East-West J. of Mathematics: Vol. 19, No 2 (2017) pp. 156-170

CLONE OF LINEAR TERMS AND CLONE OF LINEAR FORMULAS

Thodsaporn Kumduang and **Sorasak Leeratanavalee**

Dept. of Mathematics, Faculty of Science,
Chiang Mai University
Chiang Mai 50200, Thailand
e-mail: kumduang01@gmail.com, sorasak.l@cmu.ac.th

Abstract

Terms and formulas are expressed in a first-order language which are used to describe properties of algebraic systems consisting a non-empty set together with a sequence of operations and a sequence of relations on this set. In this paper we study on linear terms of type (n) for a natural number $n > 1$, this leads to define the definition of linear formulas of type $((n),(m))$ for natural numbers $n, m > 1$. To construct clone of linear terms and clone of linear formulas we give a new concept of the partial superposition operation of linear terms and the partial superposition operation of linear formulas, respectively. Moreover, we show that both of them are satisfied the superassociative law and the extension of a many-sorted mapping, which maps a generating system to clone, is an endomorphism.

1 Introduction

The concept of clone is one of the principal algebraic concepts especially a clone can be regarded to category theory. In 1963, Bill Lawvere introduced the concept of algebraic theory what is nowadays known as a Lawvere theory, which can be thought of as a category theoretical abstraction of clones (see [12]

Key words: algebraic systems, linear terms, linear formulas, clone of linear terms, clones of linear formulas.
2010 AMS Mathematics classification: 03B50, 08A05, 08A35.

for more details). Nowadays, the clone is widely accepted in this area. Some basic examples are :

(i) The set $O(A)$ of all (finitary) operations on A together with the usual composition and projections on A, called the clone of operations.

(ii) Given a topological space (X, τ), all continuous operations on X form a clone, called the clone of (X, τ). See [14] for a recent information in this area.

(iii) Given a partially ordered set $(A, \leq)$, all operations on A monotone in each variable with respect to $\leq$ form a clone, called the clone of partial order.

One of direct research in clone theory is the clone of terms which plays important role in universal algebra and computer science. To define terms one needs variables and operation symbols. Let $(f_i)_{i\in I}$ be a sequence of operation symbols, when f_i is n_i-ary and $n_i \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ is a natural number. We denote by $X := \{x_1, \ldots, x_n, \ldots\}$ is a countably infinite set of symbols called variables and for each $n \geq 1$ let $X_n := \{x_1, \ldots, x_n\}$. The sequence of $\tau := (n_i)_{i\in I}$ is called a type. Then *an n-ary term of type τ* is defined inductively as follows:

(i) Every variable $x_j \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii) and let $W_\tau(X) := \bigcup_{n\in\mathbb{N}^+} W_\tau(X_n)$ be the set of all terms of type τ.

Now, we recall the concept of superposition operation. For each $m, n \in \mathbb{N}^+$ the superposition operation $S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$ is a many-sorted mapping defined by

(i) $S^n_m(x_j, t_1, \ldots, t_{n_i}) := t_j$, if $x_j, 1 \leq j \leq n$ is a variable from X_n.

(ii) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_{n_i}) := f_i(S^n_m(s_1, t_1, \ldots, t_{n_i}), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_{n_i}))$.

Then the many-sorted algebra can be defined by

$$\text{clone}\tau = ((W_\tau(X_n))_{n\in\mathbb{N}^+}, (S^n_m)_{n,m\in\mathbb{N}^+}, (x_i)_{i\leq n\in\mathbb{N}^+}),$$

which is called *the clone of all terms of type* τ.

Here, we would like to extend all above concepts to algebraic system and thus we now recall some basic definitions. Let I, J be indexed sets and let $(f_i)_{i\in I}$, $(\gamma_j)_{j\in J}$ be sequences of operation symbols and relation symbols, respectively. Let $\tau := (n_i)_{i\in I}$ and $\tau' := (n_j)_{j\in J}$ where f_i has the arity n_i for every $i \in I$ and γ_j has the arity n_j for every $j \in J$. The pair (τ, τ') is called *the type* of an algebraic system.

Definition 1.1. ([13]) *An algebraic system of type* (τ, τ') is a triple $\mathcal{A} := (A; (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$ consisting of a non-empty set A, a sequence $(f_i^{\mathcal{A}})_{i\in I}$ of operations on A where $f_i^{\mathcal{A}}$ is n_i-ary for $i \in I$ and a sequence $(\gamma_j^{\mathcal{A}})_{j\in J}$ of relations on A where $\gamma_j^{\mathcal{A}}$ is n_j-ary for $j \in J$.

Not all of the terms in the second-order language will used to express properties of algebraic system. The one is called formulas, first introduced by A.I. Mal'cev in 1973 [13]. For approach to formulas see also [13], and we recall the definition of formula which is defined by K. Denecke and D. Phusanga in 2008.

Definition 1.2. ([6]) Let $n \in \mathbb{N}^+$. *An n-ary quantifier free formula of type* (τ, τ') *(for simply, formula)* is defined in the following way:

(i) If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary quantifier free formula of type (τ, τ').

(ii) If $j \in J$ and $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary quantifier free formula of type (τ, τ').

(iii) If F is an n-ary quantifier free formula of type (τ, τ'), then $\neg F$ is an n-ary quantifier free formula of type (τ, τ').

(iv) If F_1 and F_2 are n-ary quantifier free formulas of type (τ, τ'), then $F_1 \vee F_2$ is an n-ary quantifier free formula of type (τ, τ').

Let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all n-ary quantifier free formulas of type (τ, τ') and let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) := \bigcup_{n\in\mathbb{N}^+} \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all quantifier free formulas of type (τ, τ').

In 2012, M. Couceiro and E. Lehtonen [3] introduced the new concept of a term in which each variable occurs at most once which called a linear term (see also [2]). A linear term is a generalization of a linear expression over a vector space (see e.g.[5]). For a definition of n-ary linear terms, they replace (ii) in the definition of terms by slightly different condition.

Definition 1.3. ([3]) *An n-ary linear term of type τ* is defined in the following inductive way:

(i) Every $x_i \in X_n$ is an n-ary linear term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary linear terms of type τ with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n_i$ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary linear term of type τ.

Let $W_\tau^{lin}(X_n)$ be the set of all n-ary linear terms of type τ and let $W_\tau^{lin}(X) := \bigcup_{n\in\mathbb{N}} W_\tau^{lin}(X_n)$ be the set of all linear terms of type τ.

