, where R is as described in the theorem statement. Note that direct computation shows that any bisegment is no longer than an isosceles bisegment with the same total curvature and chordlength, and that if σ is a bisegment with endpoints on the boundary of D, then $\ell(\sigma) \leq \ell(\Gamma_{R,\kappa})$ if $\overline{\kappa}_0 \leq \kappa \leq \pi$.

Suppose now that σ is a D-admissible polysegment and that the number n of segments of σ is at least three. For any four consecutive vertices A, B, C and E of σ, we denote by $\mathbb{H}_{BC}$ the closed halfspace containing AB and CE and whose boundary contains the points B and C. Suppose A, B, C and E are the first four ordered vertices of σ, with A an endpoint. We replace the segment BC with an arc BC' of the boundary of the samelength, where C' is chosen so that it does not lie on the halfspace $\mathbb{H}_{BC}$, and then rotate the rest of σ about the center O so that the vertex C coincides with C'. If E is not an endpoint of σ we then replace the segment $C'E$ with an arc $C'E'$ of the boundary of the same length, where E' is chosen so that it does not lie on the halfspace $\mathbb{H}_{C'E}$, and then rotate the rest of σ about O so that the vertex E coincides with E'. We continue the process until all but the two ending segments of σ are replaced by arcs. We claim that each time an arc is introduced the total curvature does not increase. This is true by the Gauss-Bonnet formula for $K \geq 0$. For $K < 0$, if the original segment makes an angle δ with the replacing arc, then the total curvature is reduced by $\Delta(\delta) = 2\delta - \frac{2}{\tanh \lambda R} \tanh^{-1}(\tanh \lambda R \sin \delta)$. The condition $R \leq R_0$ implies that $\Delta'(\delta)$ is nonnegative. But $\Delta(0) = 0$, so that $\Delta(\delta)$ is nonnegative for every $\delta \in [0, \frac{\pi}{2}]$, and thus the total curvature is indeed reduced. This proves the existence of an round-tip bisegment Γ in the given disk with total curvature not exceeding $\kappa(\sigma)$ and length at least $\ell(\sigma)$. It is easily verified that $\ell(\Gamma) \leq \ell(\Lambda_{R,\kappa})$ if $0 \leq \kappa \leq \overline{\kappa}_0$ and $\ell(\Gamma) \leq \ell(\Gamma_{R,\kappa})$ if $\overline{\kappa}_0 \leq \kappa < \infty$.

This together with the facts noted in the first paragraph complete the proof of the theorem for the case of bisegments and D-admissible polysegments.

Now let γ be an arbitrary curve contained in a closed ball of radius R in an R_K space, with R as described in the theorem statement. Let σ_n be a sequence of polysegments inscribed in γ such that $\mu_\gamma(\sigma_n) \to 0$ and $\kappa(\sigma_n) \to \kappa$. Then $\ell(\sigma_n) \to \ell(\gamma)$. Theorem 2.1 and Proposition 2.2 together with the above results imply that $\ell(\sigma_n) \leq \widetilde{S}(R, \kappa(\sigma_n))$ for each n. Now taking into account the continuity of $\widetilde{S}(R,\kappa)$ in κ, it follows by taking limits as $n \to \infty$ that $\ell(\gamma) \leq \widetilde{S}(R,\kappa)$ as required. The last statement is readily verified. $\square$

3 Large Circumballs with Negative Spatial Curvature Bounds: Revealing the Unseen

Although it is impossible to obtain an estimate for $R > R_0$ when $K > 0$, an extension to the case of large R is still possible when $K < 0$. Thus in this section we assume $R > R_0$. In order to get an estimate in this case, we can still use Theorem 2.1 and Proposition 2.2 to reduce the problem into one involving only bisegments and admissible polysegments in the model spaces. However, we need a different approach to take care of the reduced case. To achieve this we introduce the following conventions, definitions and notations.

For notational convenience, let us assume without loss of generality that $K = -1$, so that $R_0 = \sinh^{-1} 1$. For each R we fix a closed disk D_R of radius R centered at a fixed origin O in S_{-1} and let C_R be its boundary. Any D_R-admissible polysegment is simply referred to as an *admissible* polysegment. For each R and each $\kappa \geq 0$ let $\widetilde{\mathcal{P}}(R,\kappa)$ denote the set of admissible polysegments in D_R with total curvature κ. Our goal is to find for each $R > R_0$ and $\kappa \geq 0$ the upper bounds

$$\widetilde{s}(R,\kappa) = \sup_{\sigma \in \widetilde{\mathcal{P}}(R,\kappa)} \ell(\sigma) \text{ and } \widetilde{S}(R,\kappa) = \begin{cases} \max\{\widetilde{s}(R,\kappa), \ell(\Lambda_{R,\kappa})\} & \text{if } \kappa \leq \pi, \\ \widetilde{s}(R,\kappa) & \text{if } \kappa > \pi, \end{cases}$$

where the latter is the upper bound for which we are looking. For any chord c of C_R, the *ending angle* $\widehat{c} \leq \frac{\pi}{2}$ of c is the angle that c makes with C_R. By an *ending angle* of an admissible polysegment we mean the ending angle of any of its segments. Now, letting $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the ordered segments of an admissible n-segment σ in D_R, we define the *excessive semi-total curvature* $\kappa_s(\sigma)$ of σ by $\kappa_s(\sigma) = \sum_{i=1}^{n} \widehat{\sigma}_i$. Let $n \geq 2$ be an integer and let σ be an admissible n-segment in D_R. The segments σ_1 and σ_n are called the *tail segments* and $\sigma_2, \ldots, \sigma_{n-1}$ the *body segments* of σ. The *linked tail* or simply the *tail* of σ, denoted by σ^T, is any of the bisegments in the congruent class of admissible bisegments in D_R consisting of segments σ_1^T of length $\ell(\sigma_1)$ and σ_2^T of length $\ell(\sigma_n)$. The *body* of σ is an admissible polysegment σ^B in D_R congruent to the concatenation $\sigma_2 * \sigma_3 * \cdots * \sigma_{n-1}$. It is easy to see that $\ell(\sigma) = \ell(\sigma^T) + \ell(\sigma^B)$, $\kappa_s(\sigma) = \kappa_s(\sigma^T) + \kappa_s(\sigma^B)$ and $\kappa(\sigma) = \kappa_s(\sigma^T) + 2\kappa_s(\sigma^B)$. Thus $\kappa_s(\sigma) = \kappa(\sigma)$ if σ is a bisegment and $\kappa_s(\sigma) - \frac{1}{2}\kappa(\sigma) = \frac{1}{2}\kappa(\sigma^T)$ in general, which justifies the term "excessive semi-total curvature." Since variation in ending angles of body segments of a given admissible polysegment doubly affects its total curvature as compared to variation in the ending angles of its tail, we will first study bodies and linked tails of admissible polysegments separately in connection with excessive semi-total curvature.

For each $\kappa_s \geq 0$, let $\mathcal{P}(R, \kappa_s)$ be the set of admissible polysegments in D_R with excessive semi-total curvature κ_s. In addition, let $s(R, \kappa_s)$ be the supremum

$$s(R, \kappa_s) = \sup_{\sigma \in \mathcal{P}(R,\kappa_s)} \ell(\sigma).$$

We are concerned at the moment to compute $s(R, \kappa_s)$ for given κ_s. Any polysegment in $\mathcal{P}(R, \kappa_s)$ can be thought of as the body of an admissible polysegment in D_R. Similarly, any bisegment in $\mathcal{P}(R, \kappa_s)$ can be thought of as the linked tail of an admissible polysegment in D_R.

