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ON THE STABILITY OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH LINEAR DRIFT

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Abstract

The aim of this paper is to prove the equivalence of a stochastic differential equation with linear drift to a stochastic integral equation in an n-dimensional space. The asymptotic p-stability and asymptotic mean square satbility of the trivial random solution for the stochastic differential equation are also investigated by using properties of the Cauchy operator.

1 Introduction

In this paper, we consider the following stochastic differential equation in an n-dimensional space

$$dx(t) = A(t)x(t)dt + f(t, x(t))dB_t, \tag{1.1}$$

where $(B_t, t \in R^+)$ is an n-dimensional Brownian motion, $A(t)$ and $f(t, x)$ are a $n \times n$-matrix function. We prove the equivalence of the equation (1.1) to the following stochastic integral equation

$$x(t) = K(t, t_0)x_0 + \int_{t_0}^{t} K(t, s)f(s, x(s))dB_s, \tag{1.2}$$

where $K(t, s)$ is the Cauchy operator generated by a homogeneous linear equation of (1.1). From the equation (1.2), we can consider the problem of asymptotic p-stability and asymptotic mean square for the random solution of (1.1) by

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using propeties of the Cauchy operator. In section 2 we give some of the basic definitions and lemmas needed for studying the stochastic differential equation (1.1). The section 3 gives main results on the equivalence between equations (1.1) and (1.2), and to give conditions for the asymptotic p-stability, asymptotic mean square stability of the random solution of (1.1). Some examples of application will be given in section 4.

2 Priliminaries

All stochastic processes in this section are supposed to be considered in a complete probability space $(\Omega, \mathcal{F}, P)$. Firstly we recall some following concepts and results needed for our main results.

Definition 2.1. A solution $x_t \equiv 0$ of Equation (1.1) is said to be p-stable $(p > 0)$ if for any $\varepsilon > 0$ there exists $r > 0$ such that $||x_0|| < r$ and $t > t_0$

$$E||x(t, \omega, t_0, x_0)||^p < \varepsilon.$$

Definition 2.2. A solution $x_t \equiv 0$ of Equation (1.1) is said to be p-asymtotically stable if it is p-stable and x_0 is small enough then

$$E||x(t, \omega, t_0, x_0)||^p \to 0 \quad \text{as} \quad t \to \infty.$$

Definition 2.3. Let $\xi(t, \omega)$ be n-dimensional measuarable random process. Define

$$\chi_p[\xi] = \chi_p[\xi(t, \omega)] = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln E||\xi(t, \omega)||^p.$$

We call $\chi_p[\xi]$ *the p-Liapunov exponent of process* $\xi(t, \omega)$.

Lemma 2.1. *For any fixed* $p > 0$*, denote* $\phi(p)$ *the set of all p-Liapunov exponent of all non-trivial solutions of the following equation*

$$\begin{aligned} &\frac{dx}{dt} = G(t, x, \xi) \\ &G(t, 0, \xi) = 0 \\ &x(t_0) = x_0, \qquad t \geq 0 \end{aligned}$$

where ξ *is a random process.*

If $\sup_{t_0, x_0} \phi(p) < 0$*, then the solution* $x \equiv 0$ *of the above equation is p-asymptotically stable.*

The proof of this lemma can be found in [5].

Lemma 2.2. *Let $\chi_p[\xi_i]$ be the p-Liapunov exponent of processes $\xi_i(t,\omega)$, $0 \leq i \leq n$, we have following assertions:*
(i)

$$\chi_p\Bigg[\sum_{0\leq i\leq n}\xi_i\Bigg] \leq \max_{0\leq i\leq n}\chi_p[\xi_i],$$

(ii) if $\xi(\omega), \eta(\omega)$ are independent random proceses, then

$$\chi_p(\xi\eta) \leq \chi_p(\xi) + \chi_p(\eta),$$

(iii) if c is a positive constant, then

$$\chi_p(c\xi) = \chi_p(\xi).$$

The proof of Lemma 2.2 can be seen directly from the above definitions. We now consider the following ordinary differential equation in an n- dimensional space

$$\begin{aligned} dx_t &= A(t)x_t dt, \\ x(t_0) &= x_0, \end{aligned} \tag{1.3}$$

where $x, x_0 \in \mathbb{R}^n, t \in [t_0, \infty), A(t) = (a_{i,j}(t))_{nxn}$ with $a_{i,j}(t)$ are continuous functions, for every $t \in [t_0, \infty)$ and $i, j = 1, 2, ..., n$.

