

ON THE STABILITY OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH LINEAR DRIFT

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Abstract

The aim of this paper is to prove the equivalence of a stochastic differential equation with linear drift to a stochastic integral equation in an n -dimensional space. The asymptotic p -stability and asymptotic mean square stability of the trivial random solution for the stochastic differential equation are also investigated by using properties of the Cauchy operator.

1 Introduction

In this paper, we consider the following stochastic differential equation in an n -dimensional space

$$dx(t) = A(t)x(t)dt + f(t, x(t))dB_t, \quad (1.1)$$

where $(B_t, t \in R^+)$ is an n -dimensional Brownian motion, $A(t)$ and $f(t, x)$ are a $n \times n$ -matrix function. We prove the equivalence of the equation (1.1) to the following stochastic integral equation

$$x(t) = K(t, t_0)x_0 + \int_{t_0}^t K(t, s)f(s, x(s))dB_s, \quad (1.2)$$

where $K(t, s)$ is the Cauchy operator generated by a homogeneous linear equation of (1.1). From the equation (1.2), we can consider the problem of asymptotic p -stability and asymptotic mean square for the random solution of (1.1) by

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using properties of the Cauchy operator. In section 2 we give some of the basic definitions and lemmas needed for studying the stochastic differential equation (1.1). The section 3 gives main results on the equivalence between equations (1.1) and (1.2), and to give conditions for the asymptotic p-stability, asymptotic mean square stability of the random solution of (1.1). Some examples of application will be given in section 4.

2 Preliminaries

All stochastic processes in this section are supposed to be considered in a complete probability space (Ω, \mathcal{F}, P) . Firstly we recall some following concepts and results needed for our main results.

Definition 2.1. A solution $x_t \equiv 0$ of Equation (1.1) is said to be p-stable ($p > 0$) if for any $\varepsilon > 0$ there exists $r > 0$ such that $\|x_0\| < r$ and $t > t_0$

$$E\|x(t, \omega, t_0, x_0)\|^p < \varepsilon.$$

Definition 2.2. A solution $x_t \equiv 0$ of Equation (1.1) is said to be p-asymptotically stable if it is p-stable and x_0 is small enough then

$$E\|x(t, \omega, t_0, x_0)\|^p \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Definition 2.3. Let $\xi(t, \omega)$ be n-dimensional measurable random process. Define

$$\chi_p[\xi] = \chi_p[\xi(t, \omega)] = \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln E\|\xi(t, \omega)\|^p.$$

We call $\chi_p[\xi]$ the *p-Liapunov exponent of process $\xi(t, \omega)$* .

Lemma 2.1. For any fixed $p > 0$, denote $\phi(p)$ the set of all p-Liapunov exponent of all non-trivial solutions of the following equation

$$\begin{aligned} \frac{dx}{dt} &= G(t, x, \xi) \\ G(t, 0, \xi) &= 0 \\ x(t_0) &= x_0, \quad t \geq 0 \end{aligned}$$

where ξ is a random process.

If $\sup_{t_0, x_0} \phi(p) < 0$, then the solution $x \equiv 0$ of the above equation is p-asymptotically stable.

The proof of this lemma can be found in [5].

Lemma 2.2. Let $\chi_p[\xi_i]$ be the p -Liapunov exponent of processes $\xi_i(t, \omega)$, $0 \leq i \leq n$, we have following assertions:

(i)

$$\chi_p \left[\sum_{0 \leq i \leq n} \xi_i \right] \leq \max_{0 \leq i \leq n} \chi_p[\xi_i],$$

(ii) if $\xi(\omega), \eta(\omega)$ are independent random processes, then

$$\chi_p(\xi\eta) \leq \chi_p(\xi) + \chi_p(\eta),$$

(iii) if c is a positive constant, then

$$\chi_p(c\xi) = \chi_p(\xi).$$

The proof of Lemma 2.2 can be seen directly from the above definitions. We now consider the following ordinary differential equation in an n -dimensional space

$$\begin{aligned} dx_t &= A(t)x_t dt, \\ x(t_0) &= x_0, \end{aligned} \tag{1.3}$$

where $x, x_0 \in \mathbb{R}^n, t \in [t_0, \infty), A(t) = (a_{i,j}(t))_{n \times n}$ with $a_{i,j}(t)$ are continuous functions, for every $t \in [t_0, \infty)$ and $i, j = 1, 2, \dots, n$.

