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ON THE SUBNORMALISER CONDITION FOR SUBGROUPS

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Abstract

A subgroup H of G is said to satisfy the *subnormaliser condition* in G if for every subgroup K of G such that $H \trianglelefteq K$, then $N_G(K) \leq N_G(H)$. In this paper, we study this embedding property of subgroups. We establish the relation between groups, whose subgroups satisfy the subnormaliser condition and the so called $\overline{T}$*-groups*, i.e., the groups, in which the normality is a transitive relation.

Let G be a group, D a subgroup, A a subset and x, y elements of G. Throughout in this paper, we denote by $y^x := x^{-1}yx, D^x := x^{-1}Dx, D^A := \langle D^a | a \in A\rangle$, the subgroup of G generated by the set $\cup_{a\in A} D^a$. Let D be a subgroup of a group G. If $D \leq H \leq G$, then we say that a subgroup H is an *intermediate subgroup of* G *with respest to* D. If D is understood from the context and there are no confusions, then we can say briefly that H is an *intermediate subgroup* of G. An intermediate subgroup H of a group G is called D*-complete* (briefly *complete* if there is no confusion) if $D^H = H$. A subgroup D is said to be *polynormal* in G if $D^{\langle x\rangle}$ is D-complete for each element x in G. We say that a subgroup D is *abnormal* (resp. *weakly abnormal*) in a group G, if for every element $x \in G$ we have $x \in \langle D, D^x\rangle$ (resp. $x \in D^{\langle x\rangle}$). A subgroup D is called *pronormal* (resp. *weakly pronormal*) in G, if for every element $x \in G$, there exists an element $u \in \langle D, D^x\rangle$ (resp. $u \in D^{\langle x\rangle}$) such that $D^x = D^u$. A subgroup D is *paranormal* in G if for every element $x \in G$ the subgroup $\langle D, D^x\rangle$ is D-complete. It is well known that all normal, abnormal,

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weakly abnormal, pronormal, weakly pronormal and paranormal subgroups are polynormal subgroups. A subgroup H of G is said to satisfy the *subnormaliser condition* in G if for every subgroup K of G such that $H \trianglelefteq K$, it follows that $N_G(K) \leq N_G(H)$. It is clear that a polynormal subgroup satisfies the subnormaliser condition. The inverse is not true. The counterexample will be given in the following:

Example

Let G be a group given by generators a, b, c, d, e, f and the following defined relations:

$$a^3 = b^2 = c^2 = d^3 = e^3 = f^3 = 1,$$
$$[b, a] = c, [c, a] = bc, [d, a] = d^2e, [e, a] = e^2f,$$
$$[f, a] = df^2, cb = bc, db = bd,$$
$$[e, b] = e, [f, b] = f, [d, c] = d, [e, c] = e,$$
$$fc = cf, ed = de, fd = df, fe = ef.$$

Consider the subgroup $D =< b, f >$ of G. Then, a simple verification shows that D satisfies the subnormaliser condition in G. On the other hand, we have

$$N_G(D) =< b, c, d, f >, D^{<a>} =< b, c, f, d, e > \text{ and } D^{D^{<a>}} =< b, f, e > \neq D^{<a>}.$$

Hence D is not polynormal in G.

It is easy to show that, if D is polynormal in G then every intermediate subgroup of G with respect to D is polynormal too (see also [1]). So, every such a subgroup satisfies the subnormaliser condition. In the following we show that the converse is also true.

Theorem 1 *Let D be a subgroup of a group G. Then D is polynormal in G if and only if every D-complete intermediate subgroup of G satisfies the subnormaliser condition in G.*

To prove this theorem, we need some auxiliary lemmas. In the following, the proofs of lemmas 1, 2 and 3 are easy and will be omitted.

Lemma 1 *Let $\varphi : G \to G'$ be a group homomorphism, D a subgroup of G containing $ker\varphi$ and F an intermediate subgroup of G. Then, the following statements hold:*

(i) $\varphi(N_G(D)) = N_{\varphi(G)}(\varphi(D))$;

(ii) D is normal in F if and only if $\varphi(D)$ is normal in $\varphi(F)$;

(iii) $\varphi(F)$ is complete in $\varphi(G)$ with respect to $\varphi(D)$ if and only if F is complete in G with respect to D;

(iv) A subgroup D satisfies the subnormaliser condition (resp. D is polynormal, paranormal, pronormal) in G if and only if $\varphi(D)$ satisfies the sub-

normaliser condition (resp. $\varphi(D)$ is polynormal, paranormal, pronormal) in $\varphi(G)$.