Recall that the formula $r_R(t) = 2\tanh^{-1}(\tanh R \sin t)$ gives the length of any chord of C_R with ending angle t. The function r (with the subscript R dropped out for convenience) and its derivative r' given by

$$r'(t) = \frac{2\tanh R \cos t}{1 - \tanh^2 R \sin^2 t}$$

defined on $[0, \frac{\pi}{2}]$ play very important roles in our discussion so we summarize their properties here. Firstly, r is strictly increasing and vanishes at 0. Its derivative r', on the other hand, is strictly increasing and decreasing respectively on the left and right hand side of the parameter value $\tau_2 = \cos^{-1}(\mathrm{csch} R) \in (0, \frac{\pi}{2})$. It vanishes at $\frac{\pi}{2}$ and gives the same value $2\tanh R$ at 0 and at $\tau_\infty = \cos^{-1}(\mathrm{csch}^2 R) \in (\tau_2, \frac{\pi}{2})$. As functions of R, τ_2 and τ_∞ are strictly increasing, approach 0 as $R \to R_0^+$ and approach $\frac{\pi}{2}$ as $R \to \infty$. The ratio $\dfrac{\tau_\infty}{\tau_2}$ is strictly decreasing with R and approaches $\sqrt{2}$ as $R \to R_0^+$. The importance of the numbers τ_2 and τ_∞ defined here is that they will appear as lower and upper bounds for ending angles of almost all segments of the polysegments that attain the maximum length in $\mathcal{P}(R, \kappa_s)$. Figure 2 shows sketches of graphs of r, r' and r''. It also locates the *dual ending angle* $d(t)$ and the *co-dual ending angle* $D(t)$ of an arbitrary $t \in [0, \tau_\infty]$, where $d(t)$ is the only $t^* \leq \tau_\infty$ on the opposite side of τ_2 as compared with t (unless $t = \tau_2$) such that $r'(t^*) = r'(t)$, while $D(t)$ is the only $\widehat{t} > \tau_2$ such that $r'(\widehat{t}\,) = \frac{1}{2}r'(t)$. We also conveniently write t^* for $d(t)$, $f^*(t)$ for $(f(t))^*$ (if f is a function) and $\widehat{t}$ for $D(t)$.

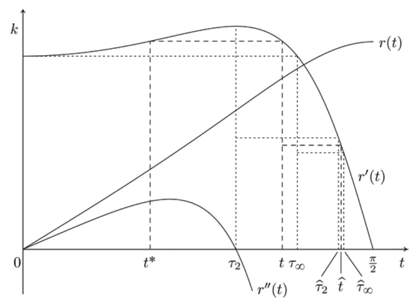

Figure 2: Sketches of graphs of r, r' and r''.

Let us now fix a point $O' = O'(R)$ on the boundary of D_R, a $\kappa_s > 0$ and a positive integer n such that the interval $I_{\kappa_s,n} = [\frac{\kappa_s}{n}, \frac{\kappa_s}{n-1}] \cap [\tau_2, \tau_\infty]$ contains more than one element. For each $t \in I_{\kappa_s,n}$ let $P^t = P^t(\kappa_s, n) \in \mathcal{P}(R, \kappa_s)$ be a (possibly degenerate) n-segment, which starts at O' and winds around the center O in the counter-clockwise direction, with $\widehat{P}^t_1 = \widehat{P}^t_2 = \cdots = \widehat{P}^t_{n-1} = t$ and $\widehat{P}^t_n = \kappa_s - (n-1)t \leq t$. An admissible polysegment whose arrangement of ordered ending angles is a rearrangement of those of $P^t(\kappa_s, n)$ for some t and n is said to be of a *generic $\mathcal{A}$-type* if it either is equilateral or has an exceptional ending angle (the one that is smaller than the others) less than τ_2. We refer to those ending angles that are at least τ_2 as *major ending angles.* For convenience, we regard $P^t(\kappa_s, n)$ as a polysegment *with* major ending angle t even though $n = 1$. In the following lemma we show that any maximizing polysegment in $\mathcal{P}(R, \kappa_s)$, where $\kappa_s > 0$, is of a generic $\mathcal{A}$-type.

Lemma 3.1. *If a polysegment $\sigma \in \mathcal{P}(R, \kappa_s)$, where $\kappa_s > 0$, is not of a generic $\mathcal{A}$-type, then there exists a generic $\mathcal{A}$-type polysegment $\eta \in \mathcal{P}(R, \kappa_s)$ such that $\ell(\sigma) < \ell(\eta)$.*

Proof. Assume that $\sigma \in \mathcal{P}(R, \kappa_s)$, where $\kappa_s > 0$, is not of a generic $\mathcal{A}$-type. If no segments of σ have ending angle greater than τ_∞ we let $\sigma' = \sigma$. Suppose otherwise. We shall show that by adding new segments to σ, each segment with ending angle more than τ_∞ can be expelled without length decrease. Without loss of generality, let $\widehat{\sigma}_1 > \tau_\infty$ be the ending angle of a segment of σ to be rid of. Let $\sigma^t \in \mathcal{P}(R, \kappa_s)$, $0 \le t \le \frac{\widehat{\sigma}_1}{2}$, be a polysegment obtained from σ by replacing σ_1 with two segments whose ending angles are t and $\widehat{\sigma}_1 - t$. Then the length of σ^t is given by

$$l(t) = \ell(\sigma) + r(t) + r(\widehat{\sigma}_1 - t) - r(\widehat{\sigma}_1),$$

and its derivative with respect to t by

$$l'(t) = r'(t) - r'(\widehat{\sigma}_1 - t).$$

Because $l'(0) > 0$, we can deform $\sigma = \sigma^0$ through σ^t in such a way that $l(t)$ increases until $t = t_0$ at which l' first vanishes. If $t_0 = (\widehat{\sigma}_1 - t_0)^*$ then both t_0 and $\widehat{\sigma}_1 - t_0$ are at most τ_∞. Otherwise, $t_0 = \dfrac{\widehat{\sigma}_1}{2}$. In the latter case, if t_0 is still greater than τ_∞ the whole process may be repeated until all the new segments have ending angles at most τ_∞. Since the number of segments of σ with ending angles greater than τ_∞ is finite, a finite number of iterations of the operation described above will result in a polysegment σ' with no segments with ending angles greater than τ_∞ and with $\ell(\sigma) < \ell(\sigma')$.

Now, if at most one of the segments of σ' has ending angle smaller than τ_2 we let $\sigma'' = \sigma'$. Suppose more than one segment of σ' have ending angles smaller than τ_2. We prove that those extra segments can be eliminated without length decrease. To see this, we assume without loss of generality that $\widehat{\sigma}'_2 \le \widehat{\sigma}'_1 < \tau_2$. Consider a deformation of σ' through a family of admissible polysegments $\sigma^t, t \ge 0$ defined by replacing the first two segments of σ' with segments whose ending angles are $\widehat{\sigma}^t_1 = \widehat{\sigma}'_1 + t$ and $\widehat{\sigma}^t_2 = \widehat{\sigma}'_2 - t$, and holding all other segments of σ' fixed. Then $\sigma^0 = \sigma'$ and

$$\frac{d}{dt}\ell(\sigma^t) = r'(\widehat{\sigma}'_1 + t) - r'(\widehat{\sigma}'_2 - t) \ge 0$$

as long as $t \le t_0$, where $t_0 = \min\{\widehat{\sigma}'_2, \tau_2 - \widehat{\sigma}'_1\}$. Note that this is true because $\tau_2 \ge \widehat{\sigma}'_1 + t \ge \widehat{\sigma}'_2 - t$ for such t. Indeed, the strict inequality holds for $0 < t < t_0$. Thus the deformation can be carried out until $t = t_0$, where we stop. The resulting polysegment σ^{t_0} has greater length and fewer segments with ending angles smaller than τ_2. Since the number of segments of σ' is finite, it is possible to perform this procedure inductively finitely many times and get a polysegment $\sigma'' \in \mathcal{P}(R, \kappa_s)$ with at most one segment having ending angle smaller than τ_2 such that $\ell(\sigma') < \ell(\sigma'')$. Notice that σ'' has no ending angle greater than τ_∞.