Lemma 2.3. Consider the matrix equation

$$\begin{aligned} du_t &= A(t)u_t dt, \\ u(t_0) &= I, \end{aligned} \tag{1.4}$$

where I is a unit matrix. If $A(t)$ is continuous matrix for every $t \in [t_0, \infty)$ then we have the following assertions

(i) there exists a unique solution $u(t)$ of (1.4).

(ii) there exists an inverse operator $u^{-1}(t)$ and it is a solution of the following matrix equation

$$\begin{aligned} dw_t &= -w_t A(t) dt, \\ w(t_0) &= I. \end{aligned} \tag{1.5}$$

Definition 2.4. We call $K(t, s) = u(t)u^{-1}(s)$, ($t \geq s \geq t_0$) the Cauchy operator of the linear equation (1.3), where $u(t)$ and $u^{-1}(s)$ are the solution of equations (1.4) and (1.5).

We see that, if A is a constant matrix, then $u(t) = e^{At}$ and $u^{-1}(s) = e^{-As}$. Therefore $K(t, s) = e^{A(t-s)}$.

Lemma 2.4. Assume that $K(t, s)$ is the Cauchy operator of the linear equation (1.3), we have the following assertions

- (i) $K(t, t) = I$ for every $t \in [t_0, \infty)$,
- (ii) $K(t, s) = K(t, \tau) \cdot K(\tau, s)$ for every $t_0 \leq s \leq \tau \leq t$.

This Lemma can be proved directly from Definition 2.4 and Lemma 2.3.

Lemma 2.5. Suppose that all eigenvalues of the constant matrix A have negative real parts. Then there exist constants $\alpha, \beta > 0$ such that:

$$|K(t, s)| = |e^{A(t-s)}| < \beta e^{-\alpha(t-s)} \quad \text{for every } t \geq t_0.$$

3 Main results

Let (Ω, \mathcal{F}, P) be a complete probability space. Suppose that $(\mathcal{F}_t, t \in \mathbb{R}^+)$ is a family of increasing subalgebras of \mathcal{F} , i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s < t$. Let B_t be an n -dimensional standard Brownian motion adapted to \mathcal{F}_t . Furthermore, we assume that the Equation (1.1) satisfies the following conditions

- (i) $A(t) = (a_{i,j}(t))_{n \times n}$ is continuous matrix for every $t \in \mathbb{R}^+$ and $i, j = 1, 2, \dots, n$, that is $a_{i,j}(t)$ are continuous functions for every $t \in \mathbb{R}^+$.
- (ii) $f(t, x)$ is matrix function defined and continuous for every $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$.
- (iii) $x(0) = x_0$ is \mathcal{F}_0 - measurable.

Theorem 3.1. With the conditions (i)-(iii), the equation (1.1) is equivalent to the following stochastic integral equation

$$x(t) = K(t, 0)x_0 + \int_0^t K(t, s)f(s, x(s))dB_s, \quad (1.2)$$

where $K(t, s)$ is the Cauchy operator generated by the homogeneous linear equation of the equation (1.1).

Proof. Suppose that $x(t)$ satisfies the equation (1.1). Put

$$y(t) = g(t, x) = K(\tau, t)x(t).$$

Then we have

$$y_q(t) = g_q(t, x(t)) = \sum_{i=1}^n K_{qi}(\tau, t)x_i(t).$$

An application of multi-dimensional Ito formula yields

$$\begin{aligned} dy_q(t) &= \frac{\partial g_q}{\partial t} dt + \sum_{i=1}^n \frac{\partial g_q}{\partial X_i} dx_i \\ &= \sum_{i=1}^n \frac{\partial K_{qi}(\tau, t)}{\partial t} x_i(t) dt + \sum_{i=1}^n K_{qi}(\tau, t) dx_i(t). \end{aligned}$$

It follows from (1.1) that

$$dx_i(t) = \sum_{i=1}^n a_{ij}(t)x_i(t)dt + \sum_{i=1}^n f_{ij}(t, x(t))dB_t^i.$$

