

## MININJECTIVITY AND KASCH MODULES

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### Abstract

Let  $R$  be an associate ring with identity. A right  $R$ -module  $M$  is called mininjective if every homomorphism from a simple right ideal of  $R$  to  $M$  can be extended to  $R$ . We now extend this notion to modules. We call a module  $N$  an  $M$ -mininjective module if every homomorphism from a simple  $M$ -cyclic submodule of  $M$  to  $N$  can be extended to  $M$ . In this note, we characterize quasi-mininjective modules and show that for a finitely generated quasi-mininjective module  $M$  which is a Kasch module, there is a bijection between the class of all maximal submodules of  $M$  and the class of all minimal left ideals of its endomorphism ring  $S = \text{End}(M)$  if and only if  $\ell_{SRM}(K) = K$  for any simple left ideal  $K$  of  $S$ . The results obtained by Nihcolson and Yousif in mininjective rings are generalized.

## 1. Introduction

Throughout this paper,  $R$  is an associative ring with identity and  $\text{Mod-}R$  denotes the category of unitary right  $R$ -modules. A right  $R$ -module  $M$  is called

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*principally injective* if any homomorphism from a principal right ideal of  $R$  to  $M$  can be extended to an  $R$ -homomorphism from  $R$  to  $M$ . This notion was first introduced by Camillo [2] for commutative rings. Nicholson and Yousif [9], [10] studied the structure of right p-injective and right mininjective rings. Harada [4] called a right  $R$ -module  $M$  mininjective if every  $R$ -homomorphism from a minimal right ideal of  $R$  to  $M$  is given by a left multiplication on an element of  $M$ . The nice structure of right mininjective and right p-injective rings have drawn our attention to extend these notions to modules. We observe that every principal right ideal  $I$  of a ring  $R$  can be considered as a homomorphic image of  $R$  and vice-versa. We therefore use this fact to generalize the notion of mininjectivity to  $M$ -mininjectivity for a given right  $R$ -module  $M$ .

Let  $M$  be a right  $R$ -module. A right  $R$ -module  $N$  is called  $M$ -principally injective (briefly,  $M$ -p-injective) if every homomorphism from an  $M$ -cyclic submodule of  $M$  to  $N$  can be extended to a homomorphism from  $M$  to  $N$  (see [12]). Equivalently, for any endomorphism  $\varepsilon$  of  $M$ , every homomorphism from  $\varepsilon(M)$  to  $N$  can be extended to a homomorphism from  $M$  to  $N$ .  $N$  is called principally injective (briefly p-injective) if  $N$  is  $R$ -principally injective. In this note, we will introduce the notion of  $M$ -mininjective modules and give some basic properties. Some recent results of Nicholson and Yousif obtained in [10] are generalized.

Let  $M$  be a right  $R$ -module. Then a module  $N$  is called  $M$ -generated if there is an epimorphism  $M^{(I)} \rightarrow N$  for some index set  $I$ . If the set  $I$  is finite, then  $N$  is called *finitely  $M$ -generated*. In particular,  $N$  is called  $M$ -cyclic if it is isomorphic to  $M/L$  for some submodule  $L \subset M$ . As usual, the socle and radical of the module  $M$  are denoted by  $\text{soc}(M)$  and  $\text{rad}(M)$ , respectively. Also, we use the notations  $\ell$  and  $r$  to stand for the left and right annihilators, respectively. All standard notations can be found in the text of Anderson and Fuller [1].

## 2. Mininjectivity

**Definition.** Let  $M$  be a right  $R$ -module. A right  $R$  module  $N$  is called  $M$ -mininjective if for every simple  $M$ -cyclic submodule  $X$  of  $M$ , any homomorphism from  $X$  to  $N$  can be extended to a homomorphism from  $M$  to  $N$ .

Examples of  $M$ -mininjective modules are plenty, for instance, any  $M$ -p-injective module is  $M$ -mininjective. If  $N$  is a module with zero socle, then  $N$  is  $M$ -mininjective and furthermore, if  $M$  has zero radical, then every right  $R$ -module  $N$  is  $M$ -mininjective.

