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FOLIATIONS FORMED BY K-ORBITS OF MAXIMAL DIMENSION OF SOME MD5-GROUPS

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Abstract

In this paper we consider some MD5-groups and MD5-algebras, i.e., five-dimensional solvable Lie algebras and groups such that their orbits in the co-adjoint representation (K-orbits) are orbits of dimension zero or maximal dimension. We describe the geometry of K-orbits of MD5-groups. The foliations formed by K-orbits of maximal dimension of these MD5-groups and their measurability are also presented in the paper.

Introduction

Let G be an n-dimensional Lie group. It is called an MDn-group (see [4]), iff its orbits in the co-adjoint representation (K-orbits) are orbits of dimension zero or maximal dimension. The corresponding Lie algebras are called MDn-algebras. All MD4-algebras were first listed by D.V Tra in 1984 (see [5])and then classified up to an isomorphism by the author in 1990 (see [8], [9]). The description of the geometry of K-orbits of all indecomposable MD4-groups, the topological classification of foliations formed by K-orbits of maximal dimension and the characterization of C*-algebras associated to these foliations by the method of K-functors were also given by the author in 1990 (see [6], [7], [8], [9]). Until now, no complete classification of MDn-algebras with $n \geq 5$ is known. In this paper we concern with a similar problem for MD5-groups and MD5-algebras.

Key words: MD5-group, MD5-algebra, foliation, K-orbit, C*-algebra, co-adjoin representation, Lie algebra, measurable foliation, transverse measure, X-invariant
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We begin our discussion in Section 2 by giving some interesting examples of MD5-algebras. Section 3 is devoted to the geometric description of K-orbits of MD5-groups corresponding to these MD5-algebras and a discussion of the foliations formed by their maximal dimensional K-orbits. At first, we recall some concepts and notations which will be used later.

1. K-Orbits of a Lie group and measurable foliations

1.1 The Co-adjoint Representation and K-orbits of a Lie Group

1. Let G be a Lie group. We denote by $\mathcal{G}$ the Lie algebra of G and by $\mathcal{G}^*$ the dual space of $\mathcal{G}$. To each element g of G we associate an automorphism

$$A_{(g)} : G \longrightarrow G$$

$$x \longmapsto A_{(g)}(x) := \; gxg^{-1}$$

(which is called the internal automorphism associated to g). $A_{(g)}$ induces the tangent map

$${A_{(g)}}_* : \mathcal{G} \longrightarrow \mathcal{G}$$

$$X \longmapsto {A_{(g)}}_*(X) := \; \frac{d}{dt}[g.exp(tX)g^{-1}] \mid_{t=0} .$$

2. The action

$$Ad : G \longrightarrow Aut(\mathcal{G})$$

$$g \longmapsto Ad(g) := \; {A_{(g)}}_*$$

is called the *adjoint representation* of G in $\mathcal{G}$.

3. The action

$$K : G \longrightarrow Aut(\mathcal{G}^*)$$

$$g \longmapsto K_{(g)}$$

such that

$$\langle K_{(g)}F, X\rangle := \langle F, Ad(g^{-1})X\rangle; \quad (F \in \mathcal{G}^*, X \in \mathcal{G})$$

is called the *co-adjoint representation* of G in $\mathcal{G}^*$.

4. Each orbit of the co-adjoint representation of G is called a *K-orbit*. The dimension of a K-orbit of G is always even (see [2]).

1.2 Measurable Foliations

1. Let V be a smooth manifold. We denote by TV its tangent bundle, so that for each $x \in V$, T_xV is the tangent space of V at x. A smooth subbundle

$\mathcal{F}$ of TV is called *integrable* iff the following condition is satisfied: every x from V is contained in a submanifold W of V such that $T_p(W) = \mathcal{F}_p(\forall p \in W)$.

2. A *foliation* $(V, \mathcal{F})$ is given by a smooth manifold V and an integrable subbundle $\mathcal{F}$ of TV. Then, V is called the *foliated manifold* and $\mathcal{F}$ is called *the subbundle defining the foliation.*

3. The *leaves* of the foliation (V, $\mathcal{F}$) are the maximal connected submanifolds L of V with $T_x(L) = \mathcal{F}_x(\forall x \in L)$.

The set of leaves with the quotient topology is denoted by $V/\mathcal{F}$ and will be called the *space of leaves* of (V, $\mathcal{F}$). It is a fairly untractable topological space.