Next we show the existence of η with the described properties. If σ'' is a geodesic we let $\eta = \sigma''$, which obviously has the required properties. Suppose without loss of generality that σ'' is an m-segment, $m \ge 2$, with ending angles $\widehat{\sigma}''_1 \ge \widehat{\sigma}''_2 \ge \cdots \ge \widehat{\sigma}''_m$. Let k, where $m - 1 \le k \le m$, be the number of segments of σ'' having ending angles at least τ_2. Let t_0 be the average of these ending angles, i.e., $t_0 = \dfrac{1}{k}\sum_{i=1}^{k}\widehat{\sigma}''_i$. Let t_i, where $1 \le i \le k$, be the deviation of $\widehat{\sigma}''_i$ from t_0: $t_i = \widehat{\sigma}''_i - t_0$ and let η be an m-segment in $\mathcal{P}(R, \kappa_s)$ with $\widehat{\eta}_i = t_0$ for all i such that $1 \le i \le k$ and $\widehat{\eta}_m = \widehat{\sigma}''_m$ if $m = k+1$. Then

$$\begin{aligned}
\ell(\eta) - \ell(\sigma'') &= \sum_{i=1}^{k}[r(t_0) - r(t_0 + t_i)] \\
&= \sum_{1\le i\le k,\, t_i<0} \int_{t_0+t_i}^{t_0} r'(t)\, dt - \sum_{1\le i\le k,\, t_i>0} \int_{t_0}^{t_0+t_i} r'(t)\, dt \\
&\ge \sum_{1\le i\le k,\, t_i<0} \int_{t_0+t_i}^{t_0} r'(t_0)\, dt - \sum_{1\le i\le k,\, t_i>0} \int_{t_0}^{t_0+t_i} r'(t_0)\, dt.
\end{aligned}$$

The last two sums are identical because the sums of the lengths of intervals of integration that appear in both sums are the same, so that $\ell(\eta) \ge \ell(\sigma'')$. Note also that the equality holds only if every t_i is zero. Thus $\ell(\eta) = \ell(\sigma)$ only if $\eta = \sigma'' = \sigma' = \sigma$. This completes the proof of the lemma. $\square$

In light of the above lemma we now focus on the class of generic $\mathcal{A}$-type polysegments. Here and below a generic $\mathcal{A}$-type polysegment whose ending angles have the same image under r' is said to be of a *specific* $\mathcal{A}$*-type*. A specific $\mathcal{A}$-type polysegment is of an $\mathcal{E}$*-type* if it is equilateral; it is of an $\mathcal{N}$*-type* if its smallest ending angle is at most τ_2. Note that $\mathcal{N}$*-type* n-segments are non-equilateral, except for those with $n = 1$ or with all ending angles equal to τ_2 or τ_∞. Now, for each positive integer n, we define a function $k_n = k_{n,R}$ on $[\tau_2, \tau_\infty]$ by

$$k_n(t) = (n-1)t + t^*.$$

Then $k_n(t)$ is the excessive semi-total curvature of an $\mathcal{N}$-type n-segment all but one ending angles equal to t. We will show in Lemma 3.2 that non-equilateral maximizing polysegments in $\mathcal{P}(R, \kappa_s)$ are of this type. An important but easy-to-verify property of k_n's is that for each n there exists a unique $\tau_n \in [\tau_2, \tau_\infty)$ such that $k'_n(\tau_n) = 0$, k_n is strictly increasing on $[\tau_2, \tau_n]$ and strictly decreasing on $[\tau_n, \tau_\infty]$ (and hence k_2 is strictly decreasing, in particular). An explicit form of τ_n can be derived (but we do not give it here), from which it follows that τ_n is strictly increasing with n and approaches τ_∞ as $n \to \infty$, justifying the use of the symbols τ_2 and τ_∞. Sketches of graphs of a few k_n's are shown in Figure 3 for typical $n > 2$.

Note in particular that for any integer $n \ge 2$ the graph of k_n is trapped in a trapezoid defined by the vertical lines $t = \tau_2$ and $t = \tau_\infty$ and the lines $k = K_n(t)$ and $k = K_{n-1}(t)$. Here K_n's are the linear functions $K_n(t) = nt$ representing the excessive semi-total curvature of $\mathcal{E}$-type polysegments.

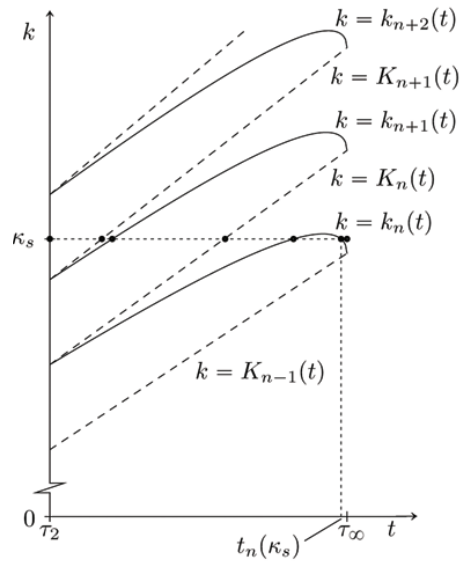

Figure 3: Sketches of graphs of k_n's and K_n's.

In what follows, the only ending angle value $t \geq \tau_n$ such that $k_n(t) = k$, if one exists, will serve as maximizers' body ending angles in certain cases and thus will be denoted by $t_n(k)$ for convenience. Our next lemma eliminates all but finitely many candidates for the maximizing polysegments in $\mathcal{P}(R, \kappa_s)$.

Lemma 3.2. *Let $\kappa_s \geq 0$ be given. Then maximizing polysegments in $\mathcal{P}(R, \kappa_s)$ exist and are of a specific $\mathcal{A}$-type.*

Proof. The assertion is clear for $\kappa_s = 0$. If $0 < \kappa_s \leq \tau_2$ then by Lemma 3.1 any σ in $\mathcal{P}(R, \kappa_s)$ is no longer than a geodesic chord with ending angle κ_s. Suppose $\kappa_s > \tau_2$. It suffices to consider, for each fixed $n \geq 2$ with nontrivial $I_{\kappa_s,n}$, the length

$$l(t) = l_{\kappa_s,n}(t) = (n-1)r(t) + r\left(\kappa_s - (n-1)t\right)$$

of $P^t(\kappa_s, n)$. Direct calculation shows that critical points of l for all possible n correspond to the points where the line $k = \kappa_s$ cuts the graphs $k = k_n(t)$ and the graphs $k = K_n(t)$, together with τ_2 and τ_∞ at the two ends. (See Figure 3.) By analyzing derivatives of l up to the fourth order, it can be shown that the critical points corresponding to the intersections of the line $k = \kappa_s$ and the increasing part of the graphs $k = k_n(t)$ as well as the points τ_2 and τ_∞ never give local maxima unless they correspond to $\mathcal{E}$-type polysegments. Local maxima can occur at other critical points, all but possibly one of which also correspond to $\mathcal{E}$-type polysegments. The only exception is the critical point $t_n(\kappa_s)$ corresponding to the intersection of the line $k = \kappa_s$ and the decreasing part of a graph $k = k_n(t)$, where at most one $n = 1 + \lfloor \frac{\kappa_s}{\tau_\infty} \rfloor$ can give rise to it. This critical point is easily seen to correspond to an $\mathcal{N}$-type polysegment since $\kappa_s = k_n(t)$ implies $\kappa_s - (n-1)t = t^*$. $\square$

Corollary 3.3. *The maximizing polysegments in $\mathcal{P}(R, \kappa_s)$ are*

(i) geodesic chords of C_R with ending angles κ_s if $0 \leq \kappa_s \leq \tau_\infty$,

(ii) non-isosceles bisegments with ending angles $t_2(\kappa_s)$ and $\kappa_s - t_2(\kappa_s)$ if $\tau_\infty < \kappa_s < 2\tau_2$, and

(iii) isosceles bisegments with ending angles $\frac{\kappa_s}{2}$ if $2\tau_2 \leq \kappa_s \leq 2\tau_\infty$.

Proof. The assertions are verified through the use of Lemmas 3.1 and 3.2, with an analysis of critical points of $l(t)$ as is done in the above lemma for case (iii). $\square$

Now we return to our original goal of maximizing the length of polysegments in $\widetilde{\mathcal{P}}(R, \kappa)$, in which the facts discovered in the case of semi-total curvature above will be utilized. In addition to the values of $\widetilde{s}(R, \kappa)$ for $\kappa \leq \tau_\infty$, the following proposition gives a lower bound for the semi-total curvature of the linked tail of maximizing polysegments in $\widetilde{\mathcal{P}}(R, \kappa)$ for $\kappa > \tau_\infty$.

