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MULTIPLICATION SEMILOCAL MODULES AND THEIR ENDOMORPHISM RINGS

P. Jaipong and **J. Sanwong**

Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
e-mail: scmti004@chiangmai.ac.th

Abstract

In this paper we give some general properties of semilocal multiplication modules. Then some characterizations of these modules are given. Also we study the endomorphism rings of semilocal multiplication modules and give necessary and sufficient conditions for the endomorphism rings of such modules to be semilocal.

1. Preliminaries

Let R be a commutative ring with identity, and let M be a unitary right R-module. The *radical* of M, $Rad(M) = \bigcap\{K \leq M/K \text{ is maximal in } M\}$, and M is called *semisimple* if it is the sum of its simple submodules. The module M is called a *semilocal module* if $M/Rad(M)$ is semisimple and M is a *multiplication module* if each submodule N of M has the form MI for some ideal I of R. The ring R is *semilocal* if R_R is a semilocal module. A submodule K of M is *superfluous* in M, abbreviated, $K \ll M$, if for every submodule L of M, $K + L = M$ implies $L = M$. For $M \neq 0$, we call M a *hollow module* if every proper submodule is superfluous in M, and M is said to have *finite hollow dimension* if there exists an exact sequence

$$M \xrightarrow{g} \bigoplus_{i=1}^{n} H_i \longrightarrow 0$$

where all the H_i are hollow and the kernel of g is superfluous in M. Then

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n is called the *hollow dimension of* M. A submodule N of M has a *weak supplement* L in M if $N + L = M$ and $N \cap L \ll M$, and M is called *weakly supplemented* if every submodule has a weak supplement in M.

Our starting point is the following result taking from Theorem 2.7 in [8].

1.1 Lemma *Consider the following properties :*
(1) M has finite hollow dimension ;
(2) M is weakly supplemented ;
(3) M is semilocal .
Then $(1) \Rightarrow (2) \Rightarrow (3)$ *holds.*
If $Rad(M) \ll M$ *then* $(3) \Rightarrow (2)$ *holds.*
If M *is finitely generated then* $(3) \Rightarrow (1)$ *holds.*

1.2 Proposition *For a multiplication module M, M is semilocal if and only if M is weakly supplemented.*

Proof $(\Rightarrow)$ Since we assume M is multiplication, we have $Rad(M) \ll M$. Thus M is weakly supplemented by Lemma 1.1.
$(\Leftarrow)$ Every weakly supplemented module is semilocal. □

1.3 Proposition *Let M be a multiplication semilocal module. Then every supplement in M and every direct summand of M are semilocal.*

Proof Since M is multiplication semilocal, M is weakly supplemented. So every supplement in M and every direct summand of M is weakly supplemented by Proposition 2.2 in [8]. Thus the assertion follows from Lemma 1.1. □

C. Lomp [8] proved that every finite direct sum of weakly supplemented modules is weakly supplemented. Next proposition shows that with multiplication the arbitrary direct sum of weakly supplemented modules is still weakly supplemented.

1.4 Proposition *Let $(M_\lambda)_{\lambda \in \Lambda}$ be a non-empty collection of modules. Assume that $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is multiplication. Then M is weakly supplemented if and only if M_λ is weakly supplemented for each $\lambda \in \Lambda$.*

Proof $(\Rightarrow)$ Suppose M is weakly supplemented. Then $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is semilocal, and thus M_λ is semilocal for all $\lambda \in \Lambda$. Since M_λ is multiplication, M_λ is weakly supplemented for all $\lambda \in \Lambda$.
$(\Leftarrow)$ Suppose that M_λ is weakly supplemented for all $\lambda \in \Lambda$. So M_λ is semilocal. From the fact that the direct sum of semilocal modules is still semilocal , hence $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is semilocal. Since we assume M is multiplication, M is weakly pupplemented by Proposition 1.2. □

1.5 Proposition *Let M be a multiplication R-module then M is cyclic if and only if $M/Rad(M)$ is cyclic.*

Proof ($\Rightarrow$)Obvious.

