

A NOTE ON PRIME IDEALS OF IFP-RINGS AND THEIR EXTENSIONS

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Abstract

Let R be a ring, σ an automorphism of R and δ a σ -derivation of R . Let further σ be such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$, where $N(R)$ is the set of nilpotent elements of R . We recall that a ring R is called an *IFP*-ring if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. In this paper we study the associated prime ideals of Ore extension $R[x; \sigma, \delta]$ and we prove the following in this direction:

Let R be a right Noetherian *IFP*-ring, which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers), σ and δ as above. Then P is an associated prime ideal of $R[x; \sigma, \delta]$ (viewed as a right module over itself) if and only if there exists an associated prime ideal U of R such that $(P \cap R)[x; \sigma, \delta] = P$ and $P \cap R = U$.

1 Introduction

Notation: We follow the notation and conventions of [3]. All rings are associative with 1. For any subset J of a right R -module M , annihilator of J is denoted by $Ann(J)$. $Spec(R)$ denotes the set of prime ideals of R , the set of associated prime ideals of R (viewed as a right module over itself) is denoted by $Ass(R_R)$. $MinSpec(R)$ denotes the set of minimal prime ideals of R . Let R

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be a right Noetherian ring. For any uniform right R -module J , the assassinator of J is denoted by $Assas(J)$. Let M be a right R -module. Consider the set $\{Assas(J) \mid J \text{ is a uniform right } R\text{-submodule of } M\}$. We denote this set by $\mathbb{A}(M_R)$.

Remark 1.1. If R is viewed as a right module over itself, we note that $Ass(R_R) = \mathbb{A}(R_R)$ (5Y of Goodearl and Warfield [5]).

Ore Extensions: Let R be a ring and σ an endomorphism of R . Recall that a σ -derivation of R is an additive map $\delta : R \rightarrow R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. In case σ is the identity map, δ is called just a derivation of R .

The Ore extension (or the skew polynomial ring) over R in an indeterminate x is: $R[x; \sigma, \delta] = \{f(x) = x^n a_n + \dots + a_0 \mid a_i \in R\}$ with $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. This definition of non-commutative polynomial rings was first introduced by Ore 1933, who combined earlier ideas of Hilbert (in the case $\delta = 0$) and Schlessinger (in the case $\sigma = 1$). We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. An ideal I of a ring R is called σ -invariant if $\sigma(I) = I$ and is called δ -invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ -invariant and δ -invariant, then $I[x; \sigma, \delta]$ is an ideal of $O(R)$ and as usual we denote it by $O(I)$.

Definition 1.2. A ring R is called 2-primal if and only if $P(R) = N(R)$, where $P(R)$ is the prime radical of R and $N(R)$ is the set of nilpotent elements of R (a familiar property of commutative rings). Some of the fundamental properties of 2-primal rings are developed in [6], [12] and [13]. (N. B. The terminology is not uniform: 2-primal rings are called “N-rings” in [6], and, under an equivalent definition, called “weakly symmetric” in [13]).

An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$, where $a \in R$.

Weak σ -rigid rings and IFP-rings:

Definition 1.3. (Kwak [8]). Let R be a ring and σ an endomorphism of R . Then R is said to be a $\sigma(*)$ -ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1.4. Let $R = \mathbb{Z}[\sqrt{-2}]$. Let $\sigma : R \rightarrow R$ be an endomorphism defined by $\sigma(a + b\sqrt{-2}) = a - b\sqrt{-2}$. Then R is a $\sigma(*)$ -ring.

Ouyang in [10] introduced weak σ -rigid rings, where σ is an endomorphism of ring R . These rings are related to 2-primal rings.

Definition 1.5. (Ouyang [10]). Let R be a ring and σ an endomorphism of R such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$. Then R is called a weak σ -rigid ring.

Example 1.6. Assume that $W_1[F]$ is the first Weyl algebra over a field F of characteristic zero. Then $W_1[F] = F[\mu, \lambda]$, the polynomial ring with indeterminates μ and λ with $\lambda\mu = \mu\lambda + 1$.

Now let R be the ring $\begin{pmatrix} W_1[F] & W_1[F] \\ 0 & W_1[F] \end{pmatrix}$. Consider the following element in R : $\begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}$. Now the prime radical $P(R)$ of R is $\begin{pmatrix} 0 & W_1[F] \\ 0 & 0 \end{pmatrix}$. Define an endomorphism $\sigma : R \rightarrow R$ by $\sigma\left(\begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}$. Then R is a weak σ -rigid ring.