Motivated and inspired by the result mentioned above, we restrictly interested in linear terms of type (n) for a natural number $n > 1$, i.e., we have only one n-ary operation symbol. In this paper we consider an algebraic system of type $((n),(m))$, i.e., we have only one n-ary operation symbol and m-ary relation symbol, say f and γ, respectively. We define the new definition of linear formulas of type $((n),(m))$ for natural numbers $n, m > 1$. Our aim, is to construct the many-sorted algebra in the same situation of $clone\tau$, we define the partial many-sorted superposition $S^{lin}\ {}^{p}_{q}$ and form the sequence of set of p-ary linear terms of type (n) for all $p \in \mathbb{N}^+$ together with this operation and projections. Furthermore, the superposition of linear formulas $R^{lin}\ {}^{p}_{q}$ is defined and we construct the clone of linear formulas. To describe some properties of these clones, the theorem of superassociative law and freely generated by a generating system are investigated.

2 Linear Terms of Type (n) and Quantifier Free Linear Formulas of Type $((n),(m))$

Let $var(t)$ be the set of all variables occurring in the term t and let $var(F)$ be the set of all variables occurring in the formula F.

In this section, we first defined the definition of a linear term and a quantifier free linear formula of type $((n),(m))$ as follows :

Definition 2.1. Let $n, p \in \mathbb{N}^+$ with $p \geq n$. *A p-ary linear term of type* (n) is defined in the following inductive way:

(i) Every $x_i \in X_p$ is a p-ary linear term of type (n).

(ii) If $t_1, \ldots, t_n$ are p-ary linear terms of type (n) with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n$ and f is an n-ary operation symbol, then $f(t_1, \ldots, t_n)$ is a p-ary linear term of type (n).

Let $W_{(n)}^{lin}(X_p)$ be the set of all p-ary linear terms of type (n) and let $W_{(n)}^{lin}(X) := \bigcup_{p\in\mathbb{N}^+} W_{(n)}^{lin}(X_p)$ be the set of all linear terms of type (n).

Remark 2.2. Indeed, if we assume that $f(t_1,\ldots,t_n) \in W_{(n)}^{lin}(X_p)$ and $p < n$, then some variables in X_p must occur more than one time in $f(t_1,\ldots,t_n)$, which is impossible. Hence, we also set this condition into Definition 2.1.

Example 2.3. Let $(n) = (2)$ be the type with a binary operation symbol f and $X_2 = \{x_1, x_2\}$. Then $x_1, x_2, f(x_1, x_2), f(x_2, x_1)$ are examples of binary linear terms of type (2).

Let $X_4 = \{x_1, x_2, x_3, x_4\}$. Then $x_1, x_2, x_3, x_4, f(x_1, x_2), f(x_2, x_1)$, $f(x_4, f(x_3, x_2)), f(f(x_1, x_2), f(x_4, x_3))$ are examples of quaternary linear terms of type (2). The example shows that every p-ary linear term of type (n) is a $p\prime$-ary linear term of type (n) for $p\prime \geq p$.

Definition 2.4. Let $m, n, p \in \mathbb{N}^+$ with $p \geq m$. *A p-ary quantifier free linear formula of type* $((n),(m))$ (for simply, linear formula) is defined as follows :

(i) If s, t are p-ary linear terms of type (n) and $var(s) \cap var(t) = \emptyset$, then the equation $s \approx t$ is a p-ary quantifier free linear formula of type $((n),(m))$.

(ii) If $t_1,\ldots,t_m$ are p-ary linear terms of type (n) with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq m$ and γ is an m-ary relation symbol, then $\gamma(t_1,\ldots,t_m)$ is a p-ary quantifier free linear formula of type $((n),(m))$.

(iii) If F is a p-ary quantifier free linear formula of type $((n),(m))$, then $\neg F$ is a p-ary quantifier free linear formula of type $((n),(m))$.

(iv) If F_1 and F_2 are p-ary quantifier free linear formulas of type $((n),(m))$ and $var(F_1) \cap var(F_2) = \emptyset$, then $F_1 \vee F_2$ is a p-ary quantifier free linear formula of type $((n),(m))$.

Let $\mathcal{F}_{((n),(m))}^{lin}(W_{(n)}^{lin}(X_p))$ be the set of all p-ary quantifier free linear formulas of type $((n),(m))$ and let $\mathcal{F}_{((n),(m))}^{lin}(W_{(n)}^{lin}(X)) := \bigcup_{p\in\mathbb{N}^+} \mathcal{F}_{((n),(m))}^{lin}(W_{(n)}^{lin}(X_p))$ be the set of all quantifier free linear formulas of type $((n),(m))$.

Remark 2.5. The linear formulas defined by (i) and (ii) are called atomic linear formulas.

Example 2.6. Let $((n),(m)) = ((2),(2))$ be a type, i.e., we have one binary operation symbol f and one binary relation symbol γ and let $X_2 = \{x_1, x_2\}$. Then the binary atomic linear formulas of type $((2),(2))$ are $x_1 \approx x_2, x_2 \approx x_1, \gamma(x_1, x_2), \gamma(x_2, x_1)$. Moreover, we obtained all other linear formulas of type $((2),(2))$ from binary atomic linear formulas of type $((2),(2))$ by using the connectives $\neg$ and $\vee$.

3 Superposition of Linear Terms of type (n) and Clone of Linear Terms

In this section, we give the concept of superposition of linear term of type (n), this leads to form the many-sorted algebra which is called the clone of linear terms.

Definition 3.1. Let $p, q \in \mathbb{N}^+$ with $p \leq q$, $t \in W_{(n)}^{lin}(X_p)$ and $s_1, \ldots, s_p \in W_{(n)}^{lin}(X_q)$ with $var(s_l) \cap var(s_k) = \emptyset$ for all $1 \leq l < k \leq p$. Then we define a superposition operation of linear terms

$$S^{lin}\,{}_q^p : W_{(n)}^{lin}(X_p) \times (W_{(n)}^{lin}(X_q))^p \multimap\!\!\rightarrow W_{(n)}^{lin}(X_q)$$

inductively by the following steps :

(i) If $t = x_i$ for $1 \leq i \leq p$, then $S^{lin}\,{}_q^p(x_i, s_1, \ldots, s_p) := s_i$.

(ii) If $t = f(t_1, \ldots, t_p)$, then
$S^{lin}\,{}_q^p(f(t_1, \ldots, t_p), s_1, \ldots, s_p) :=$
$f(S^{lin}\,{}_q^p(t_1, s_1, \ldots, s_p), \ldots, S^{lin}\,{}_q^p(t_p, s_1, \ldots, s_p))$.

