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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF LINEAR DEGENERATE DIFFERENCE EQUATIONS

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Abstract

The asymptotic properties of solutions of the linear degenerate discrete equations $Bx_{n+1} = Ax_n$ $(*)$ on a Banach space are considered, where A and B are closed unbounded linear operators from a Banach space X to a Banach space Y. Using a construction of a subspace L of exponentially bounded solutions, an operator T on L such that x_n satisfies Eq.(*) on L if and only if $x_n = T^n x_0$, and using results on almost periodicity