Proposition 3.4. *Let $\sigma \in \widetilde{\mathcal{P}}(R, \kappa)$, $\kappa > 0$, be a polysegment with $\kappa_s(\sigma^T) \leq \tau_\infty$. Then there exists a deformation of σ into a polysegment η with $\ell(\sigma) < \ell(\eta)$ such that*

(i) η is a geodesic chord with ending angle κ if $0 < \kappa \leq \tau_\infty$, and

(ii) $\eta \in \widetilde{\mathcal{P}}(R, \kappa)$ with $\kappa_s(\eta^T) > \tau_\infty$ if $\kappa > \tau_\infty$.

In particular, if $\kappa \leq \tau_\infty$, then $\widetilde{s}(R, \kappa) = r(\kappa)$. In this case, a maximizing polysegment does not exists, but a polysegment in $\widetilde{\mathcal{P}}(R, \kappa)$ can be chosen to be arbitrarily close in length to a geodesic chord with ending angle κ.

Proof. Suppose that $0 < \kappa \leq \tau_\infty$ and that $\sigma \in \widetilde{\mathcal{P}}(R, \kappa)$ is a bisegment. Since $\kappa_s(\tau) = \kappa(\tau)$ for any bisegment τ, Corollary 3.3 implies the existence of η with the asserted properties. If σ has more than two ordered segments $\sigma_1, \sigma_2, \ldots, \sigma_n$ with $\widehat{\sigma}_n \geq \widehat{\sigma}_1$, then it reduces into a bisegment case by applying the same argument on its linked tail σ^T to get a new polysegment σ' with $\ell(\sigma) < \ell(\sigma')$ and $\kappa(\sigma') = (\widehat{\sigma}_1 + \widehat{\sigma}_n) + \widehat{\sigma}_2 + 2(\widehat{\sigma}_3 + \cdots + \widehat{\sigma}_{n-1})$. Thus by increasing $\widehat{\sigma}'_1$ to $2\widehat{\sigma}_2 \leq \tau_\infty < \frac{\pi}{2}$, the newly obtained polysegment still has total curvature κ, and is longer than the original one. Since the number of segments is reduced, we can repeat the process until a bisegment is obtained.

Suppose now that $\kappa > \tau_\infty$ and σ has at least three segments with $\kappa_s(\sigma^T) \leq \tau_\infty$. The same arguments are still valid unless $2\widehat{\sigma}_2 > \frac{\pi}{2}$. In this case, instead of increasing $\widehat{\sigma}_1'$ to $2\widehat{\sigma}_2$, we increase it to $\frac{\pi}{2}$ and add another body segment for total curvature adjustment. If the semi-total curvature of the linked tail of σ' is still no greater than τ_∞, then this special case does not occur and the deformation can be repeated. If the process can be done repeatedly without adding a new segment then it must stop at or before a bisegment stage. In any case, the semi-total curvature of the linked tail of the resulting polysegment is greater than τ_∞. This proves the existence of η with the required properties. □

Next we give the values of $\widetilde{s}(R,\kappa)$ for $\tau_\infty < \kappa < 2\tau_2$.

Proposition 3.5. *If $\tau_\infty < \kappa < 2\tau_2$ then $\widetilde{s}(R,\kappa) = r(t_2(\kappa)) + r(\kappa - t_2(\kappa))$ and the maximizing polysegments in $\widetilde{\mathcal{P}}(R,\kappa)$ are non-isosceles bisegments with ending angles $t_2(\kappa)$ and $\kappa - t_2(\kappa)$.*

Proof. Suppose that $\tau_\infty < \kappa < 2\tau_2$ and that $\sigma \in \widetilde{\mathcal{P}}(R,\kappa)$. Applying Proposition 3.4, we assume without loss of generality that $\kappa_s = \kappa_s(\sigma^T) > \tau_\infty$. Notice that whenever the body of σ contains a segment that is shorter than a tail segment of it, swapping these two segments followed by lengthening appropriate segment(s) and/or adding one more body segment results in a longer polysegment having the same total curvature. With successive use of such a body-tail swapping technique together with Corollary 3.3 we get a no shorter trisegment $\delta \in \widetilde{\mathcal{P}}(R,\kappa)$ (possibly degenerate, i.e., having trivial body) with $\kappa_s' = \kappa_s(\delta^T) \geq \kappa_s > \tau_\infty$. The ending angles of δ are:

$$\widehat{\delta}_1 = t_2(\kappa_s'),\ \widehat{\delta}_2 = \frac{\kappa - \kappa_s'}{2}, \text{ and } \widehat{\delta}_3 = \kappa_s' - t_2(\kappa_s') = t_2^*(\kappa_s'),$$

where $\widehat{\delta}_2$ is the smallest (possibly zero) among the three ending angles of δ. Note that $\widehat{\delta}_3 \leq \tau_2 \leq \widehat{\delta}_1$. Let $l(k)$ be the length of a trisegment δ^k with ordered ending angles $t_2(k)$, $\frac{\kappa-k}{2}$ and $k - t_2(k) = t_2^*(k)$. Then

$$l(k) = r(t_2(k)) + r(t_2^*(k)) + r\left(\frac{\kappa - k}{2}\right)$$

and direct computation yields $l'(k) = r'(t_2(k)) - \frac{1}{2}r'\left(\frac{\kappa-k}{2}\right)$. Now $l'(k) > 0$ if $\kappa_s' \leq k \leq \kappa$ since

$$\begin{aligned} r'(t_2(k)) &\geq r'(t_2(\kappa_s')) = r'(t_2^*(\kappa_s')) = r'(\widehat{\delta}_3) \\ &\geq r'(\widehat{\delta}_2) = r'\left(\frac{\kappa - \kappa_s'}{2}\right) \geq r'\left(\frac{\kappa - k}{2}\right). \end{aligned}$$

Hence we can deform $\delta = \delta^{\kappa_s'}$ through a family of trisegments δ^k by increasing k until $k = \kappa$ without length decrease. This implies the assertion to be proved. □

Here come the roles of co-dual ending angles defined earlier. We contend below that $\widehat{\tau}_2$ and $\widehat{\tau}_\infty$ are the sharp lower and upper bounds for the ending angles of tail segments of maximizers in $\widetilde{\mathcal{P}}(R,\kappa)$ for large κ. We also note here that $\tau_2 + \widehat{\tau}_2 > \widehat{\tau}_\infty$.

Proposition 3.6. *Let $\sigma \in \widetilde{\mathcal{P}}(R,\kappa)$, $\kappa \geq 2\tau_2$, be a polysegment such that either σ^T is not isosceles or $\kappa_s(\sigma^T) < 2\tau_2$. Then there exists a longer polysegment $\eta \in \widetilde{\mathcal{P}}(R,\kappa)$, whose linked tail η^T is isosceles with $\kappa_s(\eta^T) \geq 2\tau_2$. In particular, if $2\tau_2 \leq \kappa \leq 2\widehat{\tau}_2$ then $\widetilde{s}(R,\kappa) = 2r\left(\frac{\kappa}{2}\right)$ and the maximizing polysegments in $\widetilde{\mathcal{P}}(R,\kappa)$ are isosceles bisegments $\Lambda_{R,\kappa}$.*

Proof. Let σ be as given. If $\kappa_s(\sigma^T) \geq 2\tau_2$ we set $\eta = \sigma$. Suppose $\kappa_s(\sigma^T) < 2\tau_2$. We assume without loss of generality that $\kappa_s = \kappa_s(\sigma^T) > \tau_\infty$ with σ^T having ending angles $t_2(\kappa_s) > \tau_2$ and $\kappa_s - t_2(\kappa_s)$ and that σ^B is of a specific $\mathcal{A}$-type. If there is also a body segment with ending angle at least τ_2 we perform a body-tail swapping technique to get a longer polysegment η with $\kappa_s(\eta^T) \geq 2\tau_2$. Otherwise we apply the same arguments as did in the previous proposition to get $\eta = \delta^{2\tau_2}$ so that $\kappa_s(\eta^T) \geq 2\tau_2$. Now if η^T is not isosceles a simple deformation of η^T will finish the proof.

