East-West J. of Mathematics: Vol. 5, No 2 (2003) pp. 113-122

MININJECTIVITY AND KASCH MODULES

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Abstract

Let R be an associate ring with identity. A right R-module M is called mininjective if every homomorphism from a simple right ideal of R to M can be extended to R. We now extend this notion to modules. We call a module N an M-mininjective module if every homomorphism from a simple M-cyclic submodule of M to N can be extended to M. In this note, we characterize quasi-mininjective modules and show that for a finitely generated quasi-minjective module M which is a Kasch module, there is a bijection between the class of all maximal submodules of M and the class of all minimal left ideals of its endomorphism ring $S = \mathrm{End}(M)$ if and only if $\ell_S r_M(K) = K$ for any simple left ideal K of S. The results obtained by Nihcolson and Yousif in mininjective rings are generalized.

1. Introduction

Throughout this paper, R is an associative ring with identity and Mod-R denotes the category of unitary right R-modules. A right R-module M is called

Key words: quasi-mininjective, self generator, annihilator, minannihilator, minsymmetric
2000 Mathematics Subject Classification: 16D50, 16D70, 16D80
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† This research is partially supported by UGC (CUHK) grant # 2060123 (2004-2005)

principally injective if any homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. This notion was first introduced by Camillo [2] for commutative rings. Nicholson and Yousif [9], [10] studied the structure of right p-injective and right mininjective rings. Harada [4] called a right R-module M mininjective if every R-homomorphism from a minimal right ideal of R to M is given by a left multiplication on an element of M. The nice structure of right mininjective and right p-injective rings have drawn our attention to extend these notions to modules. We observe that every principal right ideal I of a ring R can be considered as a homomorphic image of R and vice-versa. We therefore use this fact to generalize the notion of mininjectivity to M-mininjectivity for a given right R-module M.

Let M be a right R-module. A right R-module N is called M-principally injective (briefly, M-p-injective) if every homomorphism from an M-cyclic submodule of M to N can be extended to a homomorphism from M to N (see [12]). Equivalently, for any endomorphism ε of M, every homomorphism from $\varepsilon(M)$ to N can be extended to a homomorphism from M to N. N is called principally injective (briefly p-injective) if N is R-principally injective. In this note, we will introduce the notion of M-mininjective modules and give some basic properties. Some recent results of Nicholson and Yousif obtained in [10] are generalized.

Let M be a right R-module. Then a module N is called *M-generated* if there is an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. If the set I is finite, then N is called *finitely M-generated.* In particular, N is called M-cyclic if it is isomorphic to M/L for some submodule $L \subset M$. As usual, the socle and radical of the module M are denoted by $\mathrm{soc}(M)$ and $\mathrm{rad}(M)$, respectively. Also, we use the notations ℓ and r to stand for the left and right annihilators, respectively. All standard notations can be found in the text of Anderson and Fuller [1].

2. Mininjectivity

Definition. Let M be a right R-module. A right R module N is called M-mininjective if for every simple M-cyclic submodule X of M, any homomorphism from X to N can be extended to a homomorphism from M to N.

Examples of M-mininjective modules are plenty, for instance, any M-p-injective module is M-mininjective. If N is a module with zero socle, then N is M-mininjective and furthermore, if M has zero radical, then every right R-module N is M-mininjective.

The proof of the following proposition is routine. We therefore omit its proof.

Proposition 2.1 *Let M and N be R-modules.*

(1) If N is M-mininjective, then N is X-mininjective for any M-cyclic submodule X of M.
(2) If N is M-mininjective and $X \simeq N$, then X is M-mininjective.

Proposition 2.2 *Let M be a right R-module and $\{N_i | i \in I\}$ a family of M-mininjective modules. Then $\prod_{i\in I} N_i$ is M-mininjective.*

Proof Let $\varphi : s(M) \to \prod_{i\in I} N_i$ be a homomorphism with $s \in S = \mathrm{End}_R(M)$ and $s(M)$ is simple. Then $\pi_i\varphi$ is a homomorphism from $s(M)$ to N_i for each $i \in I$. By hypothesis and by the definition of product, there is $\overline{\varphi} : M \to \prod_{i\in I} N_i$ which extends φ, proving our claim. □

Proposition 2.3 *Any direct sum of any family of M-mininjective modules is again M-mininjective.*

Proof Let $\varphi : s(M) \to \oplus_{i\in I} N_i$ with $s \in S = \mathrm{End}_R(M), where s(M)$ is simple and each N_i is M-mininjective. Since $\varphi s(M)$ is simple, it is contained in a finite direct sum $\oplus_{i\in I_0} N_i$, where I_0 is a finite subset of I. Using Proposition 2.2, we can find a homomorphism $\overline{\varphi} : M \to \oplus_{i\in I} N_i$ which extends φ, as required. □

The following proposition is clear.