Thus, we have

$$\begin{aligned} dy_q(t) &= \sum_{i=1}^n \frac{\partial K_{qi}(\tau, t)}{\partial t} x_i(t)dt + \sum_{i=1}^n K_{qi}(\tau, t) \\ &\quad + \left(\sum_{i=1}^n a_{ij}(t)x_i(t)dt + \sum_{i=1}^n f_{ij}(t, x(t))dB_t^i \right). \end{aligned}$$

From the definition of $K(t, s)$, we have

$$K(\tau, t) = u(\tau)u^{-1}(t).$$

We can see that

$$\begin{aligned} \frac{\partial K(\tau, t)}{\partial t} &= u(\tau) \frac{\partial u^{-1}(t)}{\partial t} \\ &= -u(\tau)u^{-1}(t)A(t) \\ &= -K(\tau, t)A(t), \end{aligned}$$

and

$$\frac{\partial K_{qi}(\tau, t)}{\partial t} = - \sum_{j=1}^n K_{qj}(\tau, t)a_{ji}(t).$$

Hence

$$\begin{aligned} dy_q(t) &= - \sum_{i,j=1}^n K_{qi}(\tau, t)a_{ij}(t)x_i(t)dt + \sum_{j,i=1}^n K_{qi}(\tau, t)a_{ij}(t)x_j(t)dt \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n K_{qi}(\tau, t)f_{ij}(t, x(t))dB_t^j \\ &= \sum_{i=1}^n \sum_{j=1}^n K_{qi}(\tau, t)f_{ij}(t, x(t))dB_t^j, \end{aligned}$$

which implies that

$$\begin{aligned} y_q(t) &= y_q(0) + \sum_{i=1}^n \sum_{j=1}^n \int_{t_0}^t K_{qi}(\tau, s)f_{ij}(s, x(s))dB_s^j, \\ y(t) &= y(0) + \int_0^t K(\tau, s)f(s, x(s))dB_s. \end{aligned}$$

So

$$K(\tau, t)x(t) = K(\tau, 0)x_0 + \int_0^t K(\tau, s)f(s, x(s))dB_s.$$

Multiplying $K(t, \tau)$ to the above equality, we obtain

$$K(t, \tau)K(\tau, t)x(t) = K(t, \tau)K(\tau, 0)x_0 + \int_0^t K(t, \tau)K(\tau, s)f(s, x(s))dB_s.$$

By using the assertion (i) of Lemma 2.5, we have

$$x(t) = K(t, 0)x_0 + \int_0^t K(t, s)f(s, x(s))dB_s.$$

Thus, $x(t)$ is a solution of the equation (1.2).

It is easily to prove the sufficient condition of Theorem by using properties of the Cauchy operator $K(t, s)$ and the multi-dimensional Ito formula. The proof is now complete.

Corollary 3.1. *A solution of the following stochastic differential equation*

$$dx(t) = A(t)x(t)dt + f(t)dB_t,$$

can be expressed in the form

$$x(t) = K(t, 0)x_0 + \int_0^t K(t, s)f(s)dB_s.$$

Theorem 3.2. *Consider the stochastic differential equation (1.1). Assume that:*

(i) *there exist constants $\alpha, \beta > 0$ such that*

$$\|K(t, s)\| \leq \beta e^{-\alpha(t-s)}.$$

(ii) *the matrix function $f(t, x)$ satisfies Lipschitz condition*

$$\|f(t, x) - f(t, y)\| < l\|x - y\|,$$

and $\|f(t, x)\| \leq h(t)\|x\|$, where $h(t)$ is a positive function.

(iii) $\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^2(s)ds < \frac{\alpha}{\beta^2}$.

Then the trivial solution $x(t, \omega) \equiv 0$ of the equation (1.1) is asymptotically stable in mean square.