For the last assertion, let us fix $\kappa \in [2\tau_2, 2\widehat{\tau}_2]$. If $\kappa_s(\sigma^T) < 2\tau_2$, we assume without loss of generality that $\kappa_s = \kappa_s(\sigma^T) > \tau_\infty$, that the tail segments of σ have ending angles $t_2(\kappa_s)$ and $\kappa_s - t_2(\kappa_s) = t_2^*(\kappa_s)$ and that σ^B is of a specific $\mathcal{A}$-type. If no body segments of σ have ending angle at least τ_2 we set $\beta = \sigma$ and apply the arguments used in the proof of the previous proposition, with the same use of notations until a trisegment $\eta = \delta^{2\tau_2}$ is obtained. If σ has a body segment with ending angle at least τ_2 then a segment swap technique can be used to get a polysegment η with $\kappa_s(\eta^T) \geq 2\tau_2$. If σ^T is not isosceles a deformation can be performed to make the linked tail isosceles. Thus we assume now that σ is a polysegment in $\widetilde{\mathcal{P}}(R,\kappa)$ whose linked tail is isosceles with ending angles $t_0 \geq \tau_2$. We shall eliminate σ_2 by a deformation that increases arclength. Note that $t_0 < \frac{\kappa}{2} \leq \widehat{\tau}_2$. Let $l(t)$ be the length of the polysegment σ^t obtained from σ by decreasing $\widehat{\sigma}_2$ and increasing the ending angle of each of the tail segments by $t \geq 0$. Then

$$l'(t) = 2r'(t_0 + t) + r'(\widehat{\sigma}_2 - t) > 0$$

if $t < \widehat{\sigma}_2$, because for such t we have $t_0 + t < \frac{\kappa}{2} \leq \widehat{\tau}_2$ and $2r'(t_0 + t) > r'(\tau_2)$, which is the maximum value of r'. Thus we obtain by a deformation of $\sigma = \sigma^0$ through σ^t a longer polysegment σ' having a smaller number of segments. By repeating the whole process we get an isosceles bisegment of greater length while the total curvature remains fixed. $\square$

Proposition 3.7. *Let $\sigma \in \widetilde{\mathcal{P}}(R,\kappa)$, $\kappa > 2\widehat{\tau}_2$, be a polysegment such that at least one of the following conditions fails to hold:*

(i) σ^T *is isosceles,*

(ii) $\kappa_s(\sigma^T) \geq 2\widehat{\tau}_2$*, and*

(iii) $\kappa_s(\sigma^T) \leq 2\widehat{\tau}_\infty$.

Then there exists a longer polysegment $\eta \in \widetilde{\mathcal{P}}(R,\kappa)$ *which satisfies all of these conditions. In particular, if* $2\widehat{\tau}_2 \leq \kappa \leq 2\widehat{\tau}_\infty$ *then* $\widetilde{s}(R,\kappa) = 2r\left(\frac{\kappa}{2}\right)$ *and the maximizing polysegments in* $\widetilde{\mathcal{P}}(R,\kappa)$ *are isosceles bisegments* $\Lambda_{R,\kappa}$.

Proof. If σ^T is not isosceles, Proposition 3.6 implies a longer polysegment with the same total curvature and with isosceles linked tail. If $\kappa_s(\sigma^T) < 2\widehat{\tau}_2$, the same arguments as used in the previous proposition, with an appropriate modification, give rise to a longer polysegment in $\widetilde{\mathcal{P}}(R,\kappa)$ whose linked tail is isosceles and has semi-total curvature $2\widehat{\tau}_2$. Finally, if $\kappa_s(\sigma^T) > 2\widehat{\tau}_\infty$, an analogue of the proof of Proposition 3.6 above yields a polysegment whose linked tail is isosceles and has semi-total curvature $2\widehat{\tau}_\infty$. The resulting polysegment also has greater length and the same total curvature. We now complete the proof by verifying the last part.

By the previous paragraph, it suffices to consider polysegments in $\widetilde{\mathcal{P}}(R,\kappa)$ with isosceles tails whose ending angles are equal to some $t \in [\widehat{\tau}_2, \widehat{\tau}_\infty]$. The bodies of these polysegments have semi-total curvature at most $\frac{\kappa}{2} - \widehat{\tau}_2 \leq \widehat{\tau}_\infty - \widehat{\tau}_2 < \tau_2$, and hence can be replaced by chords of the circle C_R with ending angles $\frac{\kappa}{2} - t$. Consider a parametrization of representatives σ^t of admissible trisegments, of the type of our concern, by the ending angle t of their isosceles linked tails. The length of σ^t is $l(t) = 2r(t) + r\left(\frac{\kappa}{2} - t\right)$ whose derivative $l'(t) = 2r'(t) - r'\left(\frac{\kappa}{2} - t\right)$ can be shown to be strictly positive for every $t \in \left(\widehat{\tau}_2, \frac{\kappa}{2}\right)$. Indeed, by setting $F_t(k) = 2r'(t) - r'(k-t)$ on $[t, \widehat{\tau}_\infty]$ for each fixed $t \in [\widehat{\tau}_2, \widehat{\tau}_\infty]$ we have $F_t'(k) = -r''(k-t)$, which vanishes only if $k = t$ or $k = t + \tau_2 \geq \widehat{\tau}_2 + \tau_2$. Since the latter contradicts $k \leq \widehat{\tau}_\infty$, we conclude that t is the only zero of F_t'. Now, because r'' is strictly positive on $(0, \tau_2)$, F_t must be strictly decreasing on $[t, \widehat{\tau}_\infty]$. A long but elementary calculation showing that $F_t(\widehat{\tau}_\infty)$ is positive for all $t \in (\widehat{\tau}_2, \widehat{\tau}_\infty)$ completes our proof. $\square$

We now state key properties of maximizers for larger value of total curvature.

Theorem 3.8. *Any polysegment* $\sigma \in \widetilde{\mathcal{P}}(R,\kappa)$, $\kappa > 2\widehat{\tau}_\infty$*, that realizes the maximum length* $\widetilde{s}(R,\kappa)$ *satisfies the following conditions:*

(i) σ^T *is isosceles with ending angles between* $\widehat{\tau}_2$ *and* $\widehat{\tau}_\infty$.

(ii) σ^B *is of a specific* $\mathcal{A}$*-type.*

(iii) any ending angle s *of* σ^T *and* t *of* σ^B *satisfy* $s = \widehat{t}$.

Proof. (i) and (ii) have just been proved above. The derivative of the length function $l(t) = nr(t)+2r\left(\frac{\kappa}{2} - nt\right)$ for an $\mathcal{E}$-type n-segment candidate in $\widetilde{\mathcal{P}}(R,\kappa)$ satisfying (i) with body ending angles t is given by $l'(t) = nr'(t)-2nr'\left(\frac{\kappa}{2} - nt\right)$. This implies (iii) for $\mathcal{E}$-type candidates. The $\mathcal{N}$-type case can be validated in a similar manner. Note that the condition $\kappa \geq 2\widehat{\tau}_\infty$ allows the domain of l to be all of $[\tau_2, \tau_\infty]$. $\square$

A candidate for the maximizers of $\widetilde{\mathcal{P}}(R,\kappa)$, i.e., a polysegment $\sigma \in \widetilde{\mathcal{P}}(R,\kappa)$ satisfying (i)–(iii) of the above theorem, is said to be of an $\widetilde{\mathcal{E}}$-*type* if its body is of an $\mathcal{E}$-type and of an $\widetilde{\mathcal{N}}$-*type* if its body is of an $\mathcal{N}$-type. Further investigation will be based on two families of total curvature functions $\widetilde{k}_n$ and $\widetilde{K}_n$ defined by

$$\widetilde{k}_n(t) = 2(n-1)t + 2t^* + 2\widehat{t}, \qquad \widetilde{K}_n(t) = 2nt + 2\widehat{t},$$

and two families of corresponding length functions $\widetilde{l}_n$ and $\widetilde{L}_n$ defined by

$$\widetilde{l}_n(t) = (n-1)r(t) + r(t^*) + 2r(\widehat{t}), \qquad \widetilde{L}_n(t) = nr(t) + 2r(\widehat{t}).$$