Proposition 2.4 *Let M be a right R-module and N an M-mininjective module. If N is essential in a module K, then K is also M-mininjective.*

3. Quasi-mininjective modules

A module M is said to be *quasi-mininjective* if M itself is M-mininjective. A ring R is called a right self mininjective ring if R_R is a quasi mininjective module. The proof of the following lemma is straightforward.

Lemma 3.1 *Every direct summand of a quasi-mininjective module is again quasi-mininjective.*

The following theorem is a characterization theorem for quasi-mininjective modules.

Theorem 3.2 *Let M be a right R-module and $S = \mathrm{End}(M)$. Then the following conditions are equivalent.*
(1) M is quasi-mininjective;
(2) If $s(M)$ is simple, $s \in S$, then $\ell_S(\ker s) = Ss$;
(3) If $s(M)$ is simple and $\ker s \subset \ker t$, $s, t \in S, t \neq 0$ then $Ss = St$;
(4) If $s(M)$ is simple and $\gamma : s(M) \to M$ is a homomorphism, then $\gamma s \in Ss$;
(5) $\ell_S(\mathrm{Im} t \cap \ker s) = \ell_S(\mathrm{Im} t) + Ss$ for all $s, t \in S$ and $s(M)$ is simple.

Proof The proof of this theorem is similar to that given in [12]. However for the sake of completeness, we provide the proof here.

(1) $\Rightarrow$ (2). For any $t \in \ell_S(\ker s)$, we have $t(\ker s) = 0$. This implies that $\ker s \subset \ker t$. Let $s' : M \longrightarrow s(M)$ and $t' : M \longrightarrow t(M)$ be the R-homomorphisms induced by s and t respectively and $\iota_1 : s(M) \to M$, $\iota_2 : t(M) \to M$ the embeddings. Since s' is an epimorphism, there is an R-homomorphism $\varphi : s(M) \longrightarrow t(M)$ such that $\varphi s' = t'$. Furthermore, since M is quasi-mininjective, there exists an R-homomorphism $u : M \longrightarrow M$ such that $u\iota_1 = \iota_2\varphi$. Hence $t = us$ and therefore $t \in Ss$. This shows that $\ell_S(\ker(s) \subset Ss$. On the other hand, since $s \in \ell_S(\ker s)$, we have $Ss \subset \ell_S(\ker s)$. Thus we have shown that $S_S = \ell_S(\ker(s))$.

(2) $\Rightarrow$ (3). Since $\ker(s)$ is maximal and $\ker s \subset \ker t$, $\ker(t)$ is maximal if $t \neq 0$ and hence $t(M)$ must be simple. From $\ker(t) = \ker(s)$ we have $\ell_S(\ker s) = \ell_S(\ker t)$, and thereby $Ss = St$ by (2).

(3) $\Rightarrow$ (1). Let $s' : M \longrightarrow s(M)$ be an R-homomorphism induced by $s : M \longrightarrow M$ and $\iota_1 : s(M) \longrightarrow M$. Let $\varphi : s(M) \longrightarrow M$. Then it is clear to see that $\varphi s'$ is an R-endomorphism of M and $\ker(s) \subset \ker(\varphi s')$. By (3), we have $S\varphi s' = Ss$ and therefore $\varphi s' = us$ for some $u \in S$. This shows that M is quasi-mininjective.

(1) $\Leftrightarrow$ (4) This part is clear.