The proof of the following proposition is routine. We therefore omit its proof.

**Proposition 2.1** *Let  $M$  and  $N$  be  $R$ -modules.*

- (1) *If  $N$  is  $M$ -mininjective, then  $N$  is  $X$ -mininjective for any  $M$ -cyclic submodule  $X$  of  $M$ .*
- (2) *If  $N$  is  $M$ -mininjective and  $X \simeq N$ , then  $X$  is  $M$ -mininjective.*

**Proposition 2.2** *Let  $M$  be a right  $R$ -module and  $\{N_i | i \in I\}$  a family of  $M$ -mininjective modules. Then  $\prod_{i \in I} N_i$  is  $M$ -mininjective.*

**Proof** Let  $\varphi : s(M) \rightarrow \prod_{i \in I} N_i$  be a homomorphism with  $s \in S = \text{End}_R(M)$  and  $s(M)$  is simple. Then  $\pi_i \varphi$  is a homomorphism from  $s(M)$  to  $N_i$  for each  $i \in I$ . By hypothesis and by the definition of product, there is  $\bar{\varphi} : M \rightarrow \prod_{i \in I} N_i$  which extends  $\varphi$ , proving our claim.  $\square$

**Proposition 2.3** *Any direct sum of any family of  $M$ -mininjective modules is again  $M$ -mininjective.*

**Proof** Let  $\varphi : s(M) \rightarrow \bigoplus_{i \in I} N_i$  with  $s \in S = \text{End}_R(M)$ , where  $s(M)$  is simple and each  $N_i$  is  $M$ -mininjective. Since  $\varphi s(M)$  is simple, it is contained in a finite direct sum  $\bigoplus_{i \in I_0} N_i$ , where  $I_0$  is a finite subset of  $I$ . Using Proposition 2.2, we can find a homomorphism  $\bar{\varphi} : M \rightarrow \bigoplus_{i \in I} N_i$  which extends  $\varphi$ , as required.  $\square$

The following proposition is clear.

**Proposition 2.4** *Let  $M$  be a right  $R$ -module and  $N$  an  $M$ -mininjective module. If  $N$  is essential in a module  $K$ , then  $K$  is also  $M$ -mininjective.*

### 3. Quasi-mininjective modules

A module  $M$  is said to be *quasi-mininjective* if  $M$  itself is  $M$ -mininjective. A ring  $R$  is called a right self mininjective ring if  $R_R$  is a quasi mininjective module. The proof of the following lemma is straightforward.

**Lemma 3.1** *Every direct summand of a quasi-mininjective module is again quasi-mininjective.*

The following theorem is a characterization theorem for quasi-mininjective modules.

**Theorem 3.2** *Let  $M$  be a right  $R$ -module and  $S = \text{End}(M)$ . Then the following conditions are equivalent.*

- (1)  *$M$  is quasi-mininjective;*
- (2) *If  $s(M)$  is simple,  $s \in S$ , then  $\ell_S(\text{kers}) = Ss$ ;*
- (3) *If  $s(M)$  is simple and  $\text{kers} \subset \text{kert}$ ,  $s, t \in S, t \neq 0$  then  $Ss = St$ ;*
- (4) *If  $s(M)$  is simple and  $\gamma : s(M) \rightarrow M$  is a homomorphism, then  $\gamma s \in Ss$ ;*
- (5)  *$\ell_S(\text{Im}t \cap \text{kers}) = \ell_S(\text{Im}t) + Ss$  for all  $s, t \in S$  and  $s(M)$  is simple.*

**Proof** The proof of this theorem is similar to that given in [12]. However for the sake of completeness, we provide the proof here.