These are meaningfully defined for each integer $n \geq 1$ and each $t \in [\tau_2, \tau_\infty]$: $\widetilde{k}_n$ and $\widetilde{l}_n$ are respectively the total curvature and the length of an $\widetilde{\mathcal{N}}$-type $(n+2)$-segment candidate whose major body ending angles are t while $\widetilde{K}_n$ and $\widetilde{L}_n$ are those of an $\widetilde{\mathcal{E}}$-type $(n+2)$-segment candidate whose body ending angles are all t. To allow the use of differential calculus in length comparison, we consider these expressions as functions of two variables n and t and extend the domains over which the variable n ranges to all real numbers no less than 1. We state without proof here some properties of $\widetilde{k}_n$ and $\widetilde{K}_n$ analogous to those of k_n and K_n that for each $n \geq 2$ (not necessary an integer) there exists a unique $\widetilde{\tau}_n \in [\tau_2, \tau_\infty)$, with $\widetilde{\tau}_2 = \tau_2$, such that $\widetilde{k}'_n(\widetilde{\tau}_n) = 0$, $\widetilde{k}_n$ is strictly increasing on $[\tau_2, \widetilde{\tau}_n]$ and strictly decreasing on $[\widetilde{\tau}_n, \tau_\infty]$ (and hence $\widetilde{k}_2$ is strictly decreasing, in particular). Moreover, $\widetilde{\tau}_n$ is strictly increasing with n and approaches τ_∞ as $n \to \infty$. For $1 \leq n \leq 2$, $\widetilde{k}_n$ is strictly decreasing. On the other hand, $\widetilde{K}_n$, like K_n, is strictly increasing on $[\tau_2, \tau_\infty]$ for all $n \geq 1$. Sketches of graphs of $\widetilde{k}_n$'s and $\widetilde{K}_n$'s are shown in Figure 4.

Theorem 3.8 implies (in a similar manner to what is done in the proof of Lemma 3.2) that local maximizing candidates correspond to the points where the line $k = \kappa$ cuts the graphs $k = \widetilde{k}_n(t)$ and the graphs $k = \widetilde{K}_n(t)$. As is the case with k_n, the critical points corresponding to the intersection of the line $k = \kappa$ and the increasing part of the graphs $k = \widetilde{k}_n(t)$ never give local maxima. To achieve that for large κ, let us first prove the following

Lemma 3.9. *Given $t \in (\tau_2, \tau_\infty]$, there exist a unique real number $\nu_t > 2$ and a unique ending angle $\check{t} \in (\tau_2, t]$ such that $\widetilde{k}_{\nu_t}(t) = \widetilde{K}_{\nu_t}(\check{t})$ and $\widetilde{l}_{\nu_t}(t) = \widetilde{L}_{\nu_t}(\check{t})$. Moreover, ν_t is strictly increasing with t and $\widetilde{\tau}_{\nu_t} < t$.*

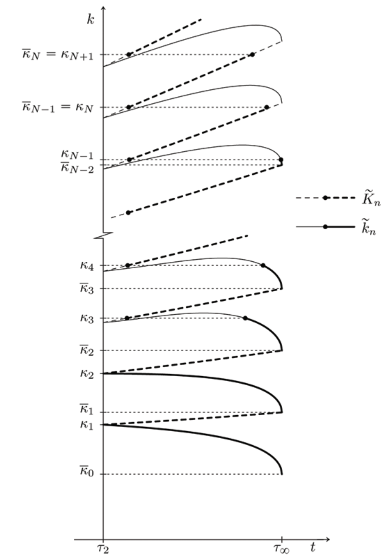

Figure 4: Sketches of graphs of $\widetilde{k}_n$'s and $\widetilde{K}_n$'s.

Proof. Fix $t \in (\tau_2, \tau_\infty]$. It is obvious from the formulas for $\widetilde{k}_n$ and $\widetilde{K}_n$ that for any given $s \in [\tau_2, t)$ there is a unique $n_c = n_c(s)$ such that $\widetilde{k}_{n_c}(t) = \widetilde{K}_{n_c}(s)$. Similarly, there is a unique $n_l = n_l(s)$ such that $\widetilde{l}_{n_l}(t) = \widetilde{L}_{n_l}(s)$. It can be shown that n_c and n_l are positive, strictly increasing and approach ∞ as s approaches t. Moreover, the difference $\Delta = n_l - n_c$ turns out to have a unique zero, which we will denote by $\breve{\mathrm{t}}$. Indeed, Δ is negative on the left side and positive on the right side of $\breve{\mathrm{t}}$. Now, by letting $\nu_t = n_l(\breve{\mathrm{t}}) = n_c(\breve{\mathrm{t}})$, we have $\widetilde{k}_{\nu_t}(t) = \widetilde{K}_{\nu_t}(\breve{\mathrm{t}})$ and $\widetilde{l}_{\nu_t}(t) = \widetilde{L}_{\nu_t}(\breve{\mathrm{t}})$. That ν_t is strictly increasing with t and approaches 2 as t approaches τ_2 ensures that $\nu_t > 2$. Uniqueness easily follows.

Putting $z(t) = \breve{\mathrm{t}}$, direct calculation with careful formulation shows that $z(t)$ is strictly increasing. On the other hand, implicit differentiation gives

$$z'(t) = \frac{-\widetilde{k}'_{\nu_t}(t)\left[\frac{r(t)-r(\breve{\mathrm{t}})}{t-\breve{\mathrm{t}}} - r'(t)\right]}{\widetilde{K}'_{\nu_t}(\breve{\mathrm{t}})\left[r'(\breve{\mathrm{t}}) - \frac{r(t)-r(\breve{\mathrm{t}})}{t-\breve{\mathrm{t}}}\right]}.$$

Since $\widetilde{K}'_{\nu_t}(\breve{\mathrm{t}})$ and the expressions in the two brackets are all positive, we conclude that $\widetilde{k}'_{\nu_t}(t) < 0$, which means $\widetilde{\tau}_{\nu_t} < t$. □

Of all the values of ν_t's the one that plays the most significant role is ν_{τ_∞}, to which we give $\widehat{n}$ as a shorthand. It can be shown that $\widehat{n} > 3$. In the proof of the following lemma and later on, the only ending angle value t such that $\widetilde{K}_n(t) = k$, if one exists, will be denoted by $\underset{\sim}{t}_n(k)$. Also, the only ending angle value $t \geq \widetilde{\tau}_n$ such that $\widetilde{k}_n(t) = k$, if one exists, will be denoted by $\widetilde{t}_n(k)$. These will turn out to be the only possible major ending angles maximizers have.

Lemma 3.10. *Let $\eta \in \widetilde{\mathcal{P}}(R, \kappa)$, where $\kappa > 2\widehat{\tau}_\infty$, be a nonequilateral $\widetilde{\mathcal{N}}$-type $(n+2)$-segment with major body ending angles t. Then each of the following implies that there is an $\widetilde{\mathcal{E}}$-type $(n+2)$-segment $\sigma \in \widetilde{\mathcal{P}}(R, \kappa)$ such that $\ell(\sigma) > \ell(\eta)$.*

(i) $t < \widetilde{\tau}_n$ and $n > 2$,

(ii) $t \geq \widetilde{\tau}_n$ and $n \geq \widehat{n}$.

Proof. Let $\kappa > 2\widehat{\tau}_\infty$ be given. For (i), we assume η with the stated properties. Let σ be the $\widetilde{\mathcal{E}}$-type $(n+2)$-segment with body ending angles $s = \underset{\sim}{t}_n(\kappa)$. Obviously, $\sigma \in \widetilde{\mathcal{P}}(R, \kappa)$. Now since $t > \tau_2$, it makes sense to let ν_t and $\breve{t}$ be as in Lemma 3.9 above. Since $\widetilde{\tau}_{\nu_t} < t < \widetilde{\tau}_n$, we have (with an imitation of notations from Lemma 3.9) $n_c(\breve{t}\,) = \nu_t < n = n_c(s)$. It follows that $\breve{t} < s$ and hence $\Delta(s) > 0$ or, equivalently, $n_l(s) > n_c(s) = n$. Thus, $\ell(\sigma) = \widetilde{L}_n(s) = \widetilde{L}_{n_l(s)}(s) - [n_l(s) - n]r(s) = \widetilde{l}_{n_l(s)}(t) - [n_l(s) - n]r(s) = \widetilde{l}_n(t) + [n_l(s) - n][(r(t) - r(s)] > \ell(\eta)$ as was to be proved.