($\Leftarrow$) Assume $M/Rad(M)$ is cyclic, say $M/Rad(M) = (m + Rad(M))R$ for some $m \in M$. Let $a + Rad(M) \in M/Rad(M)$ for any $a \in M$. Then $a + Rad(M) = mr + Rad(M)$ for some $r \in R$. So $a - mr = b$ for some $b \in Rad(M)$. Thus $a = b + mr \in Rad(M) + mR$, hence $M = Rad(M) + mR$. But $Rad(M) \ll M$, so we have $M = mR$. □

1.6 Proposition *Every finitely generated multiplication semilocal module is cyclic.*

Proof Let M be a finitely generated multiplication semilocal module. Then $M/Rad(M)$ is finitely generated, and semisimple. So $M/Rad(M) = \sum_{i=1}^{n} \bar{T}_i$, where $\bar{T}_i$ is a simple submodule of $M/Rad(M)$. Thus we can find $m \leq n$ such that $M/Rad(M) = \bigoplus_{j=1}^{m} \bar{T}_{i_j}$. Since each $\bar{T}_{i_j}$ is cyclic and $M/Rad(M)$ is multiplication, $M/Rad(M)$ is cyclic. Therefore M is cyclic as required. □

2. Multiplication Semilocal Modules and Semilocal Rings.

In this section some properties of multiplication modules over semilocal rings are studied. Also some characterizations of such rings in terms of endomorphism rings of multiplication modules are given.

2.1 Definition A ring R is said to have *stable range* 1 if whenever $a, b \in R$ and $Ra + Rb = R$, there exists $t \in R$ with $a + tb \in U(R)$ the group of invertible elements of the ring R.

Some results of a semilocal ring from Base [3] and Evans [5] are used.

(1) A semilocal ring has stable range 1.

(2) Let M be an R-module, suppose that $End(M)$ has stable range 1. If P and Q are arbitrary R-modules and $M \oplus P \cong M \oplus Q$, then $P \cong Q$.

2.2 Proposition *Let M be a multiplication module over a semilocal ring R and $S := End(M)$. Then*

(1) M is semilocal;

(2) M has finite hollow dimension ;

(3) M is weakly supplemented ;

(4) M has finitely many maximal submodules ;

(5) S is semilocal ;

(6) if P and Q are arbitrary R-modules such that $M \oplus P \cong M \oplus Q$, then $P \cong Q$.

Proof Let M be multiplication and R a semilocal ring. From the fact that every multiplication module over a semilocal ring is cyclic, so M is cyclic. Thus $S \cong R/ann_R(M)$.

(1) M is semilocal by Theorem 3.5 in [8].

(2) and (3). By Lemma 1.1, M has finite hollow dimension and weakly supplemented.

(4) Let $(K_\lambda)_{\lambda \in \Lambda}$ be the collection of all maximal submodules of M. Then there exists I_λ a maximal ideal of R such that $K_\lambda = MI_\lambda$ for each $\lambda \in \Lambda$. So Λ must be finite (see [6] page 2).

(5) From the fact that $S \cong R/ann_R(M)$ and R is semilocal, we have S is semilocal .

(6) Let P and Q be arbitrary R-modules such that $M \oplus P \cong Q \oplus P$. Since S is semilocal by (5), S has stable range 1, and hence $Q \cong P$. □

2.3 Proposition *The following conditions are equivalent :*

(1) R_R is semilocal ;

(2) every multiplication R-module has a semilocal endomorphism ring ;

(3) every cyclic R-module has a semilocal endomorphism ring.

Proof (1) $\Rightarrow$ (2) by Proposition 2.2(5).

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) is clear. □

3. The Endomorphism Rings.

Let us consider $\mathbb{Z}_{\ltimes}$. It is a semilocal $\mathbb{Z}$-module, but $\mathbb{Z}$ is not semilocal. However, we will see that if a module is faithful finitely generated and multiplication, then there are some relations among the semilocal property of the module M_R, the rings R, the ring S of R-endomorphisms of M, and the module M_S (see Corollary 3.2).

Herbera and Shamsuddin [7] gave some conditions in testing whether the endomorphism ring of a module is semilocal : Let M be a module of finite hollow dimension and every surjective endomorphism of M is an isomorphism, then $End(M)$ is semilocal. We now apply their result.