Corollary 1 *Let H be a normal subgroup of a group G and D a subgroup of G containing H. Then, D satisfies the subnormaliser condition (resp. D is polynormal, paranormal, pronormal) in G if and only if the quotient group D/H satisfies the subnormaliser condition (resp. D/H is polynormal, paranormal, pronormal) in G/H.*

Lemma 2 *Let D be a subgroup of a group G and K a subgroup of G containing D. If D satisfies the subnormaliser condition (resp. D is polynormal, paranormal, pronormal) in G, then D satisfies the subnormaliser condition (resp. D is polynormal, paranormal, pronormal) in K.*

Lemma 3 *Let D be a subgroup of a group G. Then, the following statements hold:*

(i) *If D is a subnormal subgroup, satisfying the subnormaliser condition in G, then D is normal in G*;

(ii) *If D is polynormal in G, then D satisfies the subnormaliser condition in G.*

Proof of Theorem 1 Suppose that D is polynormal in G. Then, every D-complete intermediate subgroup of G is polynormal in G, so by Lemma 3 (ii), it satisfies the subnormaliser condition in G. Conversely, suppose that every D-complete intermediate subgroup of G satisfies the subnormaliser condition in G. We will prove that D is polynormal in G. Now, for any $x \in G$, put $K := < D, x >$ and $H := D^{<x>}$. We have $H = D^K \trianglelefteq K$. Consider the following descending series of subgroups H_ν:

$$H_0 = H, H_1 = D^{H_0}, \dots, H_{\nu+1} = D^{H_\nu}, \text{ and } H_\mu = \cap_{\nu<\mu} H_\nu$$

for a limitting ordinal number μ. Clearly, in some finite or transfinite step, the series $\{H_\nu\}_\nu$ will be stable, i.e., there exists some minimal ordinal number ρ such that

$$H_{\rho+1} = D^{H_\rho} = H_\rho.$$

Put $H^* = H_\rho = H_{\rho+1}$. Clearly, H^* is a complete intermediate subgroup of G with respect to D and $H^* = H_\rho \trianglelefteq H_{\rho-1} \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 = H$. Since H^* is D-complete, by supposition H^* satisfies the subnormaliser condition in G. We will prove that H^* is normal in H. Thus, suppose that H^* is not normal in H. Then, there exists some ordinal number λ with $0 < \lambda \leq \rho$ such that H^* is not normal in $H_{\lambda-1}$, but $H^* \trianglelefteq H_\lambda \trianglelefteq H_{\lambda-1}$. Since H^* satisfies the subnormaliser condition in G, by Lemma 2, H^* satisfies the subnormaliser condition in $H_{\lambda-1}$, and it follows that H^* is normal in $H_{\lambda-1}$ (by applying Lemma 3(i)). This is a contradiction. So, H^* is normal in H. Moreover, since $H \trianglelefteq K$, by Lemma 2 and Lemma 3(i), it follows that H^* is normal in K. Since $D \leq H^* \trianglelefteq K$, $H = D^K \leq H^* \leq H$, hence $H^* = H$. Therefore, $H = D^{\langle x \rangle}$ is D-complete in G for every $x \in G$. So, D is polynormal in G. The proof of the theorem is now completed. □

Recall that a subgroup D is *weakly abnormal* in a group G if for any $x \in G$, we have $x \in D^{<x>}$. Every weakly abnormal subgroup in a group G is polynormal in G. Moreover, it is well-known that a subgroup D of a group G is weakly abnormal in G if and only if every intermediate subgroup of G with respect to D is self-normalizing. Applying Theorem 1, we can obtain some stronger result as the following:

Theorem 2 *Let D be a subgroup of a group G. Then D is weakly abnormal in G if and only if every D-complete intermediate subgroup of G is self-normalizing.*

Proof Suppose that every D-complete intermediate subgroup of G is self-normalizing. Then, clearly that, every D-complete intermediate subgroup of G satisfies the subnormaliser condition in G. By Theorem 1, D is polynormal in G. It is well- known that, D is polynormal in G iff D^H is D-complete for every intermediate subgroup H of G. So, for such a subgroup H, we have $H \leq N_G(D^H) = D^H$, hence $H = D^H$ is D-complete. By supposition, H is self-normalizing. Thus, D is weakly abnormal in G. □

Recall that a subgroup D is *paranormal* in G if for each element $x \in G$, the subgroup $< D, D^x >$ is D-complete. It is well-known that if D is a polynormal subgroup of G, then D is paranormal in G iff for every D-complete subgroup F and every $x \in G$, from the condition $D^x \leq N_G(F)$, it follows that $D^x \leq F$. In the connection with this property, we introduce the following concept:

Definition 1 A subgroup D is called *quasi-paranormal* in a group G, if for any D-complete intermediate subgroup F of G, and any $x \in G$, from the condition $D^x \leq N_G(F)$, it follows that $D^x \leq F$.