To prove (ii), we first note that $n \geq \widehat{n}$ implies $\widetilde{k}_n(\tau_2) < \widetilde{k}_n(\tau_\infty)$ and hence the existence of $\underset{\sim}{t}_n(\kappa)$ follows. We claim that the $\widetilde{\mathcal{E}}$-type $(n+2)$-segment σ with body ending angles $\underset{\sim}{t}_n(\kappa)$ will also do the job. Indeed, since $t < \tau_\infty$, we obtain $\nu_t < \widehat{n} \leq n$, i.e., $\nu_t < n$ as did in (i) above and hence the arguments used there apply. $\square$

We have now pinned down the candidates for maximizers to only at most one $\widetilde{\mathcal{N}}$-type candidate (as the line $k = \kappa$ can cut at most one of the graphs $k = \widetilde{k}_n(t)$) and some others that are all of an $\widetilde{\mathcal{E}}$-type. Let us turn our attention to the $\widetilde{\mathcal{E}}$-type candidates at the moment. To seek for the winning $\widetilde{\mathcal{E}}$-type contestants for a given total curvature κ, we see that the condition $\widetilde{K}_n(t) = \kappa$ gives n as a function of t, from which $\widetilde{L}_n(t)$ may be expressed as a function L of t only, with

$$L'(t) = \frac{1}{2t}\widetilde{K}'_n(t)(tr'(t) - r(t)).$$

Thus the critical points of L are exactly the zeros of the function $M(t) = tr'(t) - r(t)$. Elementary calculus shows that M has only one zero $\tau_\circ \in (\tau_2, \tau_\infty)$, and is positive on the left side and negative on the right side of $\tau_\circ$. Clearly, if $n(\tau_\circ)$ is an integer, the finalist will then consist of this $\widetilde{\mathcal{E}}$-type $n(\tau_\circ)$-segment and possibly the $\widetilde{\mathcal{N}}$-type candidate mentioned above. Otherwise, the finalist consists of at most three candidates, namely the $\widetilde{\mathcal{E}}$-type $\lfloor n(\tau_\circ) \rfloor$-segment, the $\widetilde{\mathcal{E}}$-type $(\lfloor n(\tau_\circ) \rfloor + 1)$-segment and the $\widetilde{\mathcal{N}}$-type candidate mentioned above. Nevertheless, identifying the winner(s) among the finalist requires further investigation.

Proposition 3.11. *Let $n > 2$ be an integer. Then upto isometry of the space and body segment rearrangement:*

(i) If $n < \widehat{n}$ then there are a unique $\widetilde{\mathcal{N}}$-type $(n+2)$-segment η and a unique $\widetilde{\mathcal{E}}$-type $(n+2)$-segment σ having the same total curvature and the same length. Moreover, the major body ending angles of η equals $\widetilde{t}_n(\kappa)$, where κ is the common total curvature.

(ii) If $n \geq \widehat{n}$ then there are a unique $\widetilde{\mathcal{E}}$-type $(n+1)$-segment σ and a unique $\widetilde{\mathcal{E}}$-type $(n+2)$-segment σ' having the same total curvature and the same length. Moreover, the body ending angles t of σ and t' of σ' satisfy $t' < \tau_\circ < t$.

Proof. Let $n > 2$ be an integer. To prove (i) for $n < \widehat{n}$, let t be the unique ending angle value such that $\nu_t = n$. Choose σ to be the $\widetilde{\mathcal{E}}$-type $(n+2)$-segment with body ending angle $\check{t}$ and choose η to be the $\widetilde{\mathcal{N}}$-type $(n+2)$-segment with major body ending angle t. The first assertion follows. By lemma 3.9, $t > \widetilde{\tau}_n$ and hence must equal $\widetilde{t}_n(\kappa)$. Uniqueness is easily verified.

To prove (ii), let us first sketch a proof of a result analogous to Lemma 3.9 that given $t \in (\tau_\circ, \tau_\infty]$ there exist a unique real number $\overline{\nu}_t \geq \widehat{n}$ and a unique ending angle $\overline{t} \in [\tau_2, t)$ such that $\widetilde{K}_{\overline{\nu}_t - 1}(t) = \widetilde{K}_{\overline{\nu}_t}(\overline{t})$ and $\widetilde{L}_{\overline{\nu}_t - 1}(t) = \widetilde{L}_{\overline{\nu}_t}(\overline{t})$: Fix $t \in (\tau_\circ, \tau_\infty]$ and define $\overline{n}_c$ and $\overline{n}_l$ on $[\tau_2, t)$ by the equations $\widetilde{K}_{\overline{n}_c - 1}(t) = \widetilde{K}_{\overline{n}_c}(s)$ and $\widetilde{L}_{\overline{n}_l - 1}(t) = \widetilde{L}_{\overline{n}_l}(s)$. Then $\overline{n}_c$ and $\overline{n}_l$, as functions of s, are positive, strictly increasing and approach ∞ as s approaches t. Letting $\overline{\nu}_t = \overline{n}_l(\overline{t}) = \overline{n}_c(\overline{t})$, where $\overline{t}$ is the only zero of the difference $\overline{\Delta} = \overline{n}_l - \overline{n}_c$, the result follows. Moreover, $\overline{\Delta}$ is always negative if $t \leq \tau_\circ$ while it is negative on the left side and positive on the right side of $\overline{t}$ if $t > \tau_\circ$. $\overline{t}$ and $\overline{\nu}_t$ are proved to be strictly decreasing with t, with the former approaching $\tau_\circ$ and the latter approaching ∞ as t approaches $\tau_\circ$, so we define $\overline{\tau}_\circ = \tau_\circ$ so that $\overline{t}$ is defined and continuous at all $t \in [\tau_\circ, \tau_\infty]$. Note also that $\overline{\nu}_{\tau_\infty} = \widehat{n}$.

Now suppose $n \geq \widehat{n}$. Then there is a unique $t \in (\tau_\circ, \tau_\infty]$ such that $\overline{\nu}_t = n$. Choose σ to be the $\widetilde{\mathcal{E}}$-type $(n+1)$-segment with body ending angles t and choose σ' to be the $\widetilde{\mathcal{E}}$-type $(n+2)$-segment with body ending angles t', where $t' = \overline{t} < \tau_\circ$. This proves existence. Uniqueness is verified in the same manner as is done in (i). ∎

In view of Proposition 3.11, for each integer $n > 2$ we let κ_n be the common total curvature asserted by (i) if $n < \widehat{n}$ and that asserted by (ii) if $n \geq \widehat{n}$. In addition, by letting $\kappa_n = \widetilde{K}_n(\tau_2)$ for $n = 1$ and 2 we now have an unbounded strictly increasing sequence κ_n, the importance of which is clearly shown in the main theorem that will follow shortly. In order to present our main result in a systematic pattern, let us also define $\overline{\kappa}_0 = 2\widehat{\tau}_\infty$ (this conforms with the definition of $\overline{\kappa}_0$ in Table 1 for $R \le R_0$), $\overline{\kappa}_n = \widetilde{K}_n(\tau_\infty)$ for each positive integer $n < \widehat{n} - 1$ and $\overline{\kappa}_n = \kappa_{n+1}$ for $n \ge \widehat{n} - 1$. Figure 4 locates κ_n's and $\overline{\kappa}_n$'s for small values of n. It is easy to see that for each $\kappa > 2\widehat{\tau}_\infty$ there exists a unique integer n such that one and only one of the following statements holds:

(i) $\overline{\kappa}_{n-1} < \kappa < \kappa_n$.

(ii) $\kappa = \kappa_n$.

(iii) $\kappa_n < \kappa < \overline{\kappa}_n$.

(iv) $\kappa = \overline{\kappa}_{n-1} \ne \kappa_n$ (equivalently, $\kappa = \overline{\kappa}_{n-1}$ and $n < \widehat{n}$).