Example 3.2. Let $\tau = (3)$ be a type, i.e., we have only one ternary operation symbol, say f. If we consider the superposition

$$S^{lin}\,{}_7^5 : W_{(3)}^{lin}(X_5) \times (W_{(3)}^{lin}(X_7))^5 \multimap\!\!\rightarrow W_{(3)}^{lin}(X_7).$$

Then we have

(1) $S^{lin}\,{}_7^5(x_4, x_1, x_6, x_7, f(x_3, x_5, x_2), x_4) = f(x_3, x_5, x_2)$.

(2)
$$\begin{aligned} & S^{lin}\,{}_7^5(f(x_2, x_4, f(x_5, x_3, x_1)), x_3, f(x_7, x_5, x_6), x_1, x_4, x_2) \\ = \quad & f(S^{lin}\,{}_7^5(x_2, x_3, f(x_7, x_5, x_6), x_1, x_4, x_2), \\ & S^{lin}\,{}_7^5(x_4, x_3, f(x_7, x_5, x_6), x_1, x_4, x_2), \\ & S^{lin}\,{}_7^5(f(x_5, x_3, x_1), x_3, f(x_7, x_5, x_6), x_1, x_4, x_2)) \\ = \quad & f(f(x_7, x_5, x_6), x_4, f(x_2, x_1, x_3)). \end{aligned}$$

Remark 3.3. 1. The condition $s_1, \ldots, s_p \in W_{(n)}^{lin}(X_q)$ with $var(s_l) \cap var(s_k) = \emptyset$ for all $1 \leq l < k \leq p$ in Definition 3.1 is necessary. Otherwise, the resulting term is not linear, as an example we let $(n) = (2)$ with a binary operation symbol f. Then $S^{lin}\,{}_2^2(f(x_2, x_1), x_2, f(x_1, x_2)) = f(f(x_1, x_2), x_2)$ is not a linear term, although x_2 and $f(x_1, x_2)$ are linear. Therefore, we must put this necessary condition.
2. According to the condition in Definition 3.1, we must set $p \leq q$. Because if $p > q$ and $s_1, \ldots, s_p \in W_{(n)}^{lin}(X_q)$, this means that there exist variables occuring more than once which is a contradiction.

On the set $W_{(n)}^{lin}(X_p)$ of all p-ary linear terms of type (n), we establish the many-sorted algebra of type $(p+1,\ldots,0,\ldots,0)$, by using the $(p+1)$-ary superposition operation $S^{lin}\,{}^p_q$ as we already defined in Definition 3.1 and adding the variables $x_1,\ldots,x_p$ as nullary operations, say projections. Then we obtain the many-sorted algebra

$$LinClone(n) := ((W_{(n)}^{lin}(X_p))_{p\in\mathbb{N}^+}; (S^{lin}\,{}^p_q)_{p\leq q,p,q\in\mathbb{N}^+}, (x_i)_{i\leq p,i\in\mathbb{N}^+}),$$

which is called *the clone of linear terms of type* (n).

Next, some properties of $LinClone(n)$ will be presented.

Theorem 3.4. *The many sorted algebra LinClone(n) satisfies the following equations :*

(LC1) $S^{lin}\,{}^p_q(S^{lin}\,{}^r_p(t,t_1,\ldots,t_r),s_1,\ldots,s_p)$
$= S^{lin}\,{}^r_q(t, S^{lin}\,{}^p_q(t_1,s_1,\ldots,s_p),\ldots,S^{lin}\,{}^p_q(t_r,s_1,\ldots,s_p)),$

(LC2) $S^{lin}\,{}^p_q(x_i,t_1,\ldots,t_p) = t_i$ *for* $1\leq i\leq p,$

(LC3) $S^{lin}\,{}^p_p(t,x_1,\ldots,x_p) = t,$

where $p,q,r\in\mathbb{N}^+$ *with* $r\leq p\leq q$, $t\in W_{(n)}^{lin}(X_r)$, $t_1,\ldots,t_r\in W_{(n)}^{lin}(X_p)$, $var(t_l)\cap var(t_k)=\emptyset$ *for all* $1\leq l<k\leq r$ *and* $s_1,\ldots,s_p\in W_{(n)}^{lin}(X_q)$, $var(s_l)\cap var(s_k)=\emptyset$ *for all* $1\leq l<k\leq p$.

Proof. (LC1) We give a proof by induction on the complexity of a linear term t.

If $t=x_i$ for all $1\leq i\leq r$, then

$$\begin{aligned}
&S^{lin}\,{}^p_q(S^{lin}\,{}^r_p(x_i,t_1,\ldots,t_r),s_1,\ldots,s_p)\\
&\quad= S^{lin}\,{}^p_q(t_i,s_1,\ldots,s_p)\\
&\quad= S^{lin}\,{}^r_q(x_i,S^{lin}\,{}^p_q(t_1,s_1,\ldots,s_p),\ldots,S^{lin}\,{}^p_q(t_r,s_1,\ldots,s_p)).
\end{aligned}$$

If $t=f(u_1,\ldots,u_r)$ and assume that $(LC1)$ satisfied for $u_1,\ldots,u_r$. Then

$$\begin{aligned}
&S^{lin}\,{}^p_q(S^{lin}\,{}^r_p(f(u_1,\ldots,u_r),t_1,\ldots,t_r),s_1,\ldots,s_p)\\
&\quad= S^{lin}\,{}^p_q(f(S^{lin}\,{}^r_p(u_1,t_1,\ldots,t_r),\ldots,S^{lin}\,{}^r_p(u_r,t_1,\ldots,t_r)),s_1,\ldots,s_p)\\
&\quad= f(S^{lin}\,{}^p_q(S^{lin}\,{}^r_p(u_1,t_1,\ldots,t_p),s_1,\ldots,s_p),\ldots,\\
&\qquad\quad S^{lin}\,{}^p_q(S^{lin}\,{}^r_p(u_r,t_1,\ldots,t_r),s_1,\ldots,s_p))\\
&\quad= f(S^{lin}\,{}^r_q(u_1,S^{lin}\,{}^p_q(t_1,s_1,\ldots,s_p),\ldots,S^{lin}\,{}^p_q(t_r,s_1,\ldots,s_p)),\ldots,\\
&\qquad\quad S^{lin}\,{}^r_q(u_r,S^{lin}\,{}^p_q(t_1,s_1,\ldots,s_n),\ldots,S^{lin}\,{}^p_q(t_r,s_1,\ldots,s_p)))\\
&\quad= S^{lin}\,{}^r_q(f(u_1,\ldots,u_r),S^{lin}\,{}^p_q(t_1,s_1,\ldots,s_p),\ldots,S^{lin}\,{}^p_q(t_r,s_1,\ldots,s_p)).
\end{aligned}$$

(LC2) is satisfied by the definition of $S^{lin}\,{}^p_q$.