4. The partition of V in leaves $V = \bigcup_{\alpha \in V/\mathcal{F}} L_\alpha$ is charaterized geometrically by the following local triviality: Every $x \in V$ has a system of local coordinates $\{U; x^1, x^2, ..., x^n\}(x \in U; n = dim\mathcal{F})$ such that for any leaf L with $L \cap U \neq \emptyset$, each connected component of $L \cap U$ (which is called a *plaque* of the leaf L) is given by the equations

$$x^{k+1} = c^1,\, x^{k+2} = c^2,\, \dots ,x^n \;=\; c^{n-k};\, k = \dim \mathcal{F},$$

where $c^1, c^2, ..., c^{n-k}$ are constants (depending on each plaque). Such a system $\{U, x^1, x^2, ..., x^n\}$ is called a *foliation chart*.

A foliation can be given by a partition of V in a family $\mathcal{C}$ of its submanifolds such that each $L \in \mathcal{C}$ is a maximal connected integral submanifold of some integrable subbundle $\mathcal{F}$ of TV. Then $\mathcal{C}$ is the family of leaves of the foliation $(V, \mathcal{F})$. Sometimes $\mathcal{C}$ is identified with $\mathcal{F}$ and we will say that $(V, \mathcal{F})$ is formed by $\mathcal{C}$.

5. A submanifold N of the foliated manifold V is called a *transversal* iff $T_xV = T_xN \oplus \mathcal{F}_x(\forall x \in N)$. Thus, $\dim N$ = n-dim $\mathcal{F}$ = codim$\mathcal{F}$.

A Borel subset B of V such that $B \cap L$ is countable for any leaf L is called a *Borel transversal* to (V, $\mathcal{F}$).

A *transverse measure* Λ for the foliation (V, $\mathcal{F}$) is σ - additive map B$\mapsto \Lambda$ (B) from the set of all Borel transversals to [0, $+\infty$] such that the following coditions are satisfied:

(i) If $\psi : B_1 \to B_2$ is a Borel bijection and ψ(x) is on the leaf of any x$\in B_1$, then $\Lambda(B_1) = \Lambda(B_2)$.

(ii) $\Lambda(K) < +\infty$ if K is any compact subset of a smooth transversal submanifold of V, then $(V, \mathcal{F})$ is called a *measurable foliation*, following A. Connes.

6. Let (V, $\mathcal{F}$) be a foliation with $\mathcal{F}$ is oriented. Then the complement of zero section of the bundle $\Lambda^k(\mathcal{F})$ (k = dim$\mathcal{F}$) has two components $\Lambda^k(\mathcal{F})^-$ and $\Lambda^k(\mathcal{F})^+$.

Let μ be a measure on V and $\{U, x^1, x^2, ..., x^n\}$ be a foliation chart of $(V, \mathcal{F})$. Then U can be identified with the direct product $N \times \Pi$ of some smooth transversal submanifold N of V and a some plaque Π. The restriction of μ on $U \equiv N \times \Pi$ becomes the product $\mu_N \times \mu_\Pi$ of measures μ_N and μ_Π respectively.

Let $X \in C^\infty\big(\Lambda^k(\mathcal{F})\big)^+$ be a smooth k-vector field and μ_X be the measure on each leaf L determined by the volume element X.

The measure μ is called *X-invariant* iff μ_X is proportional to μ_Π for an arbitrary foliation chart $\{U, x^1, x^2, ..., x^n\}$.

7. Let $(X, \mu), (Y, \nu)$ be two pairs where $X, Y \in C^\infty\big(\Lambda^k(\mathcal{F})\big)^+$ and μ, ν are measures on V such that μ is X-invariant, ν is Y-invariant. $(X, \mu), (Y, \nu)$ are *equivalent* iff $\mathrm{Y} = \varphi X$ and $\mu = \varphi\nu$ for some $\varphi \in C^\infty(V)$.

There is a bijective map between the set of transverse measures for $(V, \mathcal{F})$ and the one of equivalence classes of pairs (X, μ), where $X \in C^\infty\big(\Lambda^k(\mathcal{F})\big)^+$ and μ is a X-invariant measure on V.

Thus, to prove that $(V, \mathcal{F})$ is measurable, we only need to choose some suitable pair (X, μ) on V (see [1]).