Lemma 2.3. *Consider the matrix equation*

$$\begin{aligned} du_t &= A(t)u_t dt, \\ u(t_0) &= I, \end{aligned} \tag{1.4}$$

where I is a unit matrix. If $A(t)$ is continuous matrix for every $t \in [t_0, \infty)$ then we have the following assertions

(i) there exists a unique solution $u(t)$ of (1.4).

(ii) there exists an inverse operator $u^{-1}(t)$ and it is a solution of the following matrix equation

$$\begin{aligned} dw_t &= -w_t A(t) dt, \\ w(t_0) &= I. \end{aligned} \tag{1.5}$$

Definition 2.4. We call $K(t, s) = u(t)u^{-1}(s), (t \geq s \geq t_0)$ the Cauchy operator of the linear equation (1.3), where $u(t)$ and $u^{-1}(s)$ are the solution of equations (1.4) and (1.5).

We see that, if A is a constant matrix, then $u(t) = e^{At}$ and $u^{-1}(s) = e^{-As}$. Therefore $K(t, s) = e^{A(t-s)}$.

Lemma 2.4. *Assume that $K(t,s)$ is the Cauchy operator of the linear equation (1.3), we have the following assertions*

(i) $K(t,t) = I$ *for every* $t \in [t_0, \infty)$,

(ii) $K(t,s) = K(t,\tau).K(\tau,s)$ *for every* $t_0 \le s \le \tau \le t$.

This Lemma can be proved directly from Definition 2.4 and Lemma 2.3.

Lemma 2.5. *Suppose that all eigenvalues of the constant matrix A have negative real parts. Then there exist constants $\alpha, \beta > 0$ such that:*

$$|K(t,s)| = |e^{A(t-s)}| < \beta e^{-\alpha(t-s)} \quad \textit{for every} \quad t \ge t_0.$$

3 Main results

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose that $(\mathcal{F}_t,\, t \in R^+)$ is a family of increasing subalgebras of $\mathcal{F}$, i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s < t$. Let B_t be an n-dimensional standard Brownian motion adapted to $\mathcal{F}_t$. Futhermore, we assume that the Equation (1.1) satisfies the following conditions

(i) $A(t) = (a_{i,j}(t))_{nxn}$ is continuous matrix for every $t \in \mathbb{R}^+$ and $i, j = 1, 2, ...n$, that is $a_{i,j}(t)$ are continuous functions for every $t \in \mathbb{R}^+$.

(ii) $f(t,x)$ is matrix function defined and continuous for every $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$.

(iii) $x(0) = x_0$ is $\mathcal{F}_0 - measurable$.

Theorem 3.1. *With the conditions (i)-(iii), the equation (1.1) is equivalent to the following sochastic integral equation*

$$x(t) = K(t,0)x_0 + \int_0^t K(t,s) f(s, x(s)) dB_s, \tag{1.2}$$

where $K(t,s)$ is the Cauchy operator generated by the homogeneous linear equation of the equation (1.1).

Proof. Suppose that $x(t)$ satisfies the equation (1.1). Put

$$y(t) = g(t,x) = K(\tau,t)x(t).$$

Then we have

$$y_q(t) = g_q(t, x(t)) = \sum_{i=1}^{n} K_{qi}(\tau,t) x_i(t).$$

An application of multi-dimensional Ito formula yields

$$\begin{aligned} dy_q(t) &= \tfrac{\partial g_q}{\partial t} dt + \textstyle\sum_{i=1}^n \tfrac{\partial g_q}{\partial X_i} dx_i \\ &= \textstyle\sum_{i=1}^n \tfrac{\partial K_{qi}(\tau,t)}{\partial t} x_i(t) dt + \textstyle\sum_{i=1}^n K_{qi}(\tau,t) dx_i(t). \end{aligned}$$