(LC3) We give a proof by induction on the complexity of a linear term t. If $t=x_i$; $1\leq i\leq p$, then $S^{lin}\,{}^p_p(x_i,x_1,\ldots,x_p)=x_i$. If $t=f(t_1,\ldots,t_p)$ and we

inductively assume that $S^{lin}\,{}^{p}_{p}(t_i, x_1, \ldots, x_p) = t_i$; $1 \leq i \leq p$, then
$S^{lin}\,{}^{p}_{p}(f(t_1, \ldots, t_p), x_1, \ldots, x_p)$

$$\begin{aligned} &= f(S^{lin}\,{}^{p}_{p}(t_1, x_1, \ldots, x_p), \ldots, S^{lin}\,{}^{p}_{p}(t_p, x_1, \ldots, x_p)) \\ &= f(t_1, \ldots, t_p). \end{aligned}$$ □

An algebra is said to be *free with respect to itself* if it has a generating system and each mapping from the generating system into the universe of the algebra can be extended to an endomorphism.

We show that the many-sorted algebra $LinClone(n)$ involves this property. To get our result, let $F_p = \{f(x_1, \ldots, x_p) \mid p \in \mathbb{N}^+\}$ of so-called "the set of fundamental terms" and we need the following lemmas.

Lemma 3.5. *The many sorted algebra $LinClone(n)$ is generated by $(F_p)_{p\in\mathbb{N}^+}$.*

Proof. Let $\tau = (n)$ be a fixed type with a natural number $n > 1$. To show this, we prove that $(W^{lin}_{(n)}(X_p))_{p\in\mathbb{N}^+}$ is generated by $(F_p)_{p\in\mathbb{N}^+}$ by induction on the complexity of a linear term t. Let $p \in \mathbb{N}^+$ and $t = x_i \in X_p$. Since every variable is the projection containing in the type of $LinClone(n)$, we may consider a variable x_i is an operation symbol and so $f(x_1, \ldots, x_p) = x_i(x_1, \ldots, x_p) = x_i$ for all $i = 1, \ldots, p$. Next, let $q \in \mathbb{N}^+$ where $q \geq n$, and assume that $t_1, \ldots, t_n \in W^{lin}_{(n)}(X_q)$ are generated and $t = f(t_1, \ldots, t_n)$. By assumption, we get $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n$ and thus $S^{lin}\,{}^{n}_{q}(f(x_1, \ldots, x_n), t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$. This shows that $f(t_1, \ldots, t_n)$ is generated. Therefore $(F_p)_{p\in\mathbb{N}^+}$ is a generating system of $LinClone(n)$. □

Let $(\varphi_p)_{p\in\mathbb{N}^+} : (F_p)_{p\in\mathbb{N}^+} \to (W^{lin}_{(n)}(X_p))_{p\in\mathbb{N}^+}$. The next aim, is to prove that this mapping can be extended to an endomorphism of $LinClone(n)$. To show this, we define $(\overline{\varphi}_p)_{p\in\mathbb{N}^+} : (W^{lin}_{(n)}(X_p))_{p\in\mathbb{N}^+} \to (W^{lin}_{(n)}(X_p))_{p\in\mathbb{N}^+}$ as follows:

(1) $\overline{\varphi}_p(x_i) := x_i$ for $1 \leq i \leq p$.

(2) $\overline{\varphi}_q(f(t_1, \ldots, t_n)) := S^{lin}\,{}^{n}_{q}(\varphi_n(f(x_1, \ldots, x_n)), \overline{\varphi}_q(t_1), \ldots, \overline{\varphi}_q(t_n))$.

Lemma 3.6. *For each many-sorted mapping φ_p with $p \in \mathbb{N}^+$ and any linear term t, we have $var(\overline{\varphi}_p(t)) \subseteq var(t)$.*

Proof. We give a proof on the complexity of a linear term t. If $t = x_i$ is a variable, then $var(\overline{\varphi}_p(x_i)) = var(x_i) = \{x_i\}$. If $t = f(t_1, \ldots, t_n)$ and we inductively assume that $var(\overline{\varphi}_p(t_i)) \subseteq var(t_i)$ for $i = 1, \ldots, n$. Then
$var(\overline{\varphi}_p(f(t_1, \ldots, t_n)))$

$$\begin{array}{ll} = & var(S^{lin}\ {}^{n}_{q}(\varphi_n(f(x_1,\ldots,x_n)),\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_n))) \\ \subseteq & \bigcup\limits_{i=1}^{n} var(\overline{\varphi}_q(t_i)) \\ \subseteq & \bigcup\limits_{i=1}^{n} var(t_i) \\ = & var(f(t_1,\ldots,t_n)). \end{array}$$

□

Lemma 3.7. *For any linear term $t = f(t_1,\ldots,t_n)$ and any many-sorted mapping φ_p with $p \in \mathbb{N}^+$ we get $var(\overline{\varphi}_p(t_l)) \cap var(\overline{\varphi}_p(t_k)) = \emptyset$ for all $1 \leq l < k \leq n$.*

Proof. By assumption, we have $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n$, and by Lemma 3.6 we obtain that $var(\overline{\varphi}_p(t_l)) \cap var(\overline{\varphi}_p(t_k)) \subseteq var(t_l) \cap var(t_k) = \emptyset$. This implies that $var(\overline{\varphi}_p(t_l)) \cap var(\overline{\varphi}_p(t_k)) = \emptyset$ for all $1 \leq l < k \leq n$. □

As a result of Lemma 3.5 and 3.7, we obtain the following thorem.