3.1 Theorem *Let M be a finitely generated multiplication R-module and $S = End(M)$. Then the following statements are equivalent :*

(1) $R/ann_R(M)$ is a semilocal ring;

(2) M is semilocal as an R-module;

(3) S is a semilocal ring;

(4) M is semilocal as an S-module.

Proof (1) $\Leftrightarrow$ (3) Since M is a finitely generated multiplication R-module, $S \cong R/ann_R(M)$. Thus the assertion follows.

(2) $\Rightarrow$ (3) From M is finitely generated multiplication and semilocal, we get M has finite hollow dimension by Lemma 1.1. Because M is multiplication, so every surjective endomorphism of M is an isomorphism. Thus by Herbera and Shamsuddin [7], S is semilocal.

(3) $\Rightarrow$ (4) Since S is semilocal, Theorem 3.5 in [8] shows that M_S is semilocal.

(4) $\Rightarrow$ (2) The lattices of submodules of M_R and M_S are coincide since M_R is finitely generated multiplication. Thus M_S is semilocal if and only if M_R is semilocal . □

3.2 Corollary *Let M be a faithful finitely generated multiplication R-module and $S = End(M)$. Then the following statements are equivalent:*

(1) R is a semilocal ring;

(2) M is semilocal as an R-module;

(3) S is a semilocal ring;

(4) M is semilocal as an S-module.

Proof It is an immediate consequence of Theorem 3.1. □

3.3 Corollary *Let M be a finitely generated multiplication R-module.*

(1) M has finite hollow dimension if and only if S is semilocal;

(2) If S is semilocal, then $End(M/N)$ is semilocal for any submodule N of M.

Proof (1) A module M has finite hollow dimension if and only if it is semilocal by Lemma 1.1. Hence the assertion holds by Theorem 3.1.

(2) Assume S is semilocal. Then M has finite hollow dimension by (1). So M/N has finite hollow dimension. Since M/N is finitely generated multiplication, again by (1)we have that $End(M/N)$ is semilocal. □

The following theorem ensures that Theorem 3.1 needs finitely generated.

3.4 Theorem *Let M be a multiplication R-module. Then M is finitely generated and semilocal if and only if S is semilocal.*

Proof ($\Rightarrow$) Suppose that M is finitely generated and semilocal. Then M has finite hollow dimension by Lemma 1.1. Thus S is semilocal by Corollary 3.3.

($\Leftarrow$) Assume that S is a semilocal ring. Then M_S is a cyclic semilocal module. Since M_S is finitely generated, we must have M_R is finitely generated. And M_R is semilocal by Theorem 3.1. □

For an R-module M, we set

$$\nabla := \{s \in S \ / \ s(M) \ll M\}.$$

3.5 Proposition *Let M be a multiplication R-module. Then*
(1) $\nabla \subseteq J(S)$;
(2) $Hom(M, Rad(M)) \subseteq J(S)$.

Proof (1) Let $f \in \nabla$. Since $f(M) + (1_M - f)M = M$, where 1_M is the identity endomorphism on M, we must have $(1_M - f)M = M$, that is $1_M - f$ is surjective. By Corollary 2.2 in [10], $1_M - f$ is an isomorphism. Therefore $f \in J(S)$.

(2) If $f \in Hom(M, Rad(M))$, then $f(M) \subseteq Rad(M)$. Since M is a multiplication module, every proper submodule of M is contained in a maximal submodule of M. So $Rad(M)$ is the unique largest superfluous submodule of M. Thus $f(M)$ is a superfluous submodule of M. By (1), $f \in \nabla \subseteq J(S)$. □

Let M be a finitely generated multiplication module. Then for each $f \in S$ there is an $r \in R$ such that $f(m) = mr$ for all $m \in M$. Thus we write f_r instead of f. The following lemma is a known result (see[13]). For the sake of completeness we will present the proof here.