The unique integer n in association with a given κ as asserted above is formally denoted by n_κ. Lastly, we define $\widetilde{S}(R,\kappa)$ for $R > R_0$ and $\kappa \ge 0$, which will serve as the upper bound of the length of the curves of our concerns, by

$$\widetilde{S}(R,\kappa) = \begin{cases} \ell(\Lambda_{R,\kappa}) & \text{if } 0 \le \kappa \le \overline{\kappa}_0, \\ \ell(\mathit{\Pi}^{n_\kappa}_{R,\kappa}) & \text{if } \kappa > \overline{\kappa}_0 \text{ and } \overline{\kappa}_{n_\kappa - 1} < \kappa < \kappa_{n_\kappa}, \\ \ell(\Pi^{n_\kappa}_{R,\kappa}) & \text{otherwise,} \end{cases}$$

where the symbol $\Lambda_{R,\kappa}$ is as mentioned before in Section 2, while $\Pi^n_{R,\kappa}$ and $\mathit{\Pi}^n_{R,\kappa}$ refer respectively to any $\widetilde{\mathcal{E}}$- and $\widetilde{\mathcal{N}}$-type $(n+2)$-segment with major body ending angles respectively $\underset{\sim}{t}_n(\kappa)$ and $\widetilde{t}_n(\kappa)$ in a disk of radius R. Note that these polysegments necessarily have total curvature κ.

Lemma 3.12. *Let n be an integer and $\kappa > 2\widehat{\tau}_\infty$. Then the following hold, given that the involving polysegments exist:*

(i) If $n < \widehat{n}$ then $\ell(\Pi^n_{R,\kappa})$ is less than, equal to or greater than $\ell(\mathit{\Pi}^n_{R,\kappa})$ accordingly as κ is less than, equal to or greater than κ_n.

(ii) If $n > 2$ then $\ell(\Pi^n_{R,\kappa})$ is less than, equal to or greater than $\ell(\Pi^{n-1}_{R,\kappa})$ accordingly as κ is less than, equal to or greater than κ_n.

Proof. For (i), suppose the two polysegments exist with $n < \widehat{n}$. Putting $s = \underset{\sim}{t}_n(\kappa)$, $t = \widetilde{t}_n(\kappa)$ and $t' = \widetilde{t}_n(\kappa_n)$, we have $n = \nu_{t'}$. Then $\kappa < \kappa_n$ implies $t' < t$ and hence $s < \check{t} < \check{t}'$. It folllows, with the same use of notations as in the proof of Lemma 3.9, that $\Delta(s) < 0$, i.e., $n_l(s) < n_c(s) = n$. Thus, $\ell(\Pi^n_{R,\kappa}) = \widetilde{L}_{n_l(s)}(s) = \widetilde{l}_{n_l(s)}(t) < \widetilde{l}_n(t) = \ell(\mathit{\Pi}^n_{R,\kappa})$. If $\kappa > \kappa_n$ then all the inequalities reverse. These together with the trivial case $\kappa = \kappa_n$ amount to (i).

To prove (ii) we suppose again that the two involving polysegments exist but with $n > 2$. We now let $s = \underset{\sim}{t}_n(\kappa)$, $t = \underset{\sim}{t}_{n-1}(\kappa)$ and $t' = \underset{\sim}{t}_{n-1}(\kappa_n)$. With the same notations as those used in the proof of Proposition 3.11 (ii), we then have $n = \overline{\nu}_{t'}$. Thus, similar to the above, $\kappa < \kappa_n$ implies $t < t'$. If $t \le \tau_\circ$ then $\overline{\Delta}(s) < 0$. If $t > \tau_\circ$ then $s < \overline{t'} < \overline{t}$ and hence $\overline{\Delta}(s) < 0$ anyway. An analysis analogous to that in the proof of (i) above yields the required result. □

In conclusion, we contend that our discussion in this section can be generalized, with the usual scaling factor $\lambda = \sqrt{-K}$, to all negative values of spatial curvature K. The following theorem states a general result for arbitrary $K < 0$, with the same set of notations as used above.

Theorem 3.13. *(Length estimate for curves in large circumballs.) Let γ be a curve contained in a closed ball of radius $R < \frac{\pi}{2\sqrt{K}}$ in an R_K space, where $K < 0$, with total curvature κ. If $R > R_0$ then $\ell(\gamma) \le \widetilde{S}(R,\kappa)$. Moreover, the upper bound $\widetilde{S}(R,\kappa)$ is achieved by $\Lambda_{R,\kappa}$ in a closed disk of radius R in S_K if $\kappa \le \overline{\kappa}_0$ and, with $n = n_\kappa$, by the following polysegment(s) in S_K if $\kappa > \overline{\kappa}_0$:*

(i)	$\mathit{\Pi}^n_{R,\kappa}$	*if* $\overline{\kappa}_{n-1} < \kappa < \kappa_n$,
(ii)	$\Pi^n_{R,\kappa}$	*if* $\kappa = \kappa_n$ *and* $n = 1$ *or* 2,
(iii)	$\mathit{\Pi}^n_{R,\kappa}$ *and* $\Pi^n_{R,\kappa}$	*if* $\kappa = \kappa_n$ *and* $2 < n < \widehat{n}$,
(iv)	$\Pi^n_{R,\kappa}$ *and* $\Pi^{n-1}_{R,\kappa}$	*if* $\kappa = \kappa_n$ *and* $n \ge \widehat{n}$,
(v)	$\Pi^n_{R,\kappa}$	*if* $\kappa_n < \kappa < \overline{\kappa}_n$,
(vi)	$\Pi^n_{R,\kappa}$	*if* $\kappa = \overline{\kappa}_n$ *and* $n < \widehat{n} - 1$.

Proof. The arguments in the last paragraph of the proof of Theorem 2.4 also apply here. Thus, it suffices to verify the assertions for bisegments and D_R-admissible polysegments in S_K. We carry through that by pinpointing maximizers for each value of κ.

For $\kappa \le \overline{\kappa}_0 = 2\widehat{\tau}_\infty$, Propositions 3.4, 3.5, 3.6 and 3.7 give the supremum of curvelength in $\widetilde{\mathcal{P}}(R,\kappa)$ and comparison of it with the length of the isosceles bisegment $\Lambda_{R,\kappa}$, which is proved to be the longest among bisegments in D_R, affirms that $\ell(\Lambda_{R,\kappa})$ is the overall maximum.

Now suppose $\kappa > \overline{\kappa}_0$. We then note that the six listed alternatives are exhaustive. It is easy to check that in this case if γ is a bisegment in D_R then γ is overruled by a D_R-admissible bisegment and hence maximizers must be those of $\widetilde{\mathcal{P}}(R,\kappa)$ as asserted by Theorem 3.8. Earlier discussions conclude that maximizers of $\widetilde{\mathcal{P}}(R,\kappa)$ correspond to the points where the line $k = \kappa$ cuts the graphs $k = \widetilde{k}_n(t)$ and the graphs $k = \widetilde{K}_n(t)$. (See the paragraph preceding Lemma 3.9.) With reference to the properties of $\widetilde{K}_n$'s and $\widetilde{k}_n$'s depicted in Figure 4, the result is readily verified for $\kappa \le \kappa_2$. We therefore assume for the rest of the proof that $\kappa > \kappa_2$. For any positive integer m, let us denote by η_m

the $\widetilde{\mathcal{N}}$-type maximizing candidate $\mathit{\Pi}^m_{R,\kappa}$ when it exists, and similarly by σ_m the $\widetilde{\mathcal{E}}$-type candidate $\Pi^m_{R,\kappa}$.

Let $n = n_\kappa$. For (i), suppose that $\overline{\kappa}_{n-1} < \kappa < \kappa_n$. Then necessarily, $2 < n < \widehat{n}$ and hence η_n exists. If no $\widetilde{\mathcal{E}}$-type candidates exist, we are done. Suppose otherwise. Since $\kappa < \kappa_m$ for all $m \geq n$, we have $\ell(\sigma_n) < \ell(\eta_n)$ by Lemma 3.12 (i) and $\ell(\sigma_m) < \ell(\sigma_n)$ for all other existing σ_m by Lemma 3.12 (ii), rendering η_n as the only maximizer as well. As for (iii), let us assume that $\kappa = \kappa_n$ and $2 < n < \widehat{n}$. Then we have at least η_n and σ_n as candidates, both of the same length. If σ_m is another candidate then $m > n$ and again $\kappa < \kappa_m$ and hence $\ell(\sigma_m) < \ell(\sigma_n)$ by Lemma 3.12 (ii), so that the result follows. The other cases can be carried out in a similar manner. $\square$

Let us make a final note here that the relation between $\widehat{n}$ and the integer N in Theorem 1.1(ii) is indeed $N = \lceil \widehat{n} \rceil$.

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