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RATHER LARGE SUBSETS OF PRIME AND SEMIPRIME RINGS WITH DERIVATIONS

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Abstract

Let K be a commutative ring with unity, R a prime K-algebra of characteristic different from 2, with extended centroid C, d and δ non-zero derivations of R, $f(x_1,..,x_n)$ a polynomial over K. If $\delta(d(f(r_1,..,r_n)) - f(r_1,..,r_n)) = 0$, for all $r_1,..,r_n \in R$, then $f(x_1,..,x_n)$ is central-valued on R. We also examine the case when R is a two-torsion free semiprime ring, $n = 2$ and $f(x_1, x_2) = [x_1, x_2]_k$, the k-th commutator in two variables, for k a fixed positive integer.

Let K be a commutative ring with unity, R a prime K-algebra of characteristic different from 2, with center $Z(R)$ and extended centroid C. Recall that an additive mapping d of R into itself is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. This result is included in a line of investigation concerning the relationship between the structure of R anf the behaviour of some derivation defined on R. In this context, by considering appropriate conditions on the subset $P(d, S) = \{d(s) - s/s \in S\}$, where S is a suitable subset of R, it is possible to formulate many results obtained in literature. For istance the result of Bell and Daif in [2] states that if $S = \{[x_1, x_2]/x_1, x_2 \in I\}$, for I a non-zero ideal of a semiprime ring R, then $P(d, S) = 0$ implies that I is central in R. Later Hongan proved that the same conclusion holds if $P(d, S) \subseteq Z(R)$ [9]. Recently we proved that in a prime ring R, if for any $a \in P(d, S)$ there exists $n = n(a) \geq 1$ such that $a^n = 0$, then R is commutative [6]. In an other recent paper we also considered the following situation: let

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$P(d, f(R)) = \{d(f(x_1, .., x_n)) - f(x_1, .., x_n)/x_1, .., x_n \in R\}$, such that $a^m = 0$, for all $a \in P(d, f(I))$ and m a fixed integer. Under this assumption, we showed that $f(x_1, .., x_n)$ is an identity for R [7]. In this note we will assume that $f(x_1, .., x_n)$ is not necessarily multilinear and there exists a non-zero derivation δ of R such that $\delta(a) = 0$, for all $a \in P(d, f(R))$. We will prove that this condition forces $f(x_1, .., x_n)$ to be central in R. It is well known that this conclusion says that the set $P(d, f(R))$ is rather large in R.

In the first part we study the case $\delta(P(d, f(R))) = 0$, where both δ and d are inner derivation: more precisely there exist $a, b \in R$ such that $\delta(x) = [a, x]$ and $d(x) = [b, x]$, for all $x \in R$.

Then we extend our result to arbitrary derivations.

Finally, in the last part of the paper we examine the case when R is a two-torsion free semiprime ring, $k \geq 1$ is a fixed integer and the polynomial f is the k-th commutator $[x_1, x_2]_k$, which is defined as follows: for $k = 1$, $[x_1, x_2]_1 = [x_1, x_2] = x_1x_2 - x_2x_1$ and for $k \geq 2$, $[x_1, x_2]_k = [[x_1, x_2]_{k-1}, x_2]$.

We begin with the following easy result:

Lemma 1 *If $f(x_1, .., x_n)$ is not central in R then there exists a non-zero ideal M of R such that $\delta(d([x_1, x_2]) - [x_1, x_2]) = 0$ for all $x_1 \in M$, $x_2 \in R$.*

Proof Let G the additive subgroup generated by the set

$$f(R) = \{f(r_1, .., r_n)/r_1, .., r_n \in R\} \neq 0.$$

Of course $\delta(d(g) - g) = 0$, for all $g \in G$. Since $f(x_1, .., x_n)$ is not central in R, by [5] and $char(R) \neq 2$, it follows that there exists a non-central Lie ideal L of R such that $L \subseteq G$. Moreover, by [8, pp. 4-5] there exists a non-zero ideal M of R such that $[M, R] \subseteq L$, and we are done. □

Remark 1 In all that follows we will always assume that the polynomial f is not central in R. then there exists M an ideal of R such that $\delta(d([x_1, x_2]) - [x_1, x_2])$ is a differential identity for M. Since R and M satisfy the same differential identities (see [11]), $\delta(d([x_1, x_2]) - [x_1, x_2])$ is also a differential identity for R.