It follows from (1.1) that

$$dx_i(t) = \sum_{i=1}^{n} a_{ij}(t)x_i(t)dt + \sum_{i=1}^{n} f_{ij}(t, x(t))dB_t^i.$$

Thus, we have

$$\begin{aligned} dy_q(t) &= \textstyle\sum_{i=1}^n \frac{\partial K_{qi}(\tau,t)}{\partial t} x_i(t)dt + \sum_{i=1}^n K_{qi}(\tau,t) \\ &\quad + \Big(\textstyle\sum_{i=1}^n a_{ij}(t)x_i(t)dt + \sum_{i=1}^n f_{ij}(t,x(t))dB_t^i \Big). \end{aligned}$$

From the definition of $K(t,s)$, we have

$$K(\tau,t) = u(\tau)u^{-1}(t).$$

We can see that

$$\begin{aligned} \tfrac{\partial K(\tau,t)}{\partial t} &= u(\tau)\tfrac{\partial u^{-1}(t)}{\partial t} \\ &= -u(\tau)u^{-1}(t)A(t) \\ &= -K(\tau,t)A(t), \end{aligned}$$

and

$$\frac{\partial K_{qi}(\tau,t)}{\partial t} = -\sum_{j=1}^{n} K_{qj}(\tau,t)a_{ji}(t).$$

Hence

$$\begin{aligned} dy_q(t) &= -\textstyle\sum_{i,j=1}^n K_{qi}(\tau,t)a_{ij}(t)x_i(t)dt + \sum_{j,i=1}^n K_{qi}(\tau,t)a_{ij}(t)x_j(t)dt \\ &\quad + \textstyle\sum_{i=1}^n \sum_{j=1}^n K_{qi}(\tau,t)f_{ij}(t,x(t))dB_t^j \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^n K_{qi}(\tau,t)f_{ij}(t,x(t))dB_t^j, \end{aligned}$$

which implies that

$$\begin{aligned} y_q(t) &= y_q(0) + \textstyle\sum_{i=1}^n \sum_{j=1}^n \int_{t_0}^t K_{qi}(\tau,s)f_{ij}(s,x(s))dB_s^j, \\ y(t) &= y(0) + \textstyle\int_0^t K(\tau,s)f(s,x(s))dB_s. \end{aligned}$$

So

$$K(\tau,t)x(t) = K(\tau,0)x_0 + \int_0^t K(\tau,s)f(s,x(s))dB_s.$$

Multiplying $K(t,\tau)$ to the above equality, we obtain

$$K(t,\tau)K(\tau,t)x(t) = K(t,\tau)K(\tau,0)x_0 + \int_0^t K(t,\tau)K(\tau,s)f(s,x(s))dB_s.$$

By using the assertion (i) of Lemma 2.5, we have

$$x(t) = K(t,0)x_0 + \int_0^t K(t,s)f(s,x(s))dB_s.$$

Thus, $x(t)$ is a solution of the equation (1.2).

It is easily to prove the sufficient condition of Theorem by using properties of the Cauchy operator $K(t,s)$ and the multi-dimensional Ito formula. The proof is now complete.

Corollary 3.1. *A solution of the following stochastic differential equation*

$$dx(t) = A(t)x(t)dt + f(t)dB_t,$$

can be expressed in the form

$$x(t) = K(t,0)x_0 + \int_0^t K(t,s)f(s)dB_s.$$

Theorem 3.2. *Consider the stochastic differential equation (1.1). Assume that:*
(i) there exist constants $\alpha, \beta > 0$ such that

$$||K(t,s)|| \leq \beta e^{-\alpha(t-s)}.$$

(ii) the matrix function $f(t,x)$ satisfies Lipschitz condition

$$||f(t,x) - f(t,y)|| < l||x-y||,$$

and $||f(t,x)|| \leq h(t)||x||$, where $h(t)$ is a positive function.
(iii) $\overline{\lim}_{t\to\infty} \frac{1}{t}\int_0^t h^2(s)ds < \frac{\alpha}{\beta^2}$.
Then the trivial solution $x(t,\omega) \equiv 0$ of the equation (1.1) is asymptotically stable in mean square.