Definition 1.7. (Shin [12]). Let R be a ring. Then R is called an *IFP*-ring (or Ring with Insert Factory Property) if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. Also known as *IFP*-ring.

Example 1.8. (1) Let $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in \mathbb{Z} \right\}$.

The only matrices A and B satisfying $AB = 0$ are of the type

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}; a, b \in \mathbb{Z}.$$

$$\text{i.e., } A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

$$\text{Now for all } K = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in R,$$

$$AB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \text{implies } AKB &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right) \\ &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & db \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This implies R is an *IFP*-ring.

(2) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; a, b, c \in \mathbb{Z} \right\}$.

Then the only matrices A and B satisfying $AB = 0$ are of the type

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}; a, b \in \mathbb{Z}.$$

$$\text{Now let } a, b, c \text{ and } d \neq 0 \text{ then for all } K = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in R$$

$$\begin{aligned}
AB &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\text{But } AKB &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \\
&= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right) \\
&= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & db \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & adb \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

This implies R is not an IFP-ring.

- (3) (Example (5.3) of [12]). Let $F = \mathbb{Z}_2(y)$ be the field of rational functions over $\mathbb{Z}_2$ with y an indeterminate. Consider the ring $R = \{f(x) \in F[x] \mid xy + yx = 1\}$. Then clearly R is a domain, so it is reduced and hence an IFP-ring.

Reduced rings (i.e., rings without nonzero nilpotent elements) are obviously IFP-rings, right (left) duo rings are IFP-rings by ([12], Lemma 1.2). Shin showed that IFP-rings are 2-primal in ([12], Theorem 1.5), and so reduced rings are 2-primal.

Lemma 1.9. *Let R be a ring. Let σ be an automorphism of R .*

1. *If P is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of R and $\sigma(P \cap R) = P \cap R$.*
2. *If Q is a prime ideal of R such that $\sigma(Q) = Q$, then $S(Q)$ is a prime ideal of $S(R)$ and $S(Q) \cap R = Q$.*

Proof. The proof follows on the same lines as in Lemma (10.6.4) of [9]. $\square$

Theorem 1.10. *Let R be a Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$ -ring. Then R is a weak σ -rigid ring. Conversely a 2-primal weak σ -rigid ring is a $\sigma(*)$ -ring.*

Proof. See Theorem (5) of [2]. $\square$

Theorem 1.11. *Let R be a right Noetherian $\mathbb{Q}$ -algebra. Let σ be an automorphism and δ be a σ -derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $e^{t\delta}$ is an automorphism of $T = R[[t; \delta]]$, the skew power series ring.*

Proof. The proof is on the same lines as in [11] and in the non-commutative case on the same lines as in [4]. $\square$ Hence forth we denote $R[[t; \delta]]$ by T . Let σ be an automorphism of a ring R , and I be an ideal of R such that $\sigma(I) = I$. Then it is easy to see that $TI \subseteq IT$ and $IT \subseteq TI$. Hence $TI = IT$ is an ideal of T .

Lemma 1.12. *Let R be a right Noetherian $\mathbb{Q}$ -algebra. Let σ be an automorphism and δ be a σ -derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Let I be an ideal of R such that $\sigma(I) = I$. Then I is δ -invariant if and only if IT is $e^{t\delta}$ -invariant.*

Proof. See Lemma (2.5) of [3]. $\square$

Proposition 1.13. *Let R be a ring and T as usual. Then:*

- (1) $P \in \text{MinSpec}(T)$ implies that $P \cap R \in \text{MinSpec}(R)$ and $P = (P \cap R)T$.
- (2) $U \in \text{MinSpec}(R)$ with $\sigma(U) = U$ implies that $UT \in \text{MinSpec}(T)$.