2. Some examples of MD5-algebras and MD5-groups

From now on, G will denote a connected solvable Lie group of dimension 5. The Lie algebra of G is denoted by $\mathcal{G}$. We always choose a fixed basis (S, T, X, Y, Z) in $\mathcal{G}$. Then its Lie algebra is isomorphic to $\mathbb{R}^5$ as a real vector space. The notation $\mathcal{G}^*$ will mean the dual space of $\mathcal{G}$. Clearly $\mathcal{G}^*$ can be identified with $\mathbb{R}^5$ by fixing in it the basis $(S^*, T^*, X^*, Y^*, Z^*)$ which is dual to the basis (S, T, X, Y, Z).

Recall that a group G is called an *MD5-group* if its K-orbits are orbits of dimension zero or maximal dimension. Then its Lie algebra is called an *MD5 - algebra*. Note that for any MD4 - algebra $\mathcal{G}_0$, the direct product $\mathcal{G} = \mathcal{G}_\prime \times \mathbb{R}$ of $\mathcal{G}_0$ with the commutative Lie $\mathbb{R}$ is a MD5-algebra. It is called a *decomposable* MD5-algebra the study of which can be directly reduced to the case of MD4 - algebras. Therefore, we will restrict in the case of *indecomposable* MD5 - algebras.

2.1 Some Examples of Indecomposable MD5 - algebras and MD5 - groups

1. Denote by $\mathcal{G}_1$ the real algebra of dimension 5 with the basis (S, T, X, Y, Z) such that

$$\mathcal{G}_1{}^1 = [\mathcal{G}_1, \mathcal{G}_1] = \langle Y, Z\rangle \equiv \mathbb{R}^2; \mathrm{End}\big(\mathcal{G}_1{}^1\big) \equiv Mat(2, \mathbb{R});$$

$$[S, T] = [T, X] = [X, S] = 0; ad_X = ad_T = 0; ad_S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The simply connected Lie group associated to $\mathcal{G}_1$ is denoted by G_1.

2. Let $\mathcal{G}_2$ be the real algebra of dimension 5 with the basis (S, T, X, Y, Z) such that

$$\mathcal{G}_2{}^1 = [\mathcal{G}_2, \mathcal{G}_2] = \langle T, X, Y, Z\rangle \equiv \mathbb{R}^4;$$

$$ad_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Aut\big(\mathcal{G}_2{}^1\big) \equiv GL(4, \mathbb{R}).$$

The simply connected Lie group associated to $\mathcal{G}_2$ is denoted by G_2.

3. Let $\mathcal{G}_3$ be the real algebra of dimension 5 with the basis (S, T, X, Y, Z), such that

$$\mathcal{G}_3{}^1 = [\mathcal{G}_3, \mathcal{G}_3] = \langle T, X, Y, Z\rangle \equiv \mathbb{R}^4;$$

$$ad_S = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in Aut\big(\mathcal{G}_3{}^1\big) \equiv GL(4, \mathbb{R}).$$

The simply connected Lie group associated to $\mathcal{G}_3$ is denoted by G_3.

2.2 Remarks

1. The Lie algebras $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ are the semi-direct products of the form $\mathbb{R}\times_\varphi\mathcal{A}$ of the Lie abelian algebra $\mathcal{A} = \langle T, X, Y, Z\rangle \equiv \mathbb{R}^4$ with $\mathcal{B} = \langle S\rangle \equiv \mathbb{R}$ by the corresponding actions $\varphi = ad_S$.

2. In the next section we shall prove that $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are indecomposable MD5-algebras (see Section 3, Corollary 2). Hence, G_1, G_2, G_3 are also MD5-groups.

3. The main results

3.1 The Geometry of K-orbits of G_1, G_2, G_3

Throughout this section, G will denote one of the groups G_1, G_2, G_3, $\mathcal{G}$ for its Lie algebra, $\mathcal{G}^* = \langle S^*, T^*, X^*, Y^*, Z^*\rangle \equiv \mathbb{R}^5$ for the dual space of $\mathcal{G}$, and $F = s_F S^* + t_F T^* + x_F X^* + y_F Y^* + z_F Z^* \equiv (s_F, t_F, x_F, y_F, z_F)$ an arbitrary element of $\mathcal{G}^*$, and finally Ω_F for the K-orbit of G which contains F.