(3) $\Rightarrow$ (5). Let $u \in \ell_S(\mathrm{Im} t \cap \ker s)$. Then $u(\mathrm{Im}(t) \cap \ker(s)) = 0$. This implies that $\ker(st) \subset \ker(ut)$. However it is noted that if $st = 0$, then we have $Im(t) \subset \ker(s)$. It hence follows that $Ss \subset \ell_S(Im(t))$ and we are done. On the other hand, if $st \neq 0$, then $st(M)$ is simple and by (3), we have $ut = vst$ for some $v \in S$. It follows that $(u - vs)t = 0$, and therefore $u - vs \in \ell_S(\mathrm{Im}(t))$, i.e., $u \in \ell_S(\mathrm{Im}(t)) + Ss$. This shows that $\ell_S(\mathrm{Im} t \cap \ker s) \subset \ell_S(\mathrm{Im} t) + Ss$. Conversely, for any $x \in \ell_S(\mathrm{Im}(t)) + Ss$, we can write x in the form $x = u + v$, where $u(\mathrm{Im}(t)) = 0$ and $v(\ker(s)) = 0$. It then follows that $x \in \ell_S(\mathrm{Im}(t) \cap \ker(s))$. Thus $\ell_S(\mathrm{Im}(t)) + Ss = \ell_S(\mathrm{Im}(t) \cap \ker(s))$.

(5) $\Rightarrow$ (2). This part is obvious by taking $t = 1_M$, the identity map of M. The cycle of proofs is now complete. □

If all simple M-cyclic submodules of a module M are direct summands (for example, M has zero socle or M has zero radical), then M is quasi-mininjective. In particular, every semiprime ring is right and left mininjective.

The following corollary includes Lemma 1.1 in [10] as its special case.

Corollary 3.3 *The following conditions are equivalent for a ring R.*

(1) R is right self minijective;
(2) If kR is simple, $k \in R$, then $\ell r(k) = Rk$;
(3) If kR is simple, $r(a) \subset r(k), k, a \in R, a \neq 0$ then $Ra = Rk$;
(4) If kR is simple and $\gamma : kR \to R$ is R-linear, then $\gamma(k) \in Rk$;
(5) If kR is simple, then $\ell(aR \cap r(k)) = \ell(aR) + Rk$ for all $a, k \in R$.

The next lemma shows that the conditions (C_2') and (C_3') which are similar to that of (C_1) and (C_2) (see Mohamed and Müller, [8]) also hold in a quasi-mininjective module.

Proposition 3.4 *Let M_R be a quasi mininjective module and $s, t \in S = \mathrm{End}(M_R)$. Then*

(C_2') *If K is a submodule of M and $K \simeq s(M)$ which is simple and $s^2 = s$, then $K = t(M)$ for some $t^2 = t \in S$.*

(C_3') *If $s(M) \neq t(M)$ are simple, $s^2 = s$, $t^2 = t$, then $s(M) \oplus t(M) = u(M)$ for some $u^2 = u \in S$.*

Proof (C_2'). Since $s^2 = s$, $s(M)$ must be a direct summand of M. Hence, $s(M)$ is M-mininjective and so is K. Therefore K is a direct summand of M by Proposition 2.1.

(C_3'). Let $s(M) \neq t(M)$ be simple with $s^2 = s \in S$ and $t^2 = t \in S$. Then we have $s(M) \oplus t(M) = s(M) \oplus (1-s)t(M)$. If $(1-s)t = 0$, then we are done. Otherwise, $(1-s)t(M) \simeq t(M)$ and by the condition C_2', we have $(1-s)t(M) = u(M)$ for some $u = u^2 \in S$. Then $su = 0$ and hence $v = s + u - us$ is an idempotent of S such that $sv = s = vs$ and $uv = u = vu$. It follows that $s(M) \oplus t(M) = v(M)$, proving our proposition. □

We now explore some more properties concerning quasi-mininjective modules. Let M be a right R-module and $S = \mathrm{End}(M_R)$. Then we consider M as a left S-module. We denote $S_r(M) = \mathrm{soc}(M_R)$ and $S_\ell(M) = soc({}_SM)$. For the sake of convenience, we just write $\mathrm{soc}_K(M)$ for the homogeneous component of M containing the simple submodule K.

According to Wisbauer [13], a right R-module M is called a self generator if it generates all its submodules. The following theorem describes the properties of quasi-mininjective modules.