Clearly, every paranormal subgroup of G is quasi-paranormal. Moreover, a subgroup D is paranormal in G iff D is quasi-paranormal and polynormal in G.

Lemma 4 *Every quasi-paranormal subgroup D of a group G satisfies the subnormaliser condition in G.*

Proof Suppose that D is quasi-paranormal in G and $D \trianglelefteq K \leq G$. Then, for any $x \in N_G(K)$, we have $D^x \leq K^x = K \leq N_G(D)$. Since D is quasi-paranormal in G, it follows that $D^x \leq D$. Similarly, we have $D^{x^{-1}} \leq D$. Hence $D^x = D$ or $x \in N_G(D)$. Therefore $N_G(K) \leq N_G(D)$. Hence, D satisfies the subnormaliser condition in G. □

Lemma 5 *If D is paranormal in G, then every D-complete intermediate subgroup of G is paranormal in G.*

Proof Suppose that D is paranormal in G and F is a D-complete intemediate subgroup of G. Then, for any $x \in G$, we have $D^{<D,D^x>} = <D, D^x>$. Since $D^F = F$, it follows $(D^x)^{F^x} = F^x$. By virtue of this fact, we have

$$<F, F^x> \leq <D, D^x>^{<F,\, F^x>} = (D^{<D,D^x>})^{<F,\, F^x>} \leq F^{<F,\, F^x>} \leq <F, F^x> .$$

Hence $<F, F^x> = F^{<F,\, F^x>}$ or $<F, F^x>$ is F-complete subgroup of G. Therefore, F is paranormal in G. □

Theorem 3 *Let D be a subgroup of a group G. Then D is paranormal in G if and only if every D-complete intermediate subgroup of G is quasi-paranormal in G.*

Proof Suppose that D is paranormal in G. Then by Lemma 5, every D-complete intermediate subgroup of G is paranormal in G and hence, it is quasi-paranormal in G. Conversely, suppose that every D-complete intermediate subgroup of G is quasi-paranormal in G. By Lemma 4, it satisfies the subnormaliser condition in G. According to Theorem 1, D is polynormal in G. Hence, D is paranormal in G. □

A subgroup H of a group G is called an *$\mathcal{H}$-subgroup* if for every $g \in G$, $H^g \cap N_G(H) \leq H$. Let H be an $\mathcal{H}$-subgroup of a group G and $K \leq G$ such that $H \trianglelefteq K$. Then for any $x \in N_G(K)$, we have $H^x \leq K^x = K \leq N_G(H)$. Since H is an $\mathcal{H}$-subgroup, it follows that $H^x \leq H$. This conclusion is also true for $x^{-1} \in N_G(K)$.Therefore, $H^x = H$ or $N_G(K) \leq N_G(H)$. So, *every $\mathcal{H}$-subgroup of a group G satisfies the subnormaliser condition in G.*

Proposition 1 *If D is a subgroup of a group G such that every D-complete intermediate subgroup of G is an $\mathcal{H}$-subgroup of G, then D is paranormal in G.*

Proof Let D be such a subgroup as in the proposition. Then, as we have noted above, every D-complete intermediate subgroup of G satisfies the subnormaliser condition in G. From Theorem 1, it follows that D is polynormal in G. For any D-complete intermediate subgroup F and any $x \in G$ such that $D^x \leq N_G(F)$, we will prove that $D^x \leq F$. In fact, since F is an $\mathcal{H}$-subgroup and $D^x \leq N_G(F)$, it follows that $D^x \leq F^x \cap N_G(F) \leq F$. Therefore $D^x \leq F$.