(1)  $\Rightarrow$  (2). For any  $t \in \ell_S(\text{kers})$ , we have  $t(\text{kers}) = 0$ . This implies that  $\text{kers} \subset \text{kert}$ . Let  $s' : M \rightarrow s(M)$  and  $t' : M \rightarrow t(M)$  be the  $R$ -homomorphisms induced by  $s$  and  $t$  respectively and  $\iota_1 : s(M) \rightarrow M$ ,  $\iota_2 : t(M) \rightarrow M$  the embeddings. Since  $s'$  is an epimorphism, there is an  $R$ -homomorphism  $\varphi : s(M) \rightarrow t(M)$  such that  $\varphi s' = t'$ . Furthermore, since  $M$  is quasi-mininjective, there exists an  $R$ -homomorphism  $u : M \rightarrow M$  such that  $u\iota_1 = \iota_2\varphi$ . Hence  $t = us$  and therefore  $t \in Ss$ . This shows that  $\ell_S(\text{ker}(s)) \subset Ss$ . On the other hand, since  $s \in \ell_S(\text{kers})$ , we have  $Ss \subset \ell_S(\text{kers})$ . Thus we have shown that  $Ss = \ell_S(\text{ker}(s))$ .

(2)  $\Rightarrow$  (3). Since  $\text{ker}(s)$  is maximal and  $\text{kers} \subset \text{kert}$ ,  $\text{ker}(t)$  is maximal if  $t \neq 0$  and hence  $t(M)$  must be simple. From  $\text{ker}(t) = \text{ker}(s)$  we have  $\ell_S(\text{kers}) = \ell_S(\text{kert})$ , and thereby  $Ss = St$  by (2).

(3)  $\Rightarrow$  (1). Let  $s' : M \rightarrow s(M)$  be an  $R$ -homomorphism induced by  $s : M \rightarrow M$  and  $\iota_1 : s(M) \rightarrow M$ . Let  $\varphi : s(M) \rightarrow M$ . Then it is clear to see that  $\varphi s'$  is an  $R$ -endomorphism of  $M$  and  $\text{ker}(s) \subset \text{ker}(\varphi s')$ . By (3), we have  $S\varphi s' = Ss$  and therefore  $\varphi s' = us$  for some  $u \in S$ . This shows that  $M$  is quasi-mininjective.

(1)  $\Leftrightarrow$  (4) This part is clear.

(3)  $\Rightarrow$  (5). Let  $u \in \ell_S(\text{Im}(t) \cap \text{kers})$ . Then  $u(\text{Im}(t) \cap \text{ker}(s)) = 0$ . This implies that  $\text{ker}(st) \subset \text{ker}(ut)$ . However it is noted that if  $st = 0$ , then we have  $\text{Im}(t) \subset \text{ker}(s)$ . It hence follows that  $Ss \subset \ell_S(\text{Im}(t))$  and we are done. On the other hand, if  $st \neq 0$ , then  $st(M)$  is simple and by (3), we have  $ut = vst$  for some  $v \in S$ . It follows that  $(u - vs)t = 0$ , and therefore  $u - vs \in \ell_S(\text{Im}(t))$ , i.e.,  $u \in \ell_S(\text{Im}(t)) + Ss$ . This shows that  $\ell_S(\text{Im}(t) \cap \text{kers}) \subset \ell_S(\text{Im}(t)) + Ss$ . Conversely, for any  $x \in \ell_S(\text{Im}(t)) + Ss$ , we can write  $x$  in the form  $x = u + v$ , where  $u(\text{Im}(t)) = 0$  and  $v(\text{ker}(s)) = 0$ . It then follows that  $x \in \ell_S(\text{Im}(t) \cap \text{ker}(s))$ . Thus  $\ell_S(\text{Im}(t)) + Ss = \ell_S(\text{Im}(t) \cap \text{ker}(s))$ .

(5)  $\Rightarrow$  (2). This part is obvious by taking  $t = 1_M$ , the identity map of  $M$ . The cycle of proofs is now complete.  $\square$

If all simple  $M$ -cyclic submodules of a module  $M$  are direct summands (for example,  $M$  has zero socle or  $M$  has zero radical), then  $M$  is quasi-mininjective. In particular, every semiprime ring is right and left mininjective.

The following corollary includes Lemma 1.1 in [10] as its special case.