Lemma 2 *Let a, b be elements of R such that $[a, [b, [r_1, r_2]] - [r_1, r_2]] = 0$ for any $r_1, r_2 \in R$. Then $a \in Z(R)$.*

Proof Our assumption says that R satisfies the generalized polynomial identity

$$[a, [b, [x_1, x_2]] - [x_1, x_2]] =$$

$$[a, b[x_1, x_2] - [x_1, x_2]b - [x_1, x_2]] =$$

$$ab[x_1, x_2] - a[x_1, x_2]b - a[x_1, x_2] - b[x_1, x_2]a + [x_1, x_2]ba + [x_1, x_2]a.$$

The argument in [4] says that this generalized polynomial identity is also satisfied by Q, the Martindale quotients ring of R. It follows that $S = RC$ is a primitive ring with $soc(R) \neq 0$ and eHe is a simple central algebra finite dimensional over its center, for any minimal idempotent element $e \in S$ (see [12]). We may assume H non commutative, otherwise also R must be commutative. Moreover H satisfies the same generalized polynomial identities of R and Q. Since H is a simple ring, one of the following holds: either H does not contain any non-trivial idempotent element or H is generated by its idempotents.

Suppose $e^2 = e \in H$ and pick $x_1 = (1-e)h_1, x_2 = h_2e$, for $h_1, h_2 \in H$. By our assumption

$$0 = [a, [b, [(1-e)h_1, h_2e)]] - [(1-e)h_1, h_2e]] =$$

$$ab(1-e)h_1h_2e - a(1-e)h_1h_2eb - a(1-e)h_1h_2e - b(1-e)h_1h_2ea$$
$$+(1-e)h_1h_2eba + (1-e)h_1h_2ea.$$

Now, right multiplying by $(1-e)$ and left multiplying by e, we have

$$0 = -ea(1-e)h_1h_2eb(1-e) - eb(1-e)h_1h_2ea(1-e).$$

As a consequence of [12, theorem 2 (a)], it follows that $ea(1-e) = \alpha eb(1-e)$, for some $\alpha \in C = Z(Q)$. By the primeness of H and since $char(R) \neq 2$, $ea(1-e) = eb(1-e) = 0$. In a similar fashion one has $(1-e)ae = 0$. This implies that $[a, e] = 0$ and since H is generated by its idempotents, we have $a \in C$.

On the other hand, if H does not contain any non-trivial idempotent element, then H is a finite dimensional division algebra over C and we may consider $a, b \in H = RC = Q$. If C is finite then H is a finite division ring, that is H is commutative, as well as R.

If C is infinite then $H \otimes_C F \cong M_r(F)$, the ring of $r \times r$ matrices over F, where F is the central closure of C. In this case, a Vandermoonde determinant argument shows that in $M_r(F)$ $[a, [b, [x_1, x_2]] - [x_1, x_2]] = 0$ is still an identity. As above, if $r \geq 2$, then $M_r(F)$ contains some non-trivial idempotent elements, so $a \in F$. Of course, if $r = 1$, then H is commutative and we are done. □

Now the proof of the following theorem is a consequence of Lemmas 1 and 2:

Theorem 1 *Let a, b be elements of R such that $[a, [b, f(r_1, .., r_n)]] - f(r_1, .., r_n)] = 0$ for any $r_1, ..., r_n \in R$. Then either $a \in Z(R)$ or $f(x_1, .., x_n)$ is central-valued on R.*

Theorem 2 *Let K be a commutative ring with unity, R a prime K-algebra of characteristic different from 2, with extended centroid C, d and δ non-zero derivations of R, $f(x_1, .., x_n)$ a polynomial over K. If $\delta(d(f(r_1, .., r_n)) - f(r_1, .., r_n)) = 0$, for all $r_1, .., r_n \in R$, then $f(x_1, .., x_n)$ is central-valued on R.*

Proof Assume that $f(x_1,..,x_n)$ is not central on R. By Lemma 1 and Remark it follows that $\delta(d([x_1,x_2])-[x_1,x_2])$ is a differential identity for R.