Theorem 3.8. *The many-sorted algebra $LinClone(n)$ is free with respect to itself.*

Proof. From Lemma 3.5, $(F_p)_{p\in\mathbb{N}^+}$ is a generating system of $LinClone(n)$. Next, we prove that the many-sorted mapping which maps generating systems to $LinClone(n)$ can be extended to an endomorphism. That is, we show by induction on complexity of a linear term t that $\overline{\varphi}_q(S^{lin}\ {}^{p}_{q}(t,t_1,\ldots,t_p)) = S^{lin}\ {}^{p}_{q}(\overline{\varphi}_p(t),\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_p))$. According to the definition of superposition of linear term, we see that $t_1,\ldots,t_p \in W^{lin}_{(n)}(X_q)$ and $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq p$. It follows from Lemma 3.7 that $var(\overline{\varphi}_q t_l) \cap var(\overline{\varphi}_q t_k) = \emptyset$ for all $1 \leq l < k \leq p$. If $t = x_i$ for $i = 1,\ldots,p$, then $\overline{\varphi}_q(S^{lin}\ {}^{p}_{q}(x_i,t_1,\ldots,t_p)) = \overline{\varphi}_q(t_i) = S^{lin}\ {}^{p}_{q}(x_i,\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_p)) = S^{lin}\ {}^{p}_{q}(\overline{\varphi}_p(x_i),\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_p))$. If $t = f(s_1,\ldots,s_p)$ and we inductively assume that $\overline{\varphi}_q(S^{lin}\ {}^{p}_{q}(s_i,t_1,\ldots,t_p)) = S^{lin}\ {}^{p}_{q}(\overline{\varphi}_p(s_i),\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_p))$ for all $i = 1,\ldots,p$ and thus by the result of Theorem 3.4 and Lemma 3.7, we have that

$$\begin{array}{ll} \multicolumn{2}{l}{\overline{\varphi}_q(S^{lin}\ {}^{p}_{q}(f(s_1,\ldots,s_p),t_1,\ldots,t_p))} \\ = & \overline{\varphi}_q(f(S^{lin}\ {}^{p}_{q}(s_1,t_1,\ldots,t_p),\ldots,S^{lin}\ {}^{p}_{q}(s_p,t_1,\ldots,t_p))) \\ = & S^{lin}\ {}^{p}_{q}(\varphi_n(f(x_1,\ldots,x_n)),\overline{\varphi}_q(S^{lin}\ {}^{p}_{q}(s_1,t_1,\ldots,t_p)),\ldots,\overline{\varphi}_q(S^{lin}\ {}^{p}_{q}(s_p,t_1,\ldots,t_p))) \\ = & S^{lin}\ {}^{p}_{q}(\varphi_n(f(x_1,\ldots,x_n)),S^{lin}\ {}^{p}_{q}(\overline{\varphi}_p(s_1),\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_p)),\ldots, \\ & \quad S^{lin}\ {}^{p}_{q}(\overline{\varphi}_p(s_p),\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_p)))) \\ = & S^{lin}\ {}^{p}_{q}(S^{lin}\ {}^{n}_{p}(\varphi_n(f(x_1,\ldots,x_n)),\overline{\varphi}_q(s_1),\ldots,\overline{\varphi}_q(s_p)),\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_p)) \\ = & S^{lin}\ {}^{p}_{q}(\overline{\varphi}_p(f(s_1,\ldots,s_p)),\overline{\varphi}_q(t_1),\ldots,\overline{\varphi}_q(t_p)). \end{array}$$

This shows that the extension of φ_p is an endomorphism. Therefore $LinClone(n)$ is free with respect to itself. □

4 Superposition of Linear Formulas of type $((n),(m))$ and Clone of Linear Formulas

In this section, we extend the definition of superposition of linear terms of type (n) to quantifier free linear formulas by substituting variables occurring in a quantifier free linear formula by linear terms, then we obtain quantifier free linear formulas. We describe this by the following operations $R^{lin}\,{}^{p}_{q}$ when $p,q>1$.

Definition 4.1. Let $p,q\in\mathbb{N}^{+}$ with $p\geq q$, $F\in\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))$ and $s_1,\ldots,s_p\in W^{lin}_{(n)}(X_q)$ with $var(s_l)\cap var(s_k)=\emptyset$ for all $1\leq l<k\leq p$. Then we define the superposition operation

$$R^{lin}\,{}^{p}_{q}:\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))\times(W^{lin}_{(n)}(X_q)^{p}\multimap\!\!\!\longrightarrow\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_q))$$

by setting,

(i) If F has the form $s\approx t$, then
$R^{lin}\,{}^{p}_{q}(s\approx t,s_1,\ldots,s_p):=S^{lin}\,{}^{p}_{q}(s,s_1,\ldots,s_p)\approx S^{lin}\,{}^{p}_{q}(t,s_1,\ldots,s_p)$.

(ii) If F has the form $\gamma(t_1,\ldots,t_p)$, then
$R^{lin}\,{}^{p}_{q}(\gamma(t_1,\ldots,t_p),s_1,\ldots,s_p)$
$:=\gamma(S^{lin}\,{}^{p}_{q}(t_1,s_1,\ldots,s_p),\ldots,S^{lin}\,{}^{p}_{q}(t_p,s_1,\ldots,s_p))$.

(iii) If $F\in\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))$ and assume that $R^{lin}\,{}^{p}_{q}(F,s_1,\ldots,s_p)$ is already defined, then $R^{lin}\,{}^{p}_{q}(\neg F,s_1,\ldots,s_p):=\neg R^{lin}\,{}^{p}_{q}(F,s_1,\ldots,s_p)$.

(iv) If $F_1,F_2\in\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))$ and supposed that
$R^{lin}\,{}^{p}_{q}(F_l,s_1,\ldots,s_p)$ is already defined for all $l\in\{1,2\}$, then
$R^{lin}\,{}^{p}_{q}(F_1\vee F_2,s_1,\ldots,s_p):=R^{lin}\,{}^{p}_{q}(F_1,s_1,\ldots,s_p)\vee R^{lin}\,{}^{p}_{q}(F_2,s_1,\ldots,s_p)$.

Example 4.2. Let $((n),(m))=((3),(2))$ be a type with a ternary operation symbol and a binary relation symbol, say f and γ, respectively. If we consider the superposition

$$R^{lin}\,{}^{4}_{6}:\mathcal{F}^{lin}_{((3),(2))}(W^{lin}_{(3)}(X_4))\times(W^{lin}_{(3)}(X_6))^{4}\multimap\!\!\!\longrightarrow\mathcal{F}^{lin}_{((3),(2))}(W^{lin}_{(3)}(X_6)).$$

Then we have

(1) $R^{lin}\,{}^{4}_{5}(f(x_2,x_1,x_4)\approx x_3,x_1,x_3,x_6,x_2)$
$\quad=\ S^{lin}\,{}^{4}_{6}(f(x_2,x_1,x_4),x_1,x_3,x_6,x_2)\approx S^{lin}\,{}^{4}_{6}(x_3,x_1,x_3,x_6,x_2)$
$\quad=\ f(x_3,x_1,x_2)\approx x_6$.