**Corollary 3.3** *The following conditions are equivalent for a ring  $R$ .*

- (1)  $R$  is right self mininjective;
- (2) If  $kR$  is simple,  $k \in R$ , then  $\ell r(k) = Rk$ ;
- (3) If  $kR$  is simple,  $r(a) \subset r(k)$ ,  $k, a \in R$ ,  $a \neq 0$  then  $Ra = Rk$ ;
- (4) If  $kR$  is simple and  $\gamma : kR \rightarrow R$  is  $R$ -linear, then  $\gamma(k) \in Rk$ ;
- (5) If  $kR$  is simple, then  $\ell(aR \cap r(k)) = \ell(aR) + Rk$  for all  $a, k \in R$ .

The next lemma shows that the conditions  $(C'_2)$  and  $(C'_3)$  which are similar to that of  $(C_1)$  and  $(C_2)$  (see Mohamed and Müller, [8]) also hold in a quasi-mininjective module.

**Proposition 3.4** *Let  $M_R$  be a quasi mininjective module and  $s, t \in S = \text{End}(M_R)$ . Then*

- $(C'_2)$  *If  $K$  is a submodule of  $M$  and  $K \simeq s(M)$  which is simple and  $s^2 = s$ , then  $K = t(M)$  for some  $t^2 = t \in S$ .*
- $(C'_3)$  *If  $s(M) \neq t(M)$  are simple,  $s^2 = s$ ,  $t^2 = t$ , then  $s(M) \oplus t(M) = u(M)$  for some  $u^2 = u \in S$ .*

**Proof**  $(C'_2)$ . Since  $s^2 = s$ ,  $s(M)$  must be a direct summand of  $M$ . Hence,  $s(M)$  is  $M$ -mininjective and so is  $K$ . Therefore  $K$  is a direct summand of  $M$  by Proposition 2.1.

$(C'_3)$ . Let  $s(M) \neq t(M)$  be simple with  $s^2 = s \in S$  and  $t^2 = t \in S$ . Then we have  $s(M) \oplus t(M) = s(M) \oplus (1-s)t(M)$ . If  $(1-s)t = 0$ , then we are done. Otherwise,  $(1-s)t(M) \simeq t(M)$  and by the condition  $C'_2$ , we have  $(1-s)t(M) = u(M)$  for some  $u = u^2 \in S$ . Then  $su = 0$  and hence  $v = s+u-us$  is an idempotent of  $S$  such that  $sv = s = vs$  and  $uv = u = vu$ . It follows that  $s(M) \oplus t(M) = v(M)$ , proving our proposition.  $\square$

We now explore some more properties concerning quasi-mininjective modules. Let  $M$  be a right  $R$ -module and  $S = \text{End}(M_R)$ . Then we consider  $M$  as a left  $S$ -module. We denote  $S_r(M) = \text{soc}(M_R)$  and  $S_\ell(M) = \text{soc}({}_S M)$ . For the sake of convenience, we just write  $\text{soc}_K(M)$  for the homogeneous component of  $M$  containing the simple submodule  $K$ .

According to Wisbauer [13], a right  $R$ -module  $M$  is called a self generator if it generates all its submodules. The following theorem describes the properties of quasi-mininjective modules.

**Theorem 3.5** *Let  $M$  be a quasi-mininjective module and  $s, t \in S = \text{End}(M_R)$ . Then the following statements hold.*

- (1) *If  $s(M)$  is simple, then  $Ss$  is a simple left ideal of  $S$ .*
- (2) *If  $s(M) \simeq t(M)$  are simple, then  $Ss \simeq St$ .*
- (3) *If  $s(M)$  is simple, then  $Ss(M) = \text{soc}_{s(M)}(M_R)$ , a homogeneous component of  $M_R$  containing  $s(M)$ , and  $Ss(M)$  is a simple submodule of left  $S$ -module  $M$ .*
- (4) *If  $M$  is a self generator, then  $S_r(M) \subset S_\ell(M)$ .*

**Proof** (1). We first take any  $0 \neq t \in Ss$ . Then  $t = us$  for some  $u \in S$ . We now show that  $St = Ss$ . Since  $\ker(t) = \ker(us) = s^{-1}(\ker(u))$ , we can see that  $\ker(s) \subset \ker(t)$  and hence by Theorem 3.2, we have  $Ss = St$ . This means that  $Ss$  is a simple left ideal of  $S$ .