Theorem 1

$G = G_1$:

1. *If $x_F = y_F = 0$ then Ω_F is a K-orbit of dimension zero, i.e.,*

$$\Omega_F = \{F(s_F, t_F, 0, 0, z_F)\}.$$

2. *If $x_F^2 + y_F^2 \neq 0$ then Ω_F is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s, t_F, x, y, z_F)/x^2 + y^2 = x_F^2 + y_F^2\}$$

(a cylinder of revolution).

$G = G_2$:

1. *If $t_F = x_F = y_F = z_F = 0$ then Ω_F is a K-orbit of dimension zero, i.e.,*

$$\Omega_F = \{F(s_F, 0, 0, 0, 0)\}.$$

2. *If $t_F \neq 0 = x_F = y_F = z_F$ then Ω_F is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s, t, 0, 0, 0)/t_F t > 0\}$$

(a coordinate half - plane).

3. *If $x_F \neq 0 = t_F = y_F = z_F$ then Ω_F is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s, 0, x, 0, 0)/x_F x > 0\}$$

(a coordinate half - plane).

4. *If $y_F \neq 0 = t_F = x_F = z_F$ then Ω_F is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s, 0, 0, y, 0)/y_F y > 0\}$$

(a coordinate half - plane).

5. *If $z_F \neq 0 = t_F = x_F = y_F$ then Ω_F is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s, 0, 0, 0, z)/z_F z > 0\}$$

(a coordinate half - plane).

6. *If $t_F x_F \neq 0 = y_F z_F$ then Ω_F is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s, t, x, 0, 0)/x_F t - t_F x = 0, t_F t > 0, x_F x > 0\}$$

(a part of plane).

7. *If* $t_F y_F \neq 0 = x_F z_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,t,0,y,0)/y_F t - t_F y = 0, t_F t > 0, y_F y > 0\}$$

(a part of plane).

8. *If* $t_F z_F \neq 0 = x_F y_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,t,0,0,z)/z_F t - t_F z = 0, t_F t > 0, z_F z > 0\}$$

(a part of plane).

9. *If* $x_F y_F \neq 0 = t_F z_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,0,x,y,0)/y_F x - x_F y = 0, x_F x > 0, y_F y > 0\}$$

(a part of plane).

10. *If* $x_F z_F \neq 0 = t_F y_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,0,x,0,z)/z_F x - x_F z = 0, x_F x > 0, z_F z > 0\}$$

(a part of plane).

11. *If* $y_F z_F \neq 0 = t_F x_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,0,0,y,z)/z_F y - y_F z = 0, y_F y > 0, z_F z > 0\}$$

(a part of plane).

12. *If* $t_F x_F y_F \neq 0 = z_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,t,x,y,0)/t_F x - x_F t = 0, t_F y - y_F t = 0, t_F t > 0, x_F x > 0, y_F y > 0\}$$

(a part of plane).

13. *If* $t_F x_F z_F \neq 0 = y_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,t,x,0,z)/t_F x - x_F t = 0, t_F z - z_F t = 0, t_F t > 0, x_F x > 0, z_F z > 0\}$$

(a part of plane).

14. *If* $t_F y_F z_F \neq 0 = x_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,t,0,y,z)/t_F y - y_F t = 0, t_F z - z_F t = 0, t_F t > 0, y_F y > 0, z_F z > 0\}$$

(a part of plane).

15. *If* $x_Fy_Fz_F \neq 0 = t_F$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,0,x,y,z)/x_Fy - y_Fx = 0, x_Fz - z_Fx = 0, x_Fx > 0, y_Fy > 0, z_Fz > 0\}$$

(a part of plane).

16. *If* $t_Fx_Fy_Fz_F \neq 0$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,t,x,y,z)/t_Fx - x_Ft = 0, \quad t_Fy - y_Ft = t_Fz - z_Ft = 0, \quad t_Ft > 0, x_Fx > 0, y_Fy > 0, z_Fz > 0\}$$

(a part of plane).

$G = G_3$:

1. *If* $t_F = x_F = y_F = z_F = 0$ *then* Ω_F *is a K-orbit of dimension zero, i.e.,*

$$\Omega_F = \{F(s_F,0,0,0,0)\}.$$

2. *If* $x_F = y_F = 0 \neq t_F^2 + z_F^2$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,t,0,0,z)/t^2 + z^2 = t_F^2 + z_F^2\}$$

(a cylinder of revolution).