3.6 Lemma. *Let M be a finitely generated multiplication semilocal R-module . Then $f_r \in J(S)$ if and only if for each $y \in R$ there exists $z_y \in R$ such that $(1 - yr)z_y - 1 \in ann_R(M)$.*

Proof Since M is cyclic, we let $M = xR$ where $x \in M$. So for each $f \in S$ there exists $a \in R$ such that $f(m) = ma$ for all $m \in M$. Let $f_r : M \to M$ be defined by $f_r(m) = mr$ for all $m \in M$. Then $S \cong R/ann_R(x)$ via $f_r \mapsto r + ann_R(x)$. So we have

$$\begin{aligned}
f_r \in J(S) \quad &\Leftrightarrow \quad r + ann_R(x) \in J(R/ann_R(x));\\
&\Leftrightarrow \quad (1 - yr) + ann_R(x) \text{ has an inverse for all } y \in R;\\
&\Leftrightarrow \quad \text{for each } y \in R \text{ there exists } z_y \in R \text{ such that}\\
&\qquad\ (1 - yr) + ann_R(x))(z_y + ann_R(x)) = 1 + ann_R(x);\\
&\Leftrightarrow \quad \text{for each } y \in R \text{ there exists } z_y \in R,\ (1 - yr)z_y - 1 \in ann_R(x).
\end{aligned}$$

□

3.7 Proposition *Let M be a finitely generated multiplication semilocal R-module. Then*
(1) $J(S) = \nabla$;
(2) $J(S) = Hom(M, Rad(M))$.

Proof (1) By Proposition 3.6 (1) we have $\nabla \subseteq J(S)$. For the opposite direction we use the result from Lemma 3.7. Since M is cyclic, $M = xR$ for some $x \in M$, so we obtain that for each $f_r \in J(S)$, and each $y \in R$ there exists $z_y \in R$ such that $(1 - yr)z_y - 1 \in ann_R(x)$.

Let $f_a \in J(S)$. We will prove that $f_a(M)$ is a superfluous submodule of M. Let N be any submodule of M such that $N + f_a(M) = M$. Hence $x = n + f_a(m) = n + f_a(xt) = n + xta$ where $n \in N$, $m \in M$, $t \in R$. Thus $x(1-ta) = n$. Since $f_a \in J(S)$, there exists $z_t \in R$ such that $(1-ta)z_t - 1 \in ann_R(x)$. Then $x(1-ta)z_t - x = x((1-ta)z_t - 1) = 0$ which implies $x = x(1-ta)z_t = nz_t \in N$, so $x \in N$. Hence $M = N$.

(2) By Proposition 3.6 (2) $Hom(M, Rad(M)) \subseteq J(S)$. Now if $f \in J(S)$, then by (1) $f(M)$ is a superfluous submodule of M and thus $f(M) \subseteq Rad(M)$. □

From Theorem 3.4 we get that for a mutiplication module M, M is finitely generated semilocal if and only if S is semilocal. In this case S is weakly supplemented, and hence every ideal in S has a weak supplement in S. But Theorem 3.10 shows that for more general situation, if M is a multiplication self-projective module, then we have "Im f and Ker f are direct summands of $M/Rad(M)$ for every $f \in End(M/Rad(M))$ if and only if every principal ideal in S has a weak supplement in S".

Recall that for modules U and M, we say that U is *M-projective* in case for each epimorphism $g : M \to N$ and each homomorphism $\gamma : U \to N$ there is an R-homomorphism $\bar{\gamma} : U \to M$ such that $g\bar{\gamma} = \gamma$. And U is *self-projective* if U is U-projective.

The next proposition is motivated by R. Ware [14].

3.8 Proposition *Let M be a multiplication self-projective module. Then*
(1) $J(S) = \nabla$;
(2) $J(S) = Hom(M, Rad(M))$;
(3) there is a ring epimorphism $\phi : S \to End(M/Rad(M))$ with $ker\phi = Hom(M, Rad(M))$;
(4) $S/J(S) \cong End(M/Rad(M))$.

Proof (1) Let $f \in J(S)$ and suppose that $f(M) + K = M$. Let $\pi : M \to M/K$ be the natural epimorphism. Then πf is an epimorphism. Since M is self-projective, there exists $\varphi : M \to M$ such that $\pi f \varphi = \pi$. Thus $f\varphi(x) + K = x + K$ for all $x \in M$, and so $(1 - f\varphi)(M) \subseteq K$. But $f \in J(S)$ implies that $(1 - f\varphi)$ is invertible, i.e., $M = (1 - f\varphi)(M) \subseteq K$. So $f(M) \ll M$, hence $f \in \nabla$. Therefore $J(S) = \nabla$.