(2) $R^{lin}\,{}^{4}_{6}(\gamma(x_3,f(x_4,x_1,x_2)),x_1,x_3,f(x_4,x_5,x_6),x_2)$
$\quad=\ \gamma(f(x_4,x_5,x_6),f(x_2,x_1,x_3))$.

(3) $R^{lin}\,{}^{4}_{6}(\neg(x_1 \approx x_3), x_1, x_3, f(x_4, x_5, x_6), x_2)$
$= \neg R^{lin}\,{}^{4}_{6}(x_1 \approx x_3, x_1, x_3, f(x_4, x_5, x_6), x_2)$
$= \neg(x_1 \approx f(x_4, x_5, x_6))$.

(4) $R^{lin}\,{}^{4}_{6}(x_1 \approx x_3 \vee \gamma(x_2, x_4), x_1, x_3, x_6, x_2)$
$= R^{lin}\,{}^{4}_{6}(x_1 \approx x_3, x_1, x_3, x_6, x_2) \vee R^{lin}\,{}^{4}_{6}(\gamma(x_2, x_4), x_1, x_3, x_6, x_2)$
$= x_1 \approx x_6 \vee \gamma(x_3, x_2)$.

Now, we may consider the many-sorted algebra :

$$LinFormClone((n),(m)) := ((W^{lin}_{(n)}(X_p))_{p\in\mathbb{N}^+}, (\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p)))_{p\in\mathbb{N}^+}; (S^{lin}\,{}^{p}_{q})_{p\leq q, p,q\in\mathbb{N}^+}, (R^{lin}\,{}^{p}_{q})_{p\leq q, p,q\in\mathbb{N}^+}, (x_i)_{i\leq p, i\in\mathbb{N}^+}),$$

which is called *the clone of linear formula of type* $((n),(m))$.

Theorem 4.3. *The many sorted algebra* $LinFormClone((n),(m))$ *satisfies the following properties :*

(LFC1) $R^{lin}\,{}^{p}_{q}(R^{lin}\,{}^{r}_{p}(F, t_1, \ldots, t_r), s_1, \ldots, s_p)$
$= R^{lin}\,{}^{r}_{q}(F, S^{lin}\,{}^{p}_{q}(t_1, s_1, \ldots, s_p), \ldots, S^{lin}\,{}^{p}_{q}(t_r, s_1, \ldots, s_p))$,

(LFC2) $R^{lin}\,{}^{p}_{p}(F, x_1, \ldots, x_p) = F$,

where $p, q, r \in \mathbb{N}^+$ *with* $r \leq p \leq q$, $F \in \mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_r))$, $t_1, \ldots, t_r \in W^{lin}_{(n)}(X_p)$, $var(t_l) \cap var(t_k) = \emptyset$ *for all* $1 \leq l < k \leq r$ *and* $s_1, \ldots, s_p \in W^{lin}_{(n)}(X_q)$, $var(s_l) \cap var(s_k) = \emptyset$ *for all* $1 \leq l < k \leq p$.

Proof. We can prove similar to the proof of Thorem 3.4 follow on the definition of a linear formula. □

Our next aim is to prove that $LinFormClone((n),(m))$ is free with respect to itself. To do this, we introduce some notations which will be used throughout this aim, we now let $F^*_p = {F_p}^2 \cup \{\gamma(x_1, \ldots, x_p) \mid p \in \mathbb{N}^+\}$.

Lemma 4.4. $(F^*_p)_{p\in\mathbb{N}^+}$ *is a generating system of* $LinFormClone((n),(m))$.

Proof. We first prove that $(\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p)))_{p\in\mathbb{N}^+}$ is generated by the sequence $(F^*_p)_{p\in\mathbb{N}^+}$. To do this, let $s, t \in W^{lin}_{(n)}(X_q)$, then s, t are generated by $(F_p)_{p\in\mathbb{N}^+}$. Since the linear formulas of the form $s \approx t$ is the equation of linear terms s and t, this implies that the linear formulas in this form are generated by $({F_p}^2)_{p\in\mathbb{N}^+}$. Next, we let $q \in \mathbb{N}^+$ where $q \geq m$. By Theorm 3.5, we obtained that $t_1, \ldots, t_m \in W^{lin}_{(n)}(X_q)$ are generated. Let $\gamma(t_1, \ldots, t_m) \in \mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_q))$, then $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq m$, and thus $R^{lin}\,{}^{m}_{q}(\gamma(x_1, \ldots, x_m), t_1, \ldots, t_m) = \gamma(t_1, \ldots, t_m)$. This shows that

$\gamma(t_1,\ldots,t_m)$ is generated. Now, we get atomic linear formulas which are generated by $(F_p^*)_{p\in\mathbb{N}^+}$. By Definition 2.4, a linear formula obtained from atomic linear formula by repeated application of the connectives $\neg$ and $\vee$, so that they are generated. The result indicates that $(F_p^*)_{p\in\mathbb{N}^+}$ is a generating system of $LinFormClone((n),(m))$. $\square$

Let $(\psi_p)_{p\in\mathbb{N}^+} : (F_p^*)_{p\in\mathbb{N}^+} \to (\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p)))_{p\in\mathbb{N}^+}$ be a many-sorted mapping which maps a generating system to linear formula of type $((m),(n))$. Let $p_1 : {F_p}^2 \to F_p$ be the first projection and that $\psi'_p := p_1(\psi_p|{F_p}^2)$ is defined by $\psi'_p(f(x_1,\ldots,x_p)) := \overline{p_1}(\psi_p(f(x_1,\ldots,x_p),f(x_1,\ldots,x_p)))$ where $\overline{p_1} : W^{lin}_{(n)}(X_p)^2 \to W^{lin}_{(n)}(X_p)$.

Our next aim is to prove that$(\psi_p)_{p\in\mathbb{N}^+}$ can be extended to an endomorphism of $LinFormClone((n),(m))$. Then we define

$$(\overline{\psi}_p)_{p\in\mathbb{N}^+} : (\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p)))_{p\in\mathbb{N}^+} \to (\mathcal{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p)))_{p\in\mathbb{N}^+}$$

as follows:

(1) $\overline{\psi}_q(s\approx t) := \psi'_q(s)\approx\psi'_q(t)$.