Proof. As an application of Theorem (3.1), we can write

$$x(t) = K(t,0)x_0 + \int_0^t K(t,s)f(s,x(s))dB_s.$$

Therefore

$$\begin{aligned}
||x(t)||^2 &= ||K(t,0)x_0 + \textstyle\int_0^t K(t,s)f(s,x(s))dB_s||^2 \\
&\leq 2\Big(||K(t,0)||^2.||x_0||^2 + ||\textstyle\int_0^t K(t,s)f(s,x(s))dB_s||^2\Big), \\
E||x(t)||^2 &\leq 2||K(t,0)||^2.E||x_0||^2 + 2\textstyle\int_0^t ||K(t,s)||^2.E||f(s,x(s))||^2 ds.
\end{aligned}$$

From (i) and (ii), we have

$$E||x(t)||^2 \leq 2\beta^2 e^{-2\alpha t}.E||x_0||^2 + 2\beta^2 \int_0^t e^{-2\alpha(t-s)}.h^2(s)E||x(s))||^2 ds,$$

which implies that

$$e^{2\alpha t}E||x(t)||^2 \leq 2\beta^2.E||x_0||^2 + 2\beta^2 \int_0^t e^{2\alpha s}.h^2(s)E||x(s)||^2 ds.$$

If we put

$$\begin{aligned} \phi(t) &= e^{2\alpha t}E||x(t)||^2, \\ C &= 2\beta^2.E||x_0||^2 > 0, \end{aligned}$$

then we can see that

$$\phi(t) \leq C + 2\beta^2 \int_0^t h^2(s)\phi(s)ds.$$

An application of Gronwall lemma yields

$$\phi(t) \leq C.e^{2\beta^2 \int_0^t h^2(s)ds}.$$

Hence

$$E||x(t)||^2 \leq C.\frac{e^{2\beta^2 \int_0^t h^2(s)ds}}{e^{2\alpha t}},$$

$$\ln E||x(t)||^2 \leq \ln C + 2\beta^2 \int_0^t h^2(s)ds - 2\alpha t.$$

By using the assertion (iii) of Lemma 2.2 we get

$$\overline{\lim_{t\to\infty}} \ln \frac{1}{t} E||x(t)||^2 \leq 2\beta^2 \overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t h^2(s)ds - 2\alpha.$$

Then it follows from the condition (iii) that

$$\overline{\lim_{t\to\infty}} \ln \frac{1}{t} E||x(t)||^2 < 0.$$

Thus, the trivial solution of the equation(1.1) is asymptotically stable in mean square.

Corollary 3.2. *Consider the equation (1.1). Assume that A is a constant matrix whose all eigenvalues have negative real parts, the matrix function $f(t,x)$ satisfies condition (ii) of Theorem 3.2, and*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t h^2(s)ds = 0.$$

Then the trivial solution of the equation (1.1) is asymptotically stable in mean square.

Proof. By Lemma 2.5, there exist constants $\alpha, \beta > 0$ such that

$$||K(t,s)|| \leq \beta e^{-\alpha(t-s)}.$$

Because

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t h^2(s)ds < 0.$$

Therefore

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \int_0^t h^2(s)ds < \frac{\alpha}{\beta^2}.$$

Thus, the trivial solution of the equation (1.1) is asymptotically stable in mean square.

Theorem 3.3. *Consider the stochastic differential equation (1.1). Assume that*
(i) there exist constants $\alpha, \beta > 0$ such that

$$||K(t,s)|| \leq \beta e^{-\alpha(t-s)}.$$

(ii) the matrix function $f(t,x)$ satisfies Lipschitz condition and $||f(t,x)|| \leq h(t)||x||$, where $h(t)$ is a positive function.
(iii) $\overline{\lim}_{t\to\infty} \frac{1}{t} \int_0^t h^{2p}(s)s^{p-1}ds < \frac{2\alpha p}{2^{2p-1}\beta^{2p}[p(2p-1)]^p}$ for every $p \in \mathbb{N}$,
Then the solution $x(t,\omega) \equiv 0$ of the equation (1,1) is asymptotically stable in order 2p.