(2) Let  $f : s(M) \rightarrow t(M)$  be an isomorphism and  $\iota_1 : s(M) \rightarrow M$  and  $\iota_2 : t(M) \rightarrow M$  be embeddings. Let  $s' : M \rightarrow s(M)$  induced by  $s : M \rightarrow M$  (i.e.,  $\iota_1 s' = s$ ). Since  $M$  is quasi mininjective, it is clear that the homomorphism  $f : s(M) \rightarrow t(M)$  can be extended to  $\bar{f} : M \rightarrow M$  such that  $\bar{f}\iota_1 = \iota_2 f$ . Let  $\sigma : St \rightarrow Ss$  be defined by  $\sigma(ut) = u\bar{f}s$ , for every  $u \in S$ . Then  $\sigma$  is well defined, since  $\text{Im}(\bar{f}s) \subset t(M) = \text{Im}t$ . Moreover, it is trivial to see that  $\sigma$  is an  $S$ -homomorphism. For any  $v \in S$ ,  $v\iota_1 : s(M) \rightarrow M$  can be extended to an  $R$ -homomorphism  $\varphi : M \rightarrow M$  such that  $\varphi\iota_2 f = v\iota_1$ . Consequently, we have  $\sigma(\varphi t) = \varphi\bar{f}s = \varphi\bar{f}\iota_1 s' = \varphi\iota_2 f s' = v\iota_1 s' = vs$ . This shows that  $\sigma$  is an epimorphism. It is clear that  $\sigma$  is a monomorphism, proving (2).

(3) Let  $A = \text{soc}_{s(M)}(M_R)$ . Then we always have  $Ss(M) \subset A$ . Now, let  $Y$  be any simple submodule of  $M_R$  and  $\sigma : s(M) \rightarrow Y$  an isomorphism,  $s \in S$ . Then  $\sigma$  can be extended to  $\bar{\sigma} : M \rightarrow M$  such that  $\bar{\sigma}s(M) = \sigma s(M)$ . Since  $\ker(s) = \ker(\sigma s) = \ker(\bar{\sigma}s)$ , we have  $Ss = S\bar{\sigma}s$  by Theorem 3.2 (3). Hence  $Y = \sigma s(M) = \bar{\sigma}s(M) \subset Ss(M)$ , i.e.,  $A \subset Ss(M)$ . This shows that  $A = Ss(M)$ .

We now show that  $A = Ss(M)$  is a simple left  $S$ -module. For this purpose, we take any submodule  $B$  of  ${}_S M$  such that  $0 \neq B \subset A$ . It is easy to see that if  $X \subset B$  is a simple submodule of  $M_R$ , then  $X_R \simeq s(M)$ . Let  $Y$  be a submodule of  $M_R$  which is isomorphic to  $X$ . Then by letting  $\gamma : X \rightarrow Y$  be an isomorphism, we can find an  $R$ -homomorphism  $\varphi \in S$  such that  $Y = \gamma(X) = \varphi(X) \subset {}_S B$ . This shows that  $B = A$  and therefore  ${}_S A$  is a simple left  $S$ -module.

(4) Since  $M$  is a self generator, every simple submodule  $X$  of  $M$  is of the form  $s(M)$  for some  $s \in S$ . This implies that  $X$  is a subset of  $Ss(M)$  which is a simple left  $S$ -module contained in  $\text{soc}({}_S M)$ . This proves (4).  $\square$

As a corresponding result of Theorem 3.5, we obtain the following result for right self mininjective rings.