(2) Let $f \in J(S) = \nabla$. So $f(M) \ll M$ and that $f(M) \subseteq Rad(M)$. Thereby $J(S) \subseteq Hom(M, Rad(M))$. The opposite direction was shown in Proposition 3.5 (2).

(3) Let $f \in S$. Since M is fully invaraint, $f(Rad(M)) \subseteq Rad(M)$. Consider the natural epimorphism $\pi : M \to M/Rad(M)$, we get $ker(\pi) \subseteq ker(\pi f)$. By The Factor Theorem there exists unique $\bar{f} : M/Rad(M) \to M/Rad(M)$ such

that $\bar{f}\pi = \pi f$. We define $\phi : End(M) \to End(M/Rad(M))$ via $\phi(f) = \bar{f}$, that is $\phi(f)(m + Rad(M)) = f(m) + Rad(M)$. Since $f(Rad(M)) \subseteq Rad(M)$, ϕ is well-defined. By routine verification we have that ϕ is a ring homomorphism. Next we show that ϕ is an epimorphism. Let $h \in End(M/Rad(M))$. By the self-projectivity of M, there is $g : M \to M$ with $h\pi = \pi g$. And $\phi(g)(m + Rad(M)) = \bar{g}(m+Rad(M)) = g(m)+Rad(M) = \pi(g(m)) = h(\pi(m)) = h(m+Rad(M))$ for all $m \in M$. Thereby $\phi(g) = h$. And $ker\phi = Hom(M, Rad(M))$.

(4) By The Isomorphism Theorem we obtain that $End(M/Rad(M)) = Im\phi \cong End(M)/ker\phi = S/J(S)$. □

In order to prove our main theorem, the following lemma is needed.

3.9 Lemma. *$R/J(R)$ is a von Neumann regular ring if and only if every principal ideal of R has a weak supplement in R_R.*

Proof See [8] page 13. □

3.10 Theorem *Let M be a multiplication self-projective module and $S = End(M)$. Then $Im\, f$ and $Ker\, f$ are direct summands of $M/Rad(M)$ for every $f \in End(M/Rad(M))$ if and only if every principal ideal in S has a weak supplement in S.*

Proof ($\Rightarrow$) Suppose Imf and $Kerf$ are direct summands of $M/Rad(M)$ for every $f \in End(M/Rad(M))$. Then $End(M/Rad(M))$ is a von Neumann regular ring. So by Proposition 3.8 $End(M/Rad(M)) \cong S/J(S)$ is also regular. Thus Lemma 3.9 shows that every principal ideal in S has a weak supplement in S.

($\Leftarrow$) Assume that every principal ideal in S has a weak supplement in S. Then $S/J(S) \cong End(M/Rad(M))$ is von Neumann regular which implies Imf and $Kerf$ are direct summands of $M/Rad(M)$ for every $f \in End(M/Rad(M))$. □

3.11 Corollary *If M is a multiplication self-projective semilocal module, then every principal ideal in S has a weak supplement in S.*

Proof Since $M/Rad(M)$ is semisimple, Imf and $Kerf$ are direct summands of $M/Rad(M)$ for every $f \in End(M/Rad(M))$. Thus the corollary is a direct consequence of Theorem 3.10. □

Recall that a submodule N of M is *finitely M-generated* if there exist $n \in \mathbb{N}$ and an epimorphism from M^n onto N.

3.12 Corollary *Let M be a multiplication self-projective module and $S = End(M)$. If every principal ideal in S has a weak supplement in S, then every finitely M-generated submodule of $M/Rad(M)$ is a direct summand of $M/Rad(M)$.*

Proof Assume that every principal ideal in S has a weak supplement in S. So $S/J(S)$ is a von Neumann regular ring. Hence $End(M/Rad(M))$ is regular. Thus every finitely $M/Rad(M)$ - generated submodule of $M/Rad(M)$ is a direct summand of $M/Rad(M)$. But every finitely $M/Rad(M)$ - generated submodule is finitely M - generated, therefore we have the result. □

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