Proof. As an application of Theorem 3.1, we can write

$$x(t) = K(t,0)X_0 + \int_0^t K(t,s)f(s,x(s))dB_s,$$

which implies that

$$||x(t)||^{2p} \leq 2^{2p-1}||K(t,0)||^{2p}.||x_0||^{2p} + 2^{2p-1}||\int_0^t K(t,s)f(s,x(s))dB_s||^{2p},$$

$$E||x(t)||^{2p} \leq 2^{2p-1}||K(t,0)||^{2p}.E||x_0||^{2p} + 2^{2p-1}E||\int_0^t K(t,s)f(s,x(s))dB_s||^{2p}.$$

Therefore, we have (refer to [2])

$$\begin{aligned} E||x(t)||^{2p} &\leq 2^{2p-1}||K(t,0)||^{2p}.E||x_0||^{2p} \\ &+2^{2p-1}[p(2p-1)]t^{p-1} \int_0^t ||K(t,s)||^{2p}E||f(s,x(s))||^{2p}ds. \end{aligned}$$

From the conditions (i) and (ii), we get

$$\begin{aligned} E||x(t)||^{2p} &\leq 2^{2p-1}\beta^{2p}e^{-2\alpha pt}.E||x_0||^{2p} \\ &+2^{2p-1}\beta^{2p}[p(2p-1)]t^{p-1} \int_0^t e^{-2\alpha p(t-s)}E||x(s))||^{2p}h^{2p}(s)ds. \end{aligned}$$

Now, we can consider t on $[t_0, \infty], t_0 > 0$. If we put

$$\begin{aligned} \varphi(t) &= \tfrac{e^{2\alpha t} E||x(t)||^2 p}{t^{p-1}}. \\ C &= 2^{2p-1}\beta^{2p} E||x_0||^{2p}. \\ D &= 2^{2p-1}\beta^{2p}[p(2p-1)]^p, \end{aligned}$$

then we see that

$$\varphi(t) \leq \frac{C}{t_0^{p-1}} + D\int_{t_0}^t h^{2p}(s)s^{p-1}\varphi(s)ds.$$

An application of Gronwall lemma yields

$$\varphi(t) \leq \frac{C}{t_0^{p-1}} e^{D\int_{t_0}^t h^{2p}(s)s^{p-1}ds}.$$

Hence

$$E||x(t)||^{2p} \leq \frac{Ce^{-2\alpha t}}{t_0^{p-1}} t^{p-1} e^{D\int_{t_0}^t h^{2p}(s)s^{p-1}ds},$$

which implies that

$$\frac{1}{t}\ln E||x(t)||^{2p} \leq \frac{1}{t}\ln(C/t_0^{p-1}) + (p-1)\frac{\ln t}{t} + \frac{D}{t}\int_{t_0}^t h^{2p}(s)s^{p-1}ds - 2\alpha,$$

By applying Lemma 2.2, we obtain

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\ln E||x(t)||^{2p} \leq 2^{2p-1}\beta^{2p}[p(2p-1)]^p.\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t h^{2p}(s)s^{p-1}ds - 2\alpha.$$

The condition (iii) implies that

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\ln E||x(t)||^{2p} < 0.$$

Thus, the trivial solution of the equation (1.1) is asymptotically stable in order $2p$.

Corollary 3.3. *Consider the equation (1.1). Assume that A is a constant matrix whose all eigenvalues have negative real part, the function $f(t,x)$ satisfies the condition (ii) of Theorem (3.3), and*

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t h^{2p}(s)s^{p-1}ds = 0 \quad \textit{for every} \quad p \in \mathbb{N}.$$

Then the trivial solution of the equation (1.1) is asymptotically stable in order $2p$.

Theorem 3.4. *Consider the stochastic differential equation (1.1). Asume that,*
(i) there exist constants $\alpha, \beta > 0$ such that

$$||K(t,s)|| \leq \beta e^{-\alpha(t-s)}.$$

(ii) the matrix function $f(t,x)$ is continuous with respect to t, x, and satisfies Lipschitz condition

$$||f(t,x) - f(t,y)|| < l||x-y||.$$

(iii)

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t e^{2\alpha s}||f(s,0)||^2 ds < 2\alpha.$$

(iv)

$$\alpha > 2l^2.$$

Then the trivial solution $x(t,\omega) \equiv 0$ of the equation (1,1) is asymptotically stable in mean square.