**Corollary 3.6** ([10], Theorem 1.14). *Let  $R$  be a right self-mininjective ring. Then*

- (1) *If  $kR$  is simple, then  $Rk$  is a simple left ideal of  $R$ .*
- (2) *If  $kR \simeq mR$  are simple, then  $Rk \simeq Rm$ .*
- (3) *If  $kR$  is simple, then  $RkR$  is a homogeneous component of  $R_R$  containing  $kR$  and  $RkR$  is a simple left ideal of  $R$ .*
- (4)  $\text{soc}(R_R) \subset \text{soc}({}_R R)$ .

## 4. Mininjectivity and Kasch modules

For right  $R$ -modules  $M$  and  $N$ , let  $\text{Hom}_R(N, M)$  be a left  $S$ -module by considering the composition  $tu \in \text{Hom}_R(N, M)$  for every  $u \in \text{Hom}_R(N, M)$ , and  $t \in S$ . Then after some mild modifications of the arguments given in [10], we obtain the following lemma.

**Lemma 4.1** *If  $N = s(M)$ , ( $s \in S = \text{End}(M_R)$ ) and  $T = \ker(s)$ , then  $\text{Hom}_R(N, M) \simeq \ell_S(T) = \ell_S(\ker(s))$ .*

**Proof** Let  $b \in \ell_S(T) = \ell_S(\ker(s))$  and consider  $s$  as an  $R$ -homomorphism from  $M$  to  $s(M)$ . Then  $\ker(s) \subset \ker(b)$  and therefore there exists a unique  $R$ -homomorphism  $\xi_b : N \rightarrow M$  such that  $\xi_b s = b$ . Now, it is easy to see that  $b \mapsto \xi_b$  is an isomorphism  $\ell_S(T) \rightarrow \text{Hom}_R(N, M)$  of left  $S$ -modules.  $\square$

By using Lemma 4.1, we now give a discription for quasi mininjective modules.

**Theorem 4.2** *Let  $M$  be a right  $R$ -module which is a self generator. Then the following conditions are equivalent*

- (1)  $M$  is quasi-mininjective;
- (2)  $\text{Hom}_R(N, M)$  is a simple or zero left  $S$ -module for all simple submodule  $N$  of  $M$ ;
- (3)  $\ell_S(T)$  is simple or zero for all maximal submodule  $T$  of  $M$ .

**Proof** (1)  $\Rightarrow$  (2). Let  $\gamma, \delta \in \text{Hom}_R(N, M)$ , where  $N \simeq M/X$  is a simple submodule of  $M$  and assume that  $\gamma \neq 0$ . Then  $\delta\gamma^{-1} : \gamma(N) \rightarrow M$  is a homomorphism. Since  $\gamma(N)$  is simple,  $\delta\gamma^{-1}$  can be extended to a homomorphism  $\varphi : M \rightarrow M$  such that  $\varphi\iota = \delta\gamma^{-1}$ , where  $\iota : \gamma(N) \rightarrow M$  is the embedding. Hence  $\delta = \varphi\gamma \in \text{Hom}_R(N, M)$ . This shows that  $\text{Hom}_R(N, M)$  is a simple left  $S$ -module.

(2)  $\Rightarrow$  (3). Let  $T$  be a maximal submodule of  $M$ . Then  $M/T$  is a simple right  $R$ -module. Thus, by (2),  $\text{Hom}_R(M/T, M)$  is a simple left  $S$ -module. By Lemma 4.1, we have  $\ell_S(T) \simeq \text{Hom}_R(M/T, M)$  as a left  $S$ -modules. This proves (3).

(3)  $\Rightarrow$  (1). Let  $\gamma : N = s(M) \rightarrow M$  be a homomorphism, where  $s(M)$  is simple,  $s \in S$ ,  $\iota : s(M) \rightarrow M$  the embedding. If  $T = \ker(s)$ , then  $\text{Hom}_R(N, M) \simeq \ell_S(T)$  by Lemma 4.1. This shows that  $\text{Hom}_R(N, M)$  is simple by (3). Thus, we have  $\gamma = \varphi\iota \in \text{Hom}_R(N, M)$  for some  $\varphi \in S$ , proving (1).