Proof. See Lemma (2.5) of [1]. $\square$

2 Main Results

Proposition 2.1. *Let R be a ring. Then R is an IFP-ring implies that $P(R)$ is completely semiprime.*

Proof. Since R is an IFP-ring. So, by Proposition (1.5) of [12] R is 2-primal implies that $P(R)$ is completely semiprime. $\square$

Proposition 2.2. *Let R be a right Noetherian IFP-ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ -rigid ring and δ a σ -derivation of R . Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in \text{MinSpec}(R)$.*

Proof. Let $U \in \text{MinSpec}(R)$. Since $P(R)$ is completely semiprime by Proposition (2.1). So by Proposition (2.1) of [3] we have $\sigma(U) = U$. Now let $T = \{a \in U \mid \text{such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}$. Then T is a δ -invariant ideal of R . Hence it is easy to show that $\delta(U) \subseteq U$ by Proposition (2.1) of [3]. $\square$

Lemma 2.3. *Let R be a right Noetherian IFP-ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ -rigid ring and δ a σ -derivation of R . Then*

- (1) *If U is a minimal prime ideal of R , then $O(U)$ is a minimal prime ideal of $O(R)$ and $O(U) \cap R = U$.*
- (2) *If P is a minimal prime ideal of $O(R)$, then $P \cap R$ is a minimal prime ideal of R .*

Proof. Since every IFP-ring is 2-primal and a 2-primal weak σ -rigid ring is $\sigma(*)$ -ring by Theorem (1.10). Rest is obvious by using Lemma (2.2) of [3]. $\square$

Theorem 2.4. *Let R be a right Noetherian IFP-ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ -rigid ring and δ be a σ -derivation of R . Then $P \in \text{Ass}(O(R)_{O(R)})$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $O(P \cap R) = P$ and $P \cap R = U$.*

Proof. The proof follows on the same lines as in Theorem (A) of [3]. We give a sketch.

R being right Noetherian implies that $\text{Ass}(R_R) = \mathbb{A}(R_R)$. Now R is a weak σ -rigid IFP ring, therefore, Proposition (2.2) implies that $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in \text{MinSpec}(R)$. So $O(U)$ is an ideal of $O(R)$. Now $fU = 0$. Therefore $fO(R)U \subseteq fUO(R) = 0$, i.e. $U \subseteq P \cap R$. But it is clear that $P \cap R \subseteq U$. Thus $P \cap R = U$.

Conversely let $U = \text{Ann}(cR) = \text{Assas}(cR)$, $c \in R$ and R is right Noetherian implies that $\text{Ass}(R_R) = \mathbb{A}(R_R)$. Now it can be easily seen that $O(U) = \text{Ann}(chO(R))$ for all $h \in O(R)$. Therefore $O(U) = \text{Ann}(cO(R)) = \text{Assas}(cO(R))$.
□

References

- [1] V. K. Bhat, Transparent rings and their extensions, *New York J. Math.*, **15**, (2009), 291-299.
- [2] V. K. Bhat, Ore extensions over weak σ -rigid Rings and $\sigma(*)$ -rings, *Eur. J. Pure Appl. Math.*, **3**, 4 (2010), 695-703.
- [3] V. K. Bhat, Prime ideals of $\sigma(*)$ -rings and their extensions, *Lobachevskii J. Math.*, **32**, 1 (2011), 102-106.
- [4] W. D. Blair, L.W. Small, Embedding differential and skew-polynomial rings into Artinian rings, *Proc. Amer. Math. Soc.*, Vol. **109**, 4 (1990), 881-886.
- [5] K. R. Goodearl, R. B. Warfield, *An introduction to noncommutative Noetherian rings*, Camb. Uni. Press, England (2004).
- [6] Y. Hirano, Some studies on strongly T-regular rings, *Math. J. Okayama Uni.*, **20**, (1978), 141-149.
- [7] N. D. Hoa Nghiem, N. V. Sanh and N. T. Bac, *On Modules with Insertion Factor Property*, to appear in Southeast Asian Bull. of Mathematics.
- [8] T. K. Kwak, Prime radicals of skew-polynomial rings, *Int. J. Math. Sci.*, **2**, 2 (2003), 219-227.
- [9] J. C. McConnell, J. C. Robson, *Noncommutative Noetherian rings*, Wiley(1987); revised edition: Amer. Math. Soc., New York (2001).
- [10] L. Ouyang, Extensions of generalized α -rigid rings, *Int. Electron. J. Algebra*, **3**, (2008), 103-116.

- [11] A. Seidenberg, Differential ideals in rings of finitely generated type, *Amer. J. Math.*, **89**, (1967), 22-42.
- [12] G. Y. Shin, Prime ideals and sheaf representation of a pseudo symmetric ring, *Trans. Amer. Math. Soc.*, **184**, (1973), 43-60.
- [13] S. H. Sun, Non-commutative rings in which every prime ideal is contained in a unique maximal ideal, *J. Pure Appl. Algebra*, **76**, 2 (1991), 179-192.