3. *If* $t_F = z_F = 0 \neq x_F^2 + y_F^2$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,0,x,y,0)/x^2 + y^2 = x_F^2 + y_F^2\}$$

(a cylinder of revolution).

4. *If*$x_F^2 + y_F^2 \neq 0 \neq t_F^2 + z_F^2$ *then* Ω_F *is a K-orbit of dimension two as follows*

$$\Omega_F = \{(s,t,x,y,z)/t^2 + z^2 = t_F^2 + z_F^2; x^2 + y^2 = x_F^2 + y_F^2; tx - yz = t_Fx_F - y_Fz_F\}.$$

Corollary 2

1. G_1, G_2, G_3 *are indecomposable MD5-groups.*
2. $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ *are indecomposable MD5-algebras.*

3. Sketch the proof of Theorem 1

1. For each G, we denote the set $\{F_U \in \mathcal{G}^*/U \in \mathcal{G}\}$ by $\Omega_F(\mathcal{G})$, where F_U is the linear form on the Lie algebra $\mathcal{G}$ of G defined by

$$\langle F_U, A\rangle = \langle F, exp(ad_U)(A)\rangle, A, U \in \mathcal{G}.$$

   At first, we have to compute $exp(ad_U)$ and define F_U. After that, $\Omega_F(\mathcal{G})$ is described by the same method presented in [6], [8].

2. Note that for G $= G_2$, the map exp: $\mathcal{G} \to G$ is surjective (see [3]). Hence, $\Omega_F = \Omega_F(\mathcal{G})$.

3. For $G \in \{G_1, G_3\}$, the equation $\Omega_F = \Omega_F(\mathcal{G})$ is verified by using [10, Lemma II.1.5].

3.2 MD5-foliations associated to G_1, G_2, G_3

Theorem 3 *Let $G \in \{G_1, G_2, G_3\}$, $\mathcal{F}_G$ be the family of all its K-orbits of maximal dimension and $V_G = \bigcup\{\Omega/\Omega \in \mathcal{F}_G\}$. Then $(V_G, \mathcal{F}_G)$ is a measurable foliation in the sense of Connes. We call it MD5-foliation associated to MD5-group G.*

Sketch of the proof of Theorem 3

The proof is analogous to the case of MD4-groups in [6], [8], [10]. First, we need to define a smooth tangent 2-vector field on the manifold V_G such that each K-orbit Ω from $\mathcal{F}_G$ is a maximal connected integrable submanifold corresponding to it. As the next step, we have to show that the Lebegues measure is invariant for that 2-vector field. The last step is a simple matter and can be verified by direct computations. Now we introduce the smooth tangent 2-vector fields corresponding to each of G from $\{G_1, G_2, G_3\}$.

- $\mathcal{S}_{G_1} = \mathcal{X}_1 \wedge \mathcal{X}_2$ on the foliated manifold $V_G \equiv \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{O(0,0)\}) \times \mathbb{R}$, where
  1. $\mathcal{X}_1(s,t,x,y,z) = (0,0,y,-x,0); \forall(s,t,x,y,z) \in V_G;$
  2. $\mathcal{X}_2(s,t,x,y,z) = (1,0,0,0,0); \forall(s,t,x,y,z) \in V_G.$
- $\mathcal{S}_{G_2} = \mathcal{X}_1 \wedge \mathcal{X}_2$ on the foliated manifold $V_G \equiv \mathbb{R} \times (\mathbb{R}^4 \setminus \{O(0,0,0,0)\})$, where
  1. $\mathcal{X}_1(s,t,x,y,z) = (0,t,x,y,z); \forall(s,t,x,y,z) \in V_G;$
  2. $\mathcal{X}_2(s,t,x,y,z) = (1,0,0,0,0); \forall(s,t,x,y,z) \in V_G.$
- $\mathcal{S}_{G_3} = \mathcal{X}_1 \wedge \mathcal{X}_2$ on the foliated manifold $V_G \equiv \mathbb{R} \times (\mathbb{R}^4 \setminus \{O(0,0,0,0)\})$, where

1. $\mathcal{X}_1(s,t,x,y,z) = (0,-z,y,-x,t);\ \forall(s,t,x,y,z) \in V_G;$
2. $\mathcal{X}_2(s,t,x,y,z) = (1,0,0,0,0);\ \forall(s,t,x,y,z) \in V_G.$

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