Proof. As an application of Theorem (3.1), we can write

$$x(t) = K(t, 0)x_0 + \int_0^t K(t, s)f(s, x(s))dB_s.$$

Therefore

$$\begin{aligned} \|x(t)\|^2 &= \|K(t, 0)x_0 + \int_0^t K(t, s)f(s, x(s))dB_s\|^2 \\ &\leq 2 \left(\|K(t, 0)\|^2 \|x_0\|^2 + \left\| \int_0^t K(t, s)f(s, x(s))dB_s \right\|^2 \right), \end{aligned}$$

$$E\|x(t)\|^2 \leq 2\|K(t, 0)\|^2 \cdot E\|x_0\|^2 + 2 \int_0^t \|K(t, s)\|^2 \cdot E\|f(s, x(s))\|^2 ds.$$

From (i) and (ii), we have

$$E\|x(t)\|^2 \leq 2\beta^2 e^{-2\alpha t} \cdot E\|x_0\|^2 + 2\beta^2 \int_0^t e^{-2\alpha(t-s)} \cdot h^2(s) E\|x(s)\|^2 ds,$$

which implies that

$$e^{2\alpha t} E\|x(t)\|^2 \leq 2\beta^2 \cdot E\|x_0\|^2 + 2\beta^2 \int_0^t e^{2\alpha s} \cdot h^2(s) E\|x(s)\|^2 ds.$$

If we put

$$\phi(t) = e^{2\alpha t} E\|x(t)\|^2,$$

$$C = 2\beta^2 \cdot E\|x_0\|^2 > 0,$$

then we can see that

$$\phi(t) \leq C + 2\beta^2 \int_0^t h^2(s) \phi(s) ds.$$

An application of Gronwall lemma yields

$$\phi(t) \leq C \cdot e^{2\beta^2 \int_0^t h^2(s) ds}.$$

Hence

$$E\|x(t)\|^2 \leq C \cdot \frac{e^{2\beta^2 \int_0^t h^2(s) ds}}{e^{2\alpha t}},$$

$$\ln E\|x(t)\|^2 \leq \ln C + 2\beta^2 \int_0^t h^2(s) ds - 2\alpha t.$$

By using the assertion (iii) of Lemma 2.2 we get

$$\overline{\lim}_{t \rightarrow \infty} \ln \frac{1}{t} E\|x(t)\|^2 \leq 2\beta^2 \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^2(s) ds - 2\alpha.$$

Then it follows from the condition (iii) that

$$\overline{\lim}_{t \rightarrow \infty} \ln \frac{1}{t} E\|x(t)\|^2 < 0.$$

Thus, the trivial solution of the equation(1.1) is asymptotically stable in mean square.

Corollary 3.2. *Consider the equation (1.1). Assume that A is a constant matrix whose all eigenvalues have negative real parts, the matrix function $f(t, x)$ satisfies condition (ii) of Theorem 3.2, and*

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^2(s) ds = 0.$$

Then the trivial solution of the equation (1.1) is asymptotically stable in mean square.

Proof. By Lemma 2.5, there exist constants $\alpha, \beta > 0$ such that

$$\|K(t, s)\| \leq \beta e^{-\alpha(t-s)}.$$

Because

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^2(s) ds < 0.$$

Therefore

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^2(s) ds < \frac{\alpha}{\beta^2}.$$

Thus, the trivial solution of the equation (1.1) is asymptotically stable in mean square.

Theorem 3.3. Consider the stochastic differential equation (1.1). Assume that

(i) there exist constants $\alpha, \beta > 0$ such that

$$\|K(t, s)\| \leq \beta e^{-\alpha(t-s)}.$$

(ii) the matrix function $f(t, x)$ satisfies Lipschitz condition and $\|f(t, x)\| \leq h(t)|x|$, where $h(t)$ is a positive function.

(iii) $\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^{2p}(s) s^{p-1} ds < \frac{2\alpha p}{2^{2p-1} \beta^{2p} [p(2p-1)]^p}$ for every $p \in \mathbb{N}$,

Then the solution $x(t, \omega) \equiv 0$ of the equation (1.1) is asymptotically stable in order $2p$.