(2) $\overline{\psi}_q(\gamma(t_1,\ldots,t_m)) := R^{lin}\ {}^{m}_{q}(\psi_m(\gamma(x_1,\ldots,x_m)),\psi'_q(t_1),\ldots,\psi'_q(t_m))$.

(3) $\overline{\psi}_q(\neg F) := \neg(\overline{\psi}_q(F))$.

(3) $\overline{\psi}_q(F_1\vee F_2) := \overline{\psi}_q(F_1)\vee\overline{\psi}_q(F_2)$.

Lemma 4.5. *For each many-sorted mapping ψ_p with $p\in\mathbb{N}^+$ and any linear formula F, we have $var(\overline{\psi}_p(F))\subseteq var(F)$.*

Proof. We can prove this lemma by the definition of linear formulas and using the same arguments as we use in Lemma 3.6. If F has the form $s\approx t$, then

$$\begin{aligned} var(\overline{\psi}_q(s\approx t)) &= var(\psi'_q(s)\approx\psi'_q(t)) \\ &= var(\psi'_q(s))\cup var(\psi'_q(t)) \\ &\subseteq var(s)\cup var(t) \\ &= var(s\approx t). \end{aligned}$$

If F has the form $\gamma(t_1,\ldots,t_m)$, by the result of Lemma 3.6 we have $var(\psi'_p(t_i))\subseteq var(t_i)$ for all $i=1,\ldots,m$, and thus

$$\begin{aligned} var(\overline{\psi}_p(\gamma(t_1,\ldots,t_m))) &= var(R^{lin}\ {}^{m}_{q}(\psi_m(\gamma(x_1,\ldots,x_m)),\psi'_q(t_1),\ldots,\psi'_q(t_m))) \\ &\subseteq \bigcup_{i=1}^{m} var(\psi'_q(t_i)) \\ &\subseteq \bigcup_{i=1}^{m} var(t_i) \\ &= var(\gamma(t_1,\ldots,t_m)). \end{aligned}$$

If linear formula has the form $\neg F$, and $var(\overline{\varphi}_p(F))$ is already defined, so that

$var(\overline{\psi}_p(\neg F)) = var(\neg(\overline{\psi}_p(F))) = var(\overline{\psi}_p(F)) \subseteq var(F) = var(\neg F)$.
If linear formula has the form $F_1 \vee F_2$ and $var(\overline{\psi}_p(F_1))$ and $var(\overline{\psi}_p(F_2))$ are already defined. Then we get

$$\begin{aligned} var(\overline{\psi}_p(F_1 \vee F_2)) &= var(\overline{\psi}_p(F_1) \vee \overline{\psi}_p(F_2)) \\ &= var(\overline{\psi}_p(F_1)) \cup var(\overline{\psi}_p(F_2)) \\ &\subseteq var(F_1) \cup var(F_2) \\ &= var(F_1 \vee F_2). \end{aligned}$$

□

Lemma 4.6. *For any many-sorted mapping φ_p with $p \in \mathbb{N}^+$. Then the following statement hold :*

(1) *If linear formula $F := s \approx t$, then $var(\psi'_p(s)) \cap var(\psi'_p(t)) = \emptyset$.*

(2) *If linear formula $F := \gamma(t_1, \ldots, t_m)$, then $var(\psi'_p(t_l)) \cap var(\psi'_p(t_k)) = \emptyset$ for all for all $1 \leq l < k \leq m$.*

(3) *If linear formula $F := F_1 \vee F_2$, then $var(\overline{\psi}_p(F_1)) \cap var(\overline{\psi}_p(F_2)) = \emptyset$.*

Proof. These follow immediately from Lemma 4.5. □

As a result of Lemma 4.4 and 4.6, we obtain the following theorem.

Theorem 4.7. *The many-sorted algebra $LinFormClone((n),(m))$ is free with respect to itself.*

Proof. From Lemma 4.4, $(F^*_p)_{p\in\mathbb{N}^+}$ is a generating system of $LinFormClone((n),(m))$. Next, we show that the extension of $(\psi)_{p\in\mathbb{N}^+}$ is an endomorphism. We show that

$$\overline{\psi}_q(R^{lin}\,{}^p_q(F, t_1, \ldots, t_p)) = R^{lin}\,{}^p_q(\overline{\psi}_p(F), \psi'_q(t_1), \ldots, \psi'_q(t_p)).$$

According to the definition of superposition of linear formulas, we know that $t_1, \ldots, t_p \in W^{lin}_{(n)}(X_q)$ and $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq p$. It follow from Lemma 3.6 that $var(\psi'_q(t_l)) \cap var(\psi'_q(t_k)) = \emptyset$ for all $1 \leq l < k \leq p$. If F has the form $s \approx t$, by Lemma 4.6, we have that

$$\begin{aligned} &\overline{\psi}_q(R^{lin}\,{}^p_q(s \approx t, t_1, \ldots, t_p)) \\ &= \overline{\psi}_q(S^{lin}\,{}^p_q(s, t_1, \ldots, t_p) \approx S^{lin}\,{}^p_q(t, t_1, \ldots, t_p)) \\ &= \psi'_q(S^{lin}\,{}^p_q(s, t_1, \ldots, t_p)) \approx \psi'_q(S^{lin}\,{}^p_q(t, t_1, \ldots, t_p)) \\ &= S^{lin}\,{}^p_q(\psi'_p(s), \psi'_q(t_1), \ldots, \psi'_q(t_p)) \approx S^{lin}\,{}^p_q(\psi'_p(t), \psi'_q(t_1), \ldots, \psi'_q(t_p)) \\ &= R^{lin}\,{}^p_q(\psi'_p(s) \approx \psi'_p(t), \psi'_q(t_1), \ldots, \psi'_q(t_p)) \\ &= R^{lin}\,{}^p_q(\overline{\psi}_p(s \approx t), \psi'_q(t_1), \ldots, \psi'_q(t_p)). \end{aligned}$$