Thus, D is polynormal and quasi-paranormal in G. Hence, D is paranormal in G. □

The subgroup embedding property of the subnormaliser condition was introduced by V. I. Mysovskikh in [8] and it was investigated in [4]. For finite groups, A. Ballester-Bolinches and R. Esteban-Romero established the relation between subgroups with the embedding property above and the so called *T-groups*, groups in which every subnormal subgroup is normal. From Theorem A in [4], we see that a finite group G is a T-group if and only if every subgroup of G satisfies the subnormaliser condition in G. A group G is called a *$\overline{T}$-group* if each subgroup of G is a T-group. A finite T-group is a $\overline{T}$-group (see [11], Th. 1*). So, combining two results above we can see that *"If every subgroup of a finite group G satisfies the subnormaliser condition in G then G is a $\overline{T}$-group."* Here, we prove that in the proposition above the condition of a finiteness should be omitted. In fact, we prove the following more general result:

Theorem 4 *A group G is a $\overline{T}$-group if and only if every subgroup of G satisfies the subnormaliser condition in G.*

Proof Suppose that G is a $\overline{T}$-group and D is a subgroup of G. By Theorem 1 [5], D is polynormal in G. It follows that D satisfies the subnormaliser condition in G.

Conversely, suppose that every subgroup of G satisfies the subnormaliser condition in G. Then, for $D \leq H \leq G$, D satisfies the subnormaliser condition in H. Hence, to prove that G is a $\overline{T}$-group, it suffices to show that G is a T-group. Thus, let D be a subnormal subgroup of G and suppose that $D \trianglelefteq K \trianglelefteq L$. Since D satisfies the subnormaliser condition in G, it follows that $L \leq N_G(K) \leq N_G(D)$. Hence, $D \trianglelefteq L$. Now, by induction, we conclude that D is normal in G. The proof of our theorem is now completed. □

Recall that a group G is *an FC-group* if every element in G has only a finite number of conjugates.

Corollary 2 *Let G be a locally solvable T-group. If G is an FC-group then every subgroup of G satisfies the subnormaliser condition in G.*

Proof By Corollary 3.8 [7], G is a $\overline{T}$-group. Now, the conclusion is obtained from Theorem 4. □

We say that a finite group G *satisfies the condition C_p* (where p is a prime divisor of $|G|$) if every subgroup of a Sylow p-subgroup P of G is normal in $N_G(P)$. In [11], D.J.S. Robinson showed that a finite group G is a $\overline{T}$-group iff it satisfies the condition C_p for every prime divisor p of $|G|$. We use this fact to prove the following:

Corollary 3 *Let G be a locally finite group. Then G is a solvable $\overline{T}$-group if and only if every cyclic subgroup of G satisfies the subnormaliser condition in G.*

Proof If G is a $\overline{T}$-group then the conclusion follows from Theorem 4. Conversely, suppose that every cyclic subgroup of G satisfies the subnormaliser condition in G. Let H be a finitely generated subgroup of G. Then H is finite and every cyclic subgroup of H satisfies the subnormaliser condition in H. We show that H satisfies the C_p condition for every prime divisor p of $|H|$. Let P be a Sylow p-subgroup of H and K be a subgroup of P. Then, for every x in K, since P is nilpotent, the cyclic subgroup generated by x is subnormal and satisfies the subnormaliser condition in $N_H(P)$, so $\langle x\rangle \trianglelefteq N_H(P)$. It follows that $K \trianglelefteq N_H(P)$. Using the fact, mentioned above we conclude that H is a T-group. By Corollary 2 [10], G is a T-group. As the hypotheses are inherited by every subgroup of G, it follows that G is a $\overline{T}$-group. By the Corollary of Theorem 1* [11], G is solvable. □

Corollary 4 *Let G be a periodic FC-group. If every cyclic subgroup of G satisfies the subnormaliser condition in G then G is a solvable $\overline{T}$-group.*

Proof If G is a periodic FC-group then G is locally normal and hence it is locally finite (see 15.1.12 [12]). So, the conclusion follows from Corollary 3. □

Theorem 5 *Let G be an FC-group. Then G is a solvable T-group if and only if every its cyclic subgroup satisfies the subnormaliser condition in G.*