Proof. As an application of Theorem 3.1, we can write

$$x(t) = K(t, 0)X_0 + \int_0^t K(t, s)f(s, x(s))dB_s,$$

which implies that

$$\|x(t)\|^{2p} \leq 2^{2p-1} \|K(t, 0)\|^{2p} \|x_0\|^{2p} + 2^{2p-1} \left\| \int_0^t K(t, s)f(s, x(s))dB_s \right\|^{2p},$$

$$E\|x(t)\|^{2p} \leq 2^{2p-1} \|K(t, 0)\|^{2p} E\|x_0\|^{2p} + 2^{2p-1} E \left\| \int_0^t K(t, s)f(s, x(s))dB_s \right\|^{2p}.$$

Therefore, we have (refer to [2])

$$\begin{aligned} E\|x(t)\|^{2p} &\leq 2^{2p-1} \|K(t, 0)\|^{2p} E\|x_0\|^{2p} \\ &\quad + 2^{2p-1} [p(2p-1)] t^{p-1} \int_0^t \|K(t, s)\|^{2p} E\|f(s, x(s))\|^{2p} ds. \end{aligned}$$

From the conditions (i) and (ii), we get

$$\begin{aligned} E\|x(t)\|^{2p} &\leq 2^{2p-1} \beta^{2p} e^{-2\alpha pt} E\|x_0\|^{2p} \\ &\quad + 2^{2p-1} \beta^{2p} [p(2p-1)] t^{p-1} \int_0^t e^{-2\alpha p(t-s)} E\|x(s)\|^{2p} h^{2p}(s) ds. \end{aligned}$$

Now, we can consider t on $[t_0, \infty]$, $t_0 > 0$. If we put

$$\begin{aligned}\varphi(t) &= \frac{e^{2\alpha t} E\|x(t)\|^{2p}}{t^{p-1}}. \\ C &= 2^{2p-1} \beta^{2p} E\|x_0\|^{2p}. \\ D &= 2^{2p-1} \beta^{2p} [p(2p-1)]^p,\end{aligned}$$

then we see that

$$\varphi(t) \leq \frac{C}{t_0^{p-1}} + D \int_{t_0}^t h^{2p}(s) s^{p-1} \varphi(s) ds.$$

An application of Gronwall lemma yields

$$\varphi(t) \leq \frac{C}{t_0^{p-1}} e^{D \int_{t_0}^t h^{2p}(s) s^{p-1} ds}.$$

Hence

$$E\|x(t)\|^{2p} \leq \frac{C e^{-2\alpha t}}{t_0^{p-1}} t^{p-1} e^{D \int_{t_0}^t h^{2p}(s) s^{p-1} ds},$$

which implies that

$$\frac{1}{t} \ln E\|x(t)\|^{2p} \leq \frac{1}{t} \ln(C/t_0^{p-1}) + (p-1) \frac{\ln t}{t} + \frac{D}{t} \int_{t_0}^t h^{2p}(s) s^{p-1} ds - 2\alpha,$$

By applying Lemma 2.2, we obtain

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln E\|x(t)\|^{2p} \leq 2^{2p-1} \beta^{2p} [p(2p-1)]^p \cdot \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^{2p}(s) s^{p-1} ds - 2\alpha.$$

The condition (iii) implies that

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln E\|x(t)\|^{2p} < 0.$$

Thus, the trivial solution of the equation (1.1) is asymptotically stable in order $2p$.

Corollary 3.3. *Consider the equation (1.1). Assume that A is a constant matrix whose all eigenvalues have negative real part, the function $f(t, x)$ satisfies the condition (ii) of Theorem (3.3), and*

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^{2p}(s) s^{p-1} ds = 0 \quad \text{for every } p \in \mathbb{N}.$$

Then the trivial solution of the equation (1.1) is asymptotically stable in order $2p$.

Theorem 3.4. Consider the stochastic differential equation (1.1). Assume that,
(i) there exist constants $\alpha, \beta > 0$ such that

$$\|K(t, s)\| \leq \beta e^{-\alpha(t-s)}.$$

(ii) the matrix function $f(t, x)$ is continuous with respect to t, x , and satisfies Lipschitz condition

$$\|f(t, x) - f(t, y)\| < l\|x - y\|.$$

(iii)

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t e^{2\alpha s} \|f(s, 0)\|^2 ds < 2\alpha.$$

(iv)

$$\alpha > 2l^2.$$

Then the trivial solution $x(t, \omega) \equiv 0$ of the equation (1.1) is asymptotically stable in mean square.