First suppose that δ and d are C-independent modulo D_{int}.

By assumption, for all $r_1,r_2\in R$

$$\delta(d([r_1,r_2])-[r_1,r_2])=0$$

that is R satisfies the differential identity

$$\delta([d(x_1),x_2]+[x_1,d(x_2)]-[x_1,x_2])=$$

$$[\delta d(x_1),x_2]+[d(x_1),\delta(x_2)]+[\delta(x_1),d(x_2)]-[\delta(x_1),x_2]-[x_1,\delta(x_2)].$$

By Kharchenko's theorem [10] R satisfies the polynomial identity

$$[y_1,x_2]+[z_1,t_2]+[t_1,z_2]-[t_1,x_2]-[x_1,t_2]$$

in particular R satisfies any blendend component $[z_1,t_2]$ that is R is commutative, which contradicts the non-centrality of $f(x_1,..,x_n)$.

Let now δ and d C-dependent modulo D_{int}. There exist $\gamma_1,\gamma_2\in C$, such that $\gamma_1\delta+\gamma_2 d\in D_{\text{int}}$, and, by Theorem 1, it is clear that at most one of the two derivations can be inner.

Suppose $\gamma_1=0$ and $\gamma_2\neq 0$; then, for some non-central element $q\in Q$, $d=d_q$ is the inner derivation induced by q and δ is an outer derivation.

By the assumptions, $\delta([q,[r_1,r_2]]-[r_1,r_2])=0$, for all $r_1,r_2\in R$, that is R satisfies the differential identity

$$\delta([q,[x_1,x_2]]-[x_1,x_2])=$$

$$[\delta(q),[x_1,x_2]]+[q,[\delta(x_1),x_2]]+[q,[x_1,\delta(x_2)]]$$

$$-[\delta(x_1),x_2]-[x_1,\delta(x_2)].$$

As above, by Kharchenko's result, R satisfies the generalized polynomial identity

$$[\delta(q),[x_1,x_2]]+[q,[y_1,x_2]]+[q,[x_1,y_2]]$$

$$-[y_1,x_2]-[x_1,y_2].$$

In particular R satisfies the blended component

$$[q,[y_1,x_2]]-[y_1,x_2]$$

and by [2] (see also [6]) it follows that R is commutative, a contradiction again.

Suppose now $\gamma_2=0$ and $\gamma_1\neq 0$; then, for some non-central element $q\in Q$, $\delta=d_q$ is the inner derivation induced by q and d is an outer derivation.

In this case, for all $a \in I$, $r_1, r_2 \in R$, we have:

$$[q, d([r_1, r_2]) - [r_1, r_2]] = 0$$

that is R satisfies the differential identity

$$[q, [d(x_1), x_2] + [x_1, d(x_2)] - [x_1, x_2]]$$

and, as above using the Kharchenko's theorem, R satisfies the following generalized polynomial identity

$$[q, [y_1, x_2] + [x_1, y_2] - [x_1, x_2]]$$

as well as the blended component

$$[q, [x_1, x_2]].$$

In this situation, since $q \notin C$, many results in literature sate that R is commutative (see for example Lemma 2 in [3]), a contradiction.

Finally we may assume that both γ_1 and γ_2 are non-zero. So $\delta = \gamma d + d_q$, with $0 \neq \gamma \in C$ and $q \in Q$.