If F has the form $\gamma(s_1, \ldots, s_p)$, and by Theorem 3.8, we have $\psi'_q(S^{lin}\ {}^p_q(s_l, t_1, \ldots, t_p)) = S^{lin}\ {}^p_q(\psi'_p(s_l), \psi'_q(t_1), \ldots, \psi'_q(t_p))$ for all $l = 1, \ldots, p$. Then by Lemma 4.6, we get that $\overline{\psi}_q(R^{lin}\ {}^p_q(\gamma(s_1, \ldots, s_p), t_1, \ldots, t_p))$

$$\begin{aligned}
&= \overline{\psi}_q(\gamma(S^{lin}\ {}^p_q(s_1, t_1, \ldots, t_p), \ldots, S^{lin}\ {}^p_q(s_p, t_1, \ldots, t_p)))\\
&= R^{lin}\ {}^m_q(\psi_m(\gamma(x_1, \ldots, x_m)), \psi'_q(S^{lin}\ {}^p_q(s_1, t_1, \ldots, t_p)), \ldots, \psi'_q(S^{lin}\ {}^p_q(s_p, t_1, \ldots, t_p)))\\
&= R^{lin}\ {}^m_q(\psi_m(\gamma(x_1, \ldots, x_m)), S^{lin}\ {}^p_q(\psi'_p(s_1), \psi'_q(t_1), \ldots, \psi'_q(t_p)), \ldots,\\
&\qquad S^{lin}\ {}^p_q(\psi'_p(s_p), \psi'_q(t_1), \ldots, \psi'_q(t_p)))\\
&= R^{lin}\ {}^p_q(R^{lin}\ {}^m_p(\psi_m(\gamma(x_1, \ldots, x_m)), \psi'_p(s_1), \ldots, \psi'_p(s_p)), \psi'_p(t_1), \ldots \psi'_p(t_p))\\
&= R^{lin}\ {}^p_q(\overline{\psi}_p(\gamma(s_1, \ldots, s_p)), \psi'_q(t_1), \ldots, \psi'_q(t_p)).
\end{aligned}$$

If linear formula has the form $\neg F$ and we assume that F satisfied already. By Lemma 4.6, $\overline{\psi}_q(R^{lin}\ {}^p_q(\neg F, t_1, \ldots, t_p))$

$$\begin{aligned}
&= \overline{\psi}_q(\neg(R^{lin}\ {}^p_q(F, t_1, \ldots, t_p)))\\
&= \neg(\overline{\psi}_q(R^{lin}\ {}^p_q(F, t_1, \ldots, t_p)))\\
&= \neg(R^{lin}\ {}^p_q(\overline{\psi}_p(F), \psi'_q(t_1), \ldots, \psi'_q(t_p)))\\
&= R^{lin}\ {}^p_q(\neg(\overline{\psi}_p(F)), \psi'_q(t_1), \ldots, \psi'_q(t_p))\\
&= R^{lin}\ {}^p_q(\overline{\psi}_p(\neg F), \psi'_q(t_1), \ldots, \psi'_q(t_p)).
\end{aligned}$$

If linear formula has the form $F_1 \vee F_2$ and we assume that F_1, F_2 satisfied already. By Lemma 3.7 and 4.6, then we have that

$\overline{\psi}_q(R^{lin}\ {}^p_q(F_1 \vee F_2, t_1, \ldots, t_p))$

$$\begin{aligned}
&= \overline{\psi}_q(R^{lin}\ {}^p_q(F_1, t_1, \ldots, t_p) \vee R^{lin}\ {}^p_q(F_2, t_1, \ldots, t_p))\\
&= \overline{\psi}_q(R^{lin}\ {}^p_q(F_1, t_1, \ldots, t_p)) \vee \overline{\psi}_q(R^{lin}\ {}^p_q(F_2, t_1, \ldots, t_p))\\
&= R^{lin}\ {}^p_q(\overline{\psi}_p(F_1), \psi'_q(t_1), \ldots, \psi'_q(t_p)) \vee R^{lin}\ {}^p_q(\overline{\psi}_p(F_2), \psi'_q(t_1), \ldots, \psi'_q(t_p))\\
&= R^{lin}\ {}^p_q(\overline{\psi}_p(F_1) \vee \overline{\psi}_p(F_2), \psi'_q(t_1), \ldots, \psi'_q(t_p))\\
&= R^{lin}\ {}^p_q(\overline{\psi}_p(F_1 \vee F_2), \psi'_q(t_1), \ldots, \psi'_q(t_p)).
\end{aligned}$$

This shows that $\overline{\psi}_q$ for all $q \in \mathbb{N}^+$ is an endomorphism. Now, we can conclude that $LinFormClone((n), (m))$ is free with respect to itself. □

Acknowledgments

The first author was supported by the Human Resource Development in Science Project (Science Achievement Scholarship of Thailand, SAST).

References

[1] S. Burris, H. P. Sankappanavar, A Course in Universal Algebra, Springer Verlag, New York, 1981.

[2] T. Changphas, K. Denecke and B. Pibaljommee, Linear Terms and Linear Hypersubstitutions, to be appear in *SEAMS Bull. Math*.

[3] M. Couceiro, E. Lehtonen, Galois Theory for Sets of Operations Closed Under Permutation, Cylindrification and Composition, *Algebra Universalis*, 67 (2012), pp. 273-297.

[4] K. Denecke, Menger Algebras and Clones of Terms, *East-West J. of Mathematics*, Vol.5, No. 2 (2003), pp. 179-193.

[5] K. Denecke, The Partial Clone of Linear Terms, *Sib. Math J.*, 57 (2016), pp. 589-598.

[6] K. Denecke and D. Phusanga, Hyperformulas and Solid Algebraic Systems, *Studia Logica*, Vol. 90, No. 2 (2008), pp. 263-286.

[7] K. Denecke, S. L. Wismath, Hyperidentities and Clones, Gordon and Breach Science Publishers, The Netherlands, 2000.

[8] K. Denecke, S. L. Wismath, Universal Algebra and Applications in Theoretical Computer Science, Chapman and Hall/CRC, Boca Raton-London-New York-Washington D.C., 2002.

[9] S. Kerkhoff, R. Pschel and F. MartinSchneider, A Short Introduction to Clones, *Electronic Notes in Theoretical Computer Science*, Vol. 303 (2014), pp. 107-120.

[10] J. Koppitz, K. Denecke, M-solid Varieties, Springer-Verlag, New York, 2006.

[11] T. Kumduang, J. Joomwong and D. Phusanga, Monoid of Linear Hypersubstitution for Algebraic System of Type $((2),(2))$ and Their Green's Relation, *Proceeding for* 11^{th} *conference on science and technology for youth* (2016), pp. 122-131.

[12] W. F. Lawvere, Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the context of Functorial Semantics of Algebraic Theories, Ph.D. thesis, Columbia University, 1963.

[13] A. I. Mal'cev, Algebraic Systems, Akademie-Verlag, Berlin 1973.

[14] V. Trnková, Clones of Topological Spaces, *Topology Appl.*, 159 (2012), pp. 3122-3129.