Proof. Applying the Theorem 3.1, we can write

$$x(t) = K(t,0)X_0 + \int_0^t K(t,s)f(s,x(s))dB_s,$$

which implies that

$$E||x(t)||^2 \leq 2||K(t,0)||^2.E||x_0||^2 + 2\int_0^t ||K(t,s)||^2.E||f(s,x(s))||^2 ds.$$

We have

$$\begin{aligned} ||f(t,x)||^2 &= ||f(t,x) - f(t,0) + f(t,0)||^2 \\ &\leq 2||f(t,x) - f(t,0)||^2 + 2||f(t,0)||^2 \\ &\leq 2l^2||x||^2 + 2||f(t,0)||^2. \end{aligned}$$

Hence

$$\begin{aligned} E||x(t)||^2 \quad &\leq 2e^{-2\alpha t}E||x_0||^2 + 4l^2\textstyle\int_0^t e^{-2\alpha(t-s)}E||x(s)||^2 ds \\ &\quad +4\textstyle\int_0^t e^{-2\alpha(t-s)}||f(s,0||^2 ds. \end{aligned}$$

If we put

$$\begin{aligned} \varphi(t) &= e^{2\alpha t}E||x(t)||^2, \\ \psi(t) &= 2E||X_0||^2 + 4\textstyle\int_0^t e^{2\alpha s}||f(s,0||^2 ds. \end{aligned},$$

then we can see that

$$\varphi(t) \leq \psi(t) + 4l^2\int_0^t \varphi(s)ds.$$

By the Gronwall lemma, it yields

$$\begin{aligned}\varphi(t) &\leq \psi(t) + 4l^2 \int_0^t e^{4l^2 s}\psi(s)ds \\ &\leq 2E||x_0||^2 + 4\int_0^t e^{2\alpha s}||f(s,0||^2 ds + 8l^2 E||x_0||^2 \int_0^t e^{4l^2 s} ds \\ &\quad +16l^2 \int_0^t e^{4l^2 s}(\int_0^s e^{2\alpha\tau}||f(\tau,0)||^2 d\tau)ds.\end{aligned}$$

Therefore

$$\begin{aligned}E||x(t)||^2 &\leq 2E||x_0||^2 e^{-2\alpha t} + 8l^2 E||x_0||^2 e^{-2\alpha t}(\tfrac{1}{4l^2}e^{4l^2 t} - 1) \\ &\quad +4e^{-2\alpha t}\int_0^t e^{2\alpha s}||f(s,0||^2 ds \\ &\quad +16l^2 e^{-2\alpha t}\int_0^t e^{4l^2 s}(\int_0^s e^{2\alpha\tau}||f(\tau,0)||^2 d\tau)ds.\end{aligned}$$

Put

$$M_1 = \overline{\lim_{t\to\infty}}\frac{1}{t}\ln\left(8l^2 x_0^2 e^{-2\alpha t}(\frac{1}{4l^2}e^{4l^2 t} - 1)\right),$$

$$M_2 = \overline{\lim_{t\to\infty}}\frac{1}{t}\ln\left(4e^{-2\alpha t}\int_0^t e^{2\alpha s}||f(s,0||^2 ds\right),$$

$$M_3 = \overline{\lim_{t\to\infty}}\frac{1}{t}\ln\left(16l^2 e^{-2\alpha t}\int_0^t e^{4l^2 s}(\int_0^s e^{2\alpha\tau}||f(\tau,0)||^2 d\tau)\right)ds.$$

In taking account of conditions (iii) and (iv) of the theorem and lemma 2.2, we get

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\ln E||x(t)||^2 \leq max(-2\alpha, M_1, M_2, M_3),$$

and $M_1 \leq 4l^2 - 2\alpha < 0, M_2 < 0$. The negativity of M_1 and M_2 implies that of M_3. Therefore

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\ln E||x(t)||^2 < 0.$$

Thus, the trivial solution of the equation (1,1) is asymptotically stable in mean square.