Therefore, for all $r_1, r_2 \in R$

$$(\gamma d + d_q)(d([r_1, r_2]) - [r_1, r_2]) = 0.$$

In this case R satisfies the differential identity

$$= \gamma([d^2(x_1), x_2] + 2[d(x_1), d(x_2)] + [x_1, d^2(x_2)] - [d(x_1), x_2] - [x_1, d(x_2)]) +$$
$$[q, [d(x_1), x_2] + [x_1, d(x_2)] - [x_1, x_2]]$$

and so the Kharchenko's theorem provides that

$$= \gamma([z_1, x_2] + 2[y_1, y_2] + [x_1, z_2] - [y_1, x_2] - [x_1, y_2]) +$$
$$[q, [y_1, x_2] + [x_1, y_2] - [x_1, x_2]]$$

is a polynomial identity on R.

Hence R satisfies the blended component $2\gamma[y_1, y_2]$ and this implies that R is commutative, a contradiction.

Finally, if d is Q-inner, then δ is also Q-inner and we end up by Theorem 1.

All the previous contradictions say that $f(x_1, .., x_n)$ must be central in R.

□

We conclude this note studying the case when R is a two-torsion free semiprime ring and the polynomial f is the k-th commutator $[x_1, x_2]_k$. First we fix the following result which depends by Theorem 2:

Corollary 1 *Let R be a prime ring of characteristic different from 2, d and δ non-zero derivations of R. If $\delta(d([r_1, r_2]_k) - [r_1, r_2]_k) = 0$, for all $r_1, r_2 \in R$ and $k \geq 1$ a fixed integer, then R is commutative.*

Proof It follows trivially by the fact that if $[x_1, x_2]_k$ is central in R, then R is commutative.

Remark 2 Notice that in Theorem 2 and Corollary 1, the assumption that d is a non-zero derivation can be removed. In fact, if $d = 0$ the hypothesis $\delta(f(x_1, .., x_n)) = 0$ drives us to the same conclusion, i.e. $f(x_1, .., x_n)$ must be central in R.

Now we are ready to prove the semiprime-version of Corollary 1:

Theorem 3 *Let R be a two-torsion free semiprime ring, d and δ non-zero derivations of R. If $\delta(d([r_1, r_2]_k) - [r_1, r_2]_k) = 0$, for all $r_1, r_2 \in R$ and $k \geq 1$ a fixed integer, then $[\delta(R), R] = (0)$.*

Proof Let C the extended centroid of R and U the left Utumi quotient ring of R, then $Z(U) = C$. We need to mention that the definition, the axiomatic formulation and the properties of this quotient ring can be found in [1].

It is known that any derivation of R can be uniquely extended in U and moreover R an U satisfy the same differential identities (see [11]). Therefore $\delta(d([r_1, r_2]) - [r_1, r_2]) = 0$, for all $r_1, r_2 \in U$. Let M be any maximal ideal of the complete Boolean algebra of idempotents of C, denoted by B. We know that MU is a prime ideal of U. Let $\overline{\delta}$ and $\overline{d}$ the derivations respectively induced by δ and d in $\overline{U} = \frac{U}{MU}$. Thus $\overline{\delta}$ and $\overline{d}$ satisfy in $\overline{U}$ the same property of δ and d on U. By Corollary 1 and Remark 2, for all M maximal ideal of B, either $\delta(U) \subseteq MU$ or $[U, U] \subseteq MU$. In any case $\delta(U)[U, U] \subseteq \cap_M MU = (0)$. Without loss of generality we have $\delta(R)[R, R] = 0$. In particular

$$0 = \delta(R)[R^2, R] = \delta(R)R[R, R] + \delta(R)[R, R]R = \delta(R)R[R, R].$$

Therefore $[R, \delta(R)]R[R, \delta(R)] = 0$ and, by semiprimeness of R, $[R, \delta(R)] = 0$, that is $\delta(R) \subseteq Z(R)$. □

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