By taking  $M_R = R_R$  we can re-obtain the following result of Nicholson and Yousif on mininjective rings in [10].

**Corollary 4.3** *The following conditions are equivalent for a ring  $R$*

- (1)  $R$  is right self mininjective;
- (2)  $\text{Hom}(M, R)$  is simple or zero left ideal of  $R$  for all simple right ideal  $M$  of  $R$ ;
- (3)  $\ell_R(T)$  is a simple or zero left ideal of  $R$  for all maximal right ideal  $T$  of  $R$ .

By a subquotient of a module  $M$ , we mean a module of the form  $X/Y$ , where  $X$  and  $Y$  are submodules of  $M$  with  $Y \subset X$ . Call a right  $R$ -module  $M$

a *Kasch module* if every simple subquotient of  $M$  can be embedded in  $M$ . For a subset  $X \subset \text{Hom}(M, N)$ , we denote  $\ker(X) = \bigcap_{f \in X} \ker(f)$ . It is clear that  $\ker(X) = r_M(X) = \{m \in M \mid Xm = 0\}$ .

**Theorem 4.4** *Let  $M_R$  be a quasi-mininjective module which is a Kasch module. Consider the mapping*

$$\theta : T \mapsto \ell_S(T)$$

*from the set of maximal submodule  $T$  of  $M$  to the set of minimal left ideal of  $S = \text{End}(M_R)$ . Then we have*

- (1)  *$\theta$  is an injection.*
- (2) *If  $M$  is finitely generated, then  $\theta$  is a bijection if and only if  $\ell_S r_M(K) = K$  for all simple left ideals  $K$  of  $S$ . In this case,  $\theta^{-1}$  is given by  $K \mapsto r_M(K)$ .*

**Proof** (1) If  $T$  is a maximal submodule of  $M$ , then  $\ell_S(T) \neq 0$ , since  $M$  is a Kasch module. Hence  $\ell_S(T)$  is simple by Theorem 4.2. Since  $T \subset \ker(\ell_S(T)) \neq M$ , we have  $T = \ker(\ell_S(T))$  because  $T$  is maximal. This shows that  $\theta$  is injective.

(2) If  $\theta$  is surjective and  $K$  is a minimal left ideal of  $S$ , then we can write  $K = \ell_S(T)$ , where  $T$  is maximal in  $M$ . Then  $\ell_S r_M(K) = K$  follows. Conversely, suppose that  $\ell_S r_M(K) = K$  for all simple left ideals  $K$  of  $S$ . Since  $M$  is finitely generated,  $r_M(K) \subset T$  for some maximal submodule  $T$  of  $M$ . and hence  $K = \ell_S r_M(K) \supset \ell_S(T) \neq 0$ , since  $M$  is a Kasch module. Therefore,  $K = \ell_S(T)$  because  $K$  is simple. This leads to  $r_M(K) = r_M \ell_S(T) \supset T$ . Thereby, by the maximality of  $T$  in  $M$ , we have  $r_M(K) = T$ . In other words, we have shown that  $\theta$  is surjective.  $\square$

**Corollary 4.5** ([10], Theorem 3.2) *Let  $R$  be a right mininjective ring which is right Kasch, and consider the map*

$$\theta : T \mapsto \ell(T)$$

*from the set of maximal right ideals  $T$  of  $R$  to the set of minimal left ideals of  $R$ . Then*

- (1)  *$\theta$  is an injection.*
- (2)  *$\theta$  is a bijection if and only if  $\ell r(K) = K$  for all simple left ideals  $K$  of  $R$ . In this case,  $\theta^{-1}$  is given by  $K \mapsto r(K)$ .*

We call a right  $R$ -module *minsymmetric* if  $s(M)$  is simple, and  $s \in S$ , then  $Ss$  is simple.  $R$  is called *right minsymmetric* if  $R_R$  is symmetric as a right  $R$ -module. Clearly, every quasi-mininjective module is minsymmetric by Theorem 3.5, and hence every right self mininjective ring is right symmetric, as every right  $R$ -module with zero socle or zero radical is minsymmetric. We now formulate a characterization theorem for quasi minsymmetric modules.