Proof. Applying the Theorem 3.1, we can write

$$x(t) = K(t, 0)X_0 + \int_0^t K(t, s)f(s, x(s))dB_s,$$

which implies that

$$E\|x(t)\|^2 \leq 2\|K(t, 0)\|^2 E\|x_0\|^2 + 2 \int_0^t \|K(t, s)\|^2 E\|f(s, x(s))\|^2 ds.$$

We have

$$\begin{aligned} \|f(t, x)\|^2 &= \|f(t, x) - f(t, 0) + f(t, 0)\|^2 \\ &\leq 2\|f(t, x) - f(t, 0)\|^2 + 2\|f(t, 0)\|^2 \\ &\leq 2l^2\|x\|^2 + 2\|f(t, 0)\|^2. \end{aligned}$$

Hence

$$\begin{aligned} E\|x(t)\|^2 &\leq 2e^{-2\alpha t} E\|x_0\|^2 + 4l^2 \int_0^t e^{-2\alpha(t-s)} E\|x(s)\|^2 ds \\ &\quad + 4 \int_0^t e^{-2\alpha(t-s)} \|f(s, 0)\|^2 ds. \end{aligned}$$

If we put

$$\begin{aligned} \varphi(t) &= e^{2\alpha t} E\|x(t)\|^2, \\ \psi(t) &= 2E\|X_0\|^2 + 4 \int_0^t e^{2\alpha s} \|f(s, 0)\|^2 ds. \end{aligned}$$

then we can see that

$$\varphi(t) \leq \psi(t) + 4l^2 \int_0^t \varphi(s) ds.$$

By the Gronwall lemma, it yields

$$\begin{aligned}\varphi(t) &\leq \psi(t) + 4l^2 \int_0^t e^{4l^2 s} \psi(s) ds \\ &\leq 2E\|x_0\|^2 + 4 \int_0^t e^{2\alpha s} \|f(s, 0)\|^2 ds + 8l^2 E\|x_0\|^2 \int_0^t e^{4l^2 s} ds \\ &\quad + 16l^2 \int_0^t e^{4l^2 s} \left(\int_0^s e^{2\alpha \tau} \|f(\tau, 0)\|^2 d\tau \right) ds.\end{aligned}$$

Therefore

$$\begin{aligned}E\|x(t)\|^2 &\leq 2E\|x_0\|^2 e^{-2\alpha t} + 8l^2 E\|x_0\|^2 e^{-2\alpha t} \left(\frac{1}{4l^2} e^{4l^2 t} - 1 \right) \\ &\quad + 4e^{-2\alpha t} \int_0^t e^{2\alpha s} \|f(s, 0)\|^2 ds \\ &\quad + 16l^2 e^{-2\alpha t} \int_0^t e^{4l^2 s} \left(\int_0^s e^{2\alpha \tau} \|f(\tau, 0)\|^2 d\tau \right) ds.\end{aligned}$$

Put

$$\begin{aligned}M_1 &= \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln \left(8l^2 x_0^2 e^{-2\alpha t} \left(\frac{1}{4l^2} e^{4l^2 t} - 1 \right) \right), \\ M_2 &= \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln \left(4e^{-2\alpha t} \int_0^t e^{2\alpha s} \|f(s, 0)\|^2 ds \right), \\ M_3 &= \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln \left(16l^2 e^{-2\alpha t} \int_0^t e^{4l^2 s} \left(\int_0^s e^{2\alpha \tau} \|f(\tau, 0)\|^2 d\tau \right) ds \right).\end{aligned}$$

In taking account of conditions (iii) and (iv) of the theorem and lemma 2.2, we get

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln E\|x(t)\|^2 \leq \max(-2\alpha, M_1, M_2, M_3),$$

and $M_1 \leq 4l^2 - 2\alpha < 0$, $M_2 < 0$. The negativity of M_1 and M_2 implies that of M_3 . Therefore

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \ln E\|x(t)\|^2 < 0.$$

Thus, the trivial solution of the equation (1,1) is asymptotically stable in mean square.