4 Examples

Example 1. We consider the following equation

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}dt + \begin{pmatrix} e^{-t}x_1 & \frac{x_1+x_2}{\sqrt[4]{t+1}} \\ \frac{x_1-x_2}{\sqrt[4]{t+1}} & e^{-t}x_2 \end{pmatrix}\begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix},$$

we have

$$A = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}.$$

We see that $Re(\alpha_1, \alpha_2) = -1 < 0$, where α_1, α_2 are eigenvalues of the matrix A.

$$\|f(t,x)\|^2 = e^{-2t}\|x\|^2 + \frac{2}{\sqrt{t+1}}\|x\|^2 = \Big(e^{-2t} + \frac{2}{\sqrt{t+1}}\Big)\|x\|^2.$$

Therefore

$$h^2(t) = e^{-2t} + \frac{2}{\sqrt{t+1}},$$

which implies that

$$\begin{aligned}\textstyle\int_0^t h^2(s)ds = \int_0^t \Big(e^{-2s} + \frac{2}{\sqrt{s+1}}\Big)ds &= \Big(-\tfrac{1}{2}e^{-2s} + 4\sqrt{s+1}\Big)\Big|_0^t \\ &= -\tfrac{1}{2}e^{-2s} + 4\sqrt{s+1} - \tfrac{7}{2},\end{aligned}$$

and

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t h^2(s)ds = 0.$$

Thus, the solution $x(t,\omega) \equiv 0$ of the above equation is asymptotically stable in mean square.

Example 2. We consider the following equation

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}dt + \begin{pmatrix} e^{-\alpha t}\sqrt{|x_1|.|x_2|} & \frac{\sin t}{\sqrt{t+1}}(x_1+x_2) \\ \frac{\cos t}{\sqrt{t+1}}(x_1-x_2) & e^{-\alpha t}(x_1+x_2) \end{pmatrix}\begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}$$

$(\alpha > 0)$.
Put

$$A = \begin{pmatrix} 1 & -5 \\ 2 & -3 \end{pmatrix}.$$

Let λ_1, λ_2 be eigenvalues of the matrix A. We have

$$Re(\lambda_1, \lambda_2) < 0,$$

$$\|f(t,x)\|^2 \le \Big(\frac{3}{2}e^{-2\alpha t} + \frac{1}{t+1}\Big)\|x\|^2.$$

Therefore

$$h^2(t) = \frac{3}{2}e^{-2\alpha t} + \frac{1}{t+1}.$$

We see that

$$\int_0^t h^2(s)ds = \int_0^t \Big(\frac{3}{2}e^{-2\alpha t} + \frac{1}{t+1}\Big)ds = -\frac{3}{4}e^{-2\alpha t} + \ln(t+1) + \frac{3}{4},$$

hence

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t h^2(s)ds = 0,$$

which implies the asymptotical stability in mean square of the solution $x(t,\omega) \equiv 0$.

Example 3.

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + \begin{pmatrix} e^{-kt}\sin|x_1| & \ln\left(\frac{|x_1+x_2|}{\sqrt{t^n+1}}+1\right) \\ \ln\left(\frac{|x_1-x_2|}{\sqrt{t^n+1}}+1\right) & \sin\left(|x_2|e^{-kt}\right) \end{pmatrix}\begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}$$

$(k>0, n>1)$,
we have

$$\|f(t,x)\|^2 = e^{-2kt}\sin^2|x_1| + \sin^2\left(|x_2|e^{-kt}\right) + \ln^2\left(\frac{|x_1-x_2|}{\sqrt{t^n+1}}+1\right) + \ln^2\left(\frac{|x_1+x_2|}{\sqrt{t^n+1}}+1\right)$$

$$\|f(t,x)\|^2 \le \left(e^{-2kt} + \frac{4}{t^n+1}\right)\|x\|^2.$$

Hence

$$h^2(t) = e^{-2kt} + \frac{4}{t^n+1}.$$

Because $n>1$, we see that

$$\int_0^\infty h^2(s)ds = \int_0^\infty e^{-2ks}ds + 4\int_0^\infty \frac{1}{s^n+1}ds < \infty,$$

which implies that

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t h^2(s)ds = 0.$$

So, the $x(t,\omega) \equiv 0$ of the above equation is asymptotically stable in mean square.

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