**Theorem 4.6** *Let  $M$  be a right  $R$ -module. Then  $M$  is minsymmetric if and only if  $s(M)$  is simple, for  $s \in S$  implies that  $\ell_S(s(M) \cap \ker(t)) = \ell_S(s) + St$  for all  $t \in S$ .*

**Proof**  $\Rightarrow$ . Suppose that  $s(M)$  is simple and  $t \in S$ . If  $ts = 0$ , then  $t \in \ell_S(s) = \ell_S(s(M))$ , hence  $St \subset \ell_S(s(M))$ . On the other hand, by  $ts = 0$  we see that  $s(M) \subset \ker(t)$  and therefore  $\ell_S(s(M) \cap \ker(t)) = \ell_S(s(M)) = \ell_S(s)$ . Since  $M$  is minsymmetric,  $Ss$  is simple, and so  $\ell_S(s)$  is a maximal left ideal of  $S$ .

If  $ts \neq 0$ , then  $t \notin \ell_S(s)$  and hence  $\ell_S(s) + St = S$ . But in this case we have  $s(M) \cap \ker(t) = 0$ , since  $s(M)$  is simple. This shows that  $\ell_S(s(M) \cap \ker(t)) = \ell_S(s) + St$  for all  $t \in S$ .

(2)  $\Rightarrow$  (1). Let  $s \in S$  such that  $s(M)$  is simple. Then for any  $t \notin \ell_S(s)$ , we have  $s(M) \cap \ker(t) = 0$ . Since  $\ell_S(s(M) \cap \ker(t)) = \ell_S(s) + St$  for all  $t \in S$ , we have  $\ell_S(s) + St = S$  by (2). This shows that  $\ell_S(s)$  is maximal and hence  $M$  is quasi-mininjective by Theorem 4.2. Now by Theorem 3.5,  $M$  is minsymmetric. This completes the proof.  $\square$

By taking  $M_R = R_R$  again, we see that a ring  $R$  is right minsymmetric if and only if  $\ell_R(kR \cap r_R(a)) = \ell_R(k) + Ra$  for all  $k, a \in R$  with  $kR$  is simple (see [10]).

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## References

- [1] F.W. Anderson and K.R. Fuller, "Rings and Categories of Modules", Graduate Texts in Math. No.13, Springer-Verlag, New York, Heidelberg, Berlin, 1974.
- [2] V. P. Camillo, *Commutative rings whose principal ideals are annihilators* Portugal. Math. **46** (1) (1989), 33-37.
- [3] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, "Extending modules", Pitman, London, 1994.
- [4] M. Harada, *Self mini-injective rings*, Osaka J. Math, **19** (1982), 587-597.
- [5] F. Kasch, "Moduln und Ringe", Stuttgart, 1977.
- [6] S.K. Jain, S.R. López-Permouth and S. Singh, *On a class of QI-rings*, Glasgow Math. J., **34** (1992), 75-81.
- [7] S.R. López-Permouth, *Rings characterized by their weakly-injective modules*, Glasgow Math. J. **34** (1992), 349-353.
- [8] S. H. Mohamed and B. J. Müller, "Continuous and Discrete Modules," Cambridge Univ. Press, 1990.
- [9] W. K. Nicholson and M. F. Yousif, *On principally injective rings*, J. of Algebra, **174** (1995), 77-93.

- [10] W. K. Nicholson and M. F. Yousif, *On mininjective rings*, J. of Algebra, **187** (1997), 548-578.
- [11] E.A. Rutter, Jr., *Ring with the principal extension property*, Comm. in Algebra, **3**(3) (1975), 203-212.
- [12] N.V. Sanh, K.P. Shum, S. Dhompongsa and S. Wongwai, *On quasi-principally injective modules*, Algebra Colloquium **6:3** (1999), 269-276.
- [13] R. Wisbauer, "Foundations of Module and Ring Theory", Gordon and Breach, London, Tokyo, e.a. 1991.