4 Examples

Example 1. We consider the following equation

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + \begin{pmatrix} e^{-t} x_1 & \frac{x_1 + x_2}{\sqrt[3]{t+1}} \\ \frac{x_1 - x_2}{\sqrt[3]{t+1}} & e^{-t} x_2 \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix},$$

we have

$$A = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}.$$

We see that $Re(\alpha_1, \alpha_2) = -1 < 0$, where α_1, α_2 are eigenvalues of the matrix A.

$$\|f(t, x)\|^2 = e^{-2t}\|x\|^2 + \frac{2}{\sqrt{t+1}}\|x\|^2 = \left(e^{-2t} + \frac{2}{\sqrt{t+1}}\right)\|x\|^2.$$

Therefore

$$h^2(t) = e^{-2t} + \frac{2}{\sqrt{t+1}},$$

which implies that

$$\begin{aligned} \int_0^t h^2(s)ds &= \int_0^t \left(e^{-2s} + \frac{2}{\sqrt{s+1}}\right)ds = \left(-\frac{1}{2}e^{-2s} + 4\sqrt{s+1}\right)\Big|_0^t \\ &= -\frac{1}{2}e^{-2s} + 4\sqrt{s+1} - \frac{7}{2}, \end{aligned}$$

and

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^2(s)ds = 0.$$

Thus, the solution $x(t, \omega) \equiv 0$ of the above equation is asymptotically stable in mean square.

Example 2. We consider the following equation

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + \begin{pmatrix} e^{-\alpha t} \sqrt{|x_1| \cdot |x_2|} & \frac{\sin t}{\sqrt{t+1}}(x_1 + x_2) \\ \frac{\cos t}{\sqrt{t+1}}(x_1 - x_2) & e^{-\alpha t}(x_1 + x_2) \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}$$

($\alpha > 0$).

Put

$$A = \begin{pmatrix} 1 & -5 \\ 2 & -3 \end{pmatrix}.$$

Let λ_1, λ_2 be eigenvalues of the matrix A. We have

$$Re(\lambda_1, \lambda_2) < 0,$$

$$\|f(t, x)\|^2 \leq \left(\frac{3}{2}e^{-2\alpha t} + \frac{1}{t+1}\right)\|x\|^2.$$

Therefore

$$h^2(t) = \frac{3}{2}e^{-2\alpha t} + \frac{1}{t+1}.$$

We see that

$$\int_0^t h^2(s)ds = \int_0^t \left(\frac{3}{2}e^{-2\alpha s} + \frac{1}{t+1}\right)ds = -\frac{3}{4}e^{-2\alpha t} + \ln(t+1) + \frac{3}{4},$$

hence

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^2(s)ds = 0,$$

which implies the asymptotical stability in mean square of the solution $x(t, \omega) \equiv 0$.

Example 3.

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + \begin{pmatrix} e^{-kt} \sin |x_1| & \ln \left(\frac{|x_1+x_2|}{\sqrt{t^n+1}} + 1 \right) \\ \ln \left(\frac{|x_1-x_2|}{\sqrt{t^n+1}} + 1 \right) & \sin (|x_2|e^{-kt}) \end{pmatrix} \begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}$$

($k > 0, n > 1$),
we have

$$\begin{aligned} \|f(t, x)\|^2 &= e^{-2kt} \sin^2 |x_1| + \sin^2 (|x_2|e^{-kt}) + \ln^2 \left(\frac{|x_1-x_2|}{\sqrt{t^n+1}} + 1 \right) + \ln^2 \left(\frac{|x_1+x_2|}{\sqrt{t^n+1}} + 1 \right) \\ \|f(t, x)\|^2 &\leq \left(e^{-2kt} + \frac{4}{t^n+1} \right) \|x\|^2. \end{aligned}$$

Hence

$$h^2(t) = e^{-2kt} + \frac{4}{t^n+1}.$$

Because $n > 1$, we see that

$$\int_0^\infty h^2(s) ds = \int_0^\infty e^{-2ks} ds + 4 \int_0^\infty \frac{1}{s^n+1} ds < \infty,$$

which implies that

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t h^2(s) ds = 0.$$

So, the $x(t, \omega) \equiv 0$ of the above equation is asymptotically stable in mean square.

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