Proof If G is a solvable T-group then by Corollary 2, every its cyclic subgroup satisfies the subnormaliser condition in G. Conversely, suppose that every cyclic subgroup of G satisfies the subnormaliser condition in G. We have to prove that G is a solvable T-group. If G is periodic or nilpotent then the result is clear by argument above and Corollary 4. So, we can assume that G is not periodic nor nilpotent. Denote by Z the center of G. By 15.1.16 [12], it follows that there exists a non-periodic element z in Z. By 15.1.7 [12], G' is periodic. It follows from Corollary 4 and Theorem 2.3.1 [9] that G'' is abelian and periodic. If G" is not contained in Z then there exists a non-central periodic element a in G' such that the cyclic subgroup generated by a is subnormal in G and hence it is normal in G. If G" is contained in Z then G' is nilpotent. Since G is not nilpotent, it follows that G' has nontrivial intersection with Z. So, there exists an element in G' having the same propety as a in the case above. Thus, there always exists a non-central periodic element a such that $\langle a\rangle$ is normal in G. If $b = az$ then b is not periodic and $\langle b\rangle$ is normal in G. Since a is not central, there exists g in G such that $a^g := g^{-1}ag \neq a$. Since $\langle a \rangle$ and $\langle b \rangle$ are normal in G, it follows that there exist integer numbers $i, j > 1$ such that $a^g = a^i$, $b^g = b^j$. Hence $z^{j-1} = a^{i-j}$. Since a is an element of a finite order, it follows that z is an element of a finite order. This contradiction proves our theorem. □

Corollary 5 *Let G be an FC-group. Then G is a solvable T-group if and only if G is a $\overline{T}$-group.*

Proof By applying Theorem 5 and Corollary 2. □

Note that Theorem 1* [11] and Corollary 3.8 [7] are particular cases of Corollary 5. In [4], the authors proved that a finite group G is a solvable T-group iff all its subgroups are $\mathcal{H}$-subgroups. The following theorem shows that this result is also true for FC-groups.

Theorem 6 *Let G be an FC-group. Then the following conditions are equivalent:*

(i) G is a solvable T-group;

(ii) every subgroup of G is an $\mathcal{H}$-subgroup;

(iii) every cyclic subgroup of G is an $\mathcal{H}$-subgroup.

To prove this theorem, we need the following lemma:

Lemma 6 *Let G be a periodic FC-group. If G is a solvable T-group then every cyclic subgroup of G is an $\mathcal{H}$-subgroup.*

Proof Let H be a subgroup generated by $a \in G$. We will prove that for any $g \in G, H^g \cap N_G(H) \leq H$. For any $1 \neq x \in H^g \cap N_G(H), x = (a^i)^g$ for some positive integer number i. Put $L = \langle a^i \rangle, K = \langle a, g \rangle$. Then, as we have mentioned in the proof of Corollary 4 above, K is a finite subgroup of G. Therefore, K is a finite solvable T-group and $x \in L^g \cap N_K(L)$. By Theorem 1 [4], L is an $\mathcal{H}$-subgroup of G, hence $x \in L \leq H$. □

Proof of Theorem 6 If G is abelian then the conclusions are clear. So, we can assume that G is nonabelian.

$(i) \Rightarrow (iii)$ By Theorem 6.1.1 [9] and Corollary 5, G is periodic. The conclusion is now obtained by applying Lemma 6.

$(iii) \Rightarrow (ii)$ Since every $\mathcal{H}$-subgroup of G satisfies the subnormaliser condition in G, by Theorem 5, G is a solvable T-group. So, Theorem 6.1.1 [9], G is a periodic solvable T-group. By Theorem 3.9 [7], every subgroup of G is pronormal in G. Now, let H be an arbitrary subgroup of G. We have to show that $H^g \cap N_G(H) \leq H$, for any $g \in G$. In fact, let us consider an arbitrary element $x \in H^g \cap N_G(H)$. Then, there exists some element $a \in H$ such that $x = a^g$. Put $L := \langle a \rangle^H$ and $K := \langle L, g \rangle$. Then $L \trianglelefteq H, L \leq K$ and K is a finite solvable T-group. By supposition, every cyclic subgroup of L is an $\mathcal{H}$-subgroup of G, so it is also an $\mathcal{H}$-subgroup of K. Since L is pronormal in G, L is also pronormal in K. It follows from Theorem 5 [4] that L is an $\mathcal{H}$-subgroup of K. Hence, $L^g \cap N_K(L) \leq L$. Now, we show that $x \in L^g \cap N_K(L)$. In fact, since $x = a^g \in L^g, L \trianglelefteq H$ and L is pronormal in K, it follows that $N_K(H) \leq N_K(L)$. Hence $x \in L^g \cap N_K(L) \leq L \leq H$. Thus, we have proved that H is an $\mathcal{H}$-subgroup of G.

$(ii) \Rightarrow (i)$. By Theorem 5. □

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