Theorem 3.5 *Let M be a quasi-mininjective module and $s, t \in S = \mathrm{End}(M_R)$. Then the following statements hold.*

1. *If $s(M)$ is simple, then Ss is a simple left ideal of S.*
2. *If $s(M) \simeq t(M)$ are simple, then $Ss \simeq St$.*
3. *If $s(M)$ is simple, then $Ss(M) = \mathrm{soc}_{s(M)}(M_R)$, a homogeneous component of M_R containing $s(M)$, and $Ss(M)$ is a simple submodule of left S-module M.*
4. *If M is a self generator, then $S_r(M) \subset S_\ell(M)$.*

Proof (1). We first take any $0 \neq t \in Ss$. Then $t = us$ for some $u \in S$. We now show that $St = Ss$. Since $\ker(t) = \ker(us) = s^{-1}(\ker(u))$, we can see that $\ker(s) \subset \ker(t)$ and hence by Theorem 3.2, we have $Ss = St$. This means that Ss is a simple left ideal of S.

(2) Let $f : s(M) \longrightarrow t(M)$ be an isomorphism and $\iota_1 : s(M) \longrightarrow M$ and $\iota_2 : t(M) \longrightarrow M$ be embeddings. Let $s' : M \longrightarrow s(M)$ induced by $s : M \longrightarrow M$ (i.e., $\iota_1 s' = s$). Since M is quasi mininjective, it is clear that the homomorphism $f : s(M) \longrightarrow t(M)$ can be extended to $\bar{f} : M \longrightarrow M$ such that $\bar{f}\iota_1 = \iota_2 f$. Let $\sigma : St \longrightarrow Ss$ be defined by $\sigma(ut) = u\bar{f}s$, for every $u \in S$. Then σ is well defined, since $\mathrm{Im}(\bar{f}s) \subset t(M) = \mathrm{Im}t$. Moreover, it is trivial to see that σ is an S-homomorphism. For any $v \in S$, $v\iota_1 : s(M) \longrightarrow M$ can be extended to an R-homomorphism $\varphi : M \longrightarrow M$ such that $\varphi\iota_2 f = v\iota_1$. Consequently, we have $\sigma(\varphi t) = \varphi\bar{f}s = \varphi\bar{f}\iota_1 s' = \varphi\iota_2 f s' = v\iota_1 s' = vs$. This shows that σ is an epimorphism. It is clear that σ is a monomorphism, proving (2).

(3) Let $A = \mathrm{soc}_{s(M)}(M_R)$. Then we always have $Ss(M) \subset A$. Now, let Y be any simple submodule of M_R and $\sigma : s(M) \to Y$ an isomorphism, $s \in S$. Then σ can be extended to $\bar{\sigma} : M \to M$ such that $\bar{\sigma}s(M) = \sigma s(M)$. Since $\ker(s) = \ker(\sigma s) = \ker(\bar{\sigma}s)$, we have $Ss = S\bar{\sigma}s$ by Theorem 3.2 (3). Hence $Y = \sigma s(M) = \bar{\sigma}s(M) \subset Ss(M)$, i.e., $A \subset Ss(M)$. This shows that $A = Ss(M)$.

We now show that $A = Ss(M)$ is a simple left S-module. For this purpose, we take any submodule B of ${}_SM$ such that $0 \neq B \subset A$. It is easy to see that if $X \subset B$ is a simple submodule of M_R, then $X_R \simeq s(M)$. Let Y be a submodule of M_R which is isomorphic to X. Then by letting $\gamma : X \to Y$ be an isomorphism, we can find an R-homomorphism $\varphi \in S$ such that $Y = \gamma(X) = \varphi(X) \subset {}_SB$. This shows that $B = A$ and therefore ${}_SA$ is a simple left S-module.

(4) Since M is a self generator, every simple submodule X of M is of the form $s(M)$ for some $s \in S$. This implies that X is a subset of $Ss(M)$ which is a simple left S-module contained in $\mathrm{soc}({}_SM)$. This proves (4). □

As a corresponding result of Theorem 3.5, we obtain the following result for right self mininjective rings.

Corollary 3.6 ([10], Theorem 1.14). *Let R be a right self-mininjective ring. Then*

(1) If kR is simple, then Rk is a simple left ideal of R.
(2) If $kR \simeq mR$ are simple, then $Rk \simeq Rm$.
(3) If kR is simple, then RkR is a homogeneous component of R_R containing kR and RkR is a simple left ideal of R.
(4) $\mathrm{soc}(R_R) \subset \mathrm{soc}({}_RR)$.

4. Mininjectivity and Kasch modules

For right R-modules M and N, let $\mathrm{Hom}_R(N, M)$ be a left S-module by considering the composition $tu \in \mathrm{Hom}_R(N, M)$ for every $u \in \mathrm{Hom}_R(N, M)$, and $t \in S$. Then after some mild modifications of the arguments given in [10], we obtain the following lemma.

Lemma 4.1 *If* $N = s(M)$, *(*$s \in S = \mathrm{End}(M_R)$*) and* $T = \ker(s)$, *then* $\mathrm{Hom}_R(N, M) \simeq \ell_S(T) = \ell_S(\ker(s))$.

Proof Let $b \in \ell_S(T) = \ell_S(\ker(s))$ and consider s as an R-homomorphism from M to $s(M)$. Then $\ker(s) \subset \ker(b)$ and therefore there exists a unique R-homomorphism $\xi_b : N \to M$ such that $\xi_b s = b$. Now, it is easy to see that $b \mapsto \xi_b$ is an isomorphism $\ell_S(T) \to \mathrm{Hom}_R(N, M)$ of left S-modules. □

By using Lemma 4.1, we now give a discription for quasi mininjective modules.

Theorem 4.2 *Let* M *be a right* R*-module which is a self generator. Then the following conditions are equivalent*

(1) M *is quasi-mininjective;*

(2) $\mathrm{Hom}_R(N, M)$ *is a simple or zero left* S*-module for all simple submodule* N *of* M*;*

(3) $\ell_S(T)$ *is simple or zero for all maximal submodule* T *of* M*.*

Proof (1) ⇒ (2). Let $\gamma, \delta \in \mathrm{Hom}_R(N, M)$, where $N \simeq M/X$ is a simple submodule of M and assume that $\gamma \neq 0$. Then $\delta\gamma^{-1} : \gamma(N) \to M$ is a homomorphism. Since $\gamma(N)$ is simple, $\delta\gamma^{-1}$ can be extended to a homomorphism $\varphi : M \to M$ such that $\varphi\iota = \delta\gamma^{-1}$, where $\iota : \gamma(N) \to M$ is the embedding. Hence $\delta = \varphi\gamma \in \mathrm{Hom}_R(N, M)$. This shows that $\mathrm{Hom}_R(N, M)$ is a simple left S-module.

(2) ⇒ (3). Let T be a maximal submodule of M. Then M/T is a simple right R-module. Thus, by (2), $\mathrm{Hom}_R(M/T, M)$ is a simple left S-module. By Lemma 4.1, we have $\ell_S(T) \simeq \mathrm{Hom}_R(M/T, M)$ as a left S-modules. This proves (3).

(3) ⇒ (1). Let $\gamma : N = s(M) \to M$ be a homomorphism, where $s(M)$ is simple, $s \in S$, $\iota : s(M) \to M$ the embedding. If $T = \ker(s)$, then $\mathrm{Hom}_R(N, M) \simeq \ell_S(T)$ by Lemma 4.1. This shows that $\mathrm{Hom}_R(N, M)$ is simple by (3). Thus, we have $\gamma = \varphi\iota \in \mathrm{Hom}_R(N, M)$ for some $\varphi \in S$, proving (1).

By taking $M_R = R_R$ we can re-obtain the following result of Nicholson and Yousif on mininjective rings in [10].

Corollary 4.3 *The following conditions are equivalent for a ring* R

(1) R *is right self mininjective;*

(2) $Hom(M, R)$ *is simple or zero left ideal of* R *for all simple right ideal* M *of* R;

(3) $\ell_R(T)$ *is a simple or zero left ideal of* R *for all maximal right ideal* T *of* R*.*

By a subquotient of a module M, we mean a module of the form X/Y, where X and Y are submodules of M with $Y \subset X$. Call a right R-module M

a *Kasch module* if every simple subquotient of M can be embedded in M. For a subset $X \subset Hom(M, N)$, we denote $\ker(X) = \cap_{f \in X} \ker(f)$. It is clear that $\ker(X) = r_M(X) = \{m \in M | Xm = 0\}$.

Theorem 4.4 *Let M_R be a quasi-mininjective module which is a Kasch module. Consider the mapping*

$$\theta : T \mapsto \ell_S(T)$$

from the set of maximal submodule T of M to the set of minimal left ideal of $S = \mathrm{End}(M_R)$. Then we have

(1) θ is an injection.

(2) If M is finitely generated, then θ is a bijection if and only if $\ell_S r_M(K) = K$ for all simple left ideals K of S. In this case, θ^{-1} is given by $K \mapsto r_M(K)$.

Proof (1) If T is a maximal submodule of M, then $\ell_S(T) \neq 0$, since M is a Kasch module. Hence $\ell_S(T)$ is simple by Theorem 4.2. Since $T \subset \ker(\ell_S(T)) \neq M$, we have $T = \ker(\ell_S(T))$ because T is maximal. This shows that θ is injective.

(2) If θ is surjective and K is a minimal left ideal of S, then we can write $K = \ell_S(T)$, where T is maximal in M. Then $\ell_S r_M(K) = K$ follows. Conversely, suppose that $\ell_S r_M(K) = K$ for all simple left ideals K of S. Since M is finitely generated, $r_M(K) \subset T$ for some maximal submodule T of M. and hence $K = \ell_S r_M(K) \supset \ell_S(T) \neq 0$, since M is a Kasch module. Therefore, $K = \ell_S(T)$ because K is simple. This leads to $r_M(K) = r_M \ell_S(T) \supset T$. Thereby, by the maximality of T in M, we have $r_M(K) = T$. In other words, we have shown that θ is surjective. □

Corollary 4.5 ([10], Theorem 3.2) *Let R be a right mininjective ring which is right Kasch, and consider the map*

$$\theta : T \mapsto \ell(T)$$

from the set of maximal right ideals T of R to the set of minimal left ideals of R. Then

(1) θ is an injection.

(2) θ is a bijection if and only if $\ell r(K) = K$ for all simple left ideals K of R. In this case, θ^{-1} is given by $K \mapsto r(K)$.

We call a right R-module *minsymmetric* if $s(M)$ is simple, and $s \in S$, then Ss is simple. R is called *right minsymmetric* if R_R is symmetric as a right R-module. Clearly, every quasi-mininjective module is minsymmetric by Theorem 3.5, and hence every right self mininjective ring is right symmetric, as every right R-module with zero socle or zero radical is minsymmetric. We now formulate a characterization theorem for quasi minsymmetric modules.

Theorem 4.6 *Let M be a right R-module. Then M is minsymmetric if and only if $s(M)$ is simple, for $s \in S$ implies that $\ell_S(s(M) \cap \ker(t)) = \ell_S(s) + St$ for all $t \in S$.*

Proof $\Rightarrow$. Suppose that $s(M)$ is simple and $t \in S$. If $ts = 0$, then $t \in \ell_S(s) = \ell_S(s(M))$, hence $St \subset \ell_S(s(M))$. On the other hand, by $ts = 0$ we see that $s(M) \subset \ker(t)$ and therefore $\ell_S(s(M) \cap \ker(t)) = \ell_S(s(M)) = \ell_S(s)$. Since M is minsymmetric, Ss is simple, and so $\ell_S(s)$ is a maximal left ideal of S.

If $ts \neq 0$, then $t \notin \ell_S(s)$ and hence $\ell_S(s) + St = S$. But in this case we have $s(M) \cap \ker(t) = 0$, since $s(M)$ is simple. This shows that $\ell_S(s(M) \cap \ker(t)) = \ell_S(s) + St$ for all $t \in S$.

(2) $\Rightarrow$ (1). Let $s \in S$ such that $s(M)$ is simple. Then for any $t \notin \ell_S(s)$, we have $s(M) \cap \ker(t) = 0$. Since $\ell_S(s(M) \cap \ker(t)) = \ell_S(s) + St$ for all $t \in S$, we have $\ell_S(s) + St = S$ by (2). This shows that $\ell_S(s)$ is maximal and hence M is quiasi-mininjective by Theorem 4.2. Now by Theorem 3.5, M is minsymmetric. This completes the proof. □

By taking $M_R = R_R$ again, we see that a ring R is right minsymmetric if and only if $\ell_R(kR \cap r_R(a)) = \ell_R(k) + Ra$ for all $k, a \in R$ with kR is simple (see [10]).

Acknowledgment. Nguyen Van Sanh gratefully acknowledges the support of the Faculty of Science, The Chinese University of Hong Kong, for his visit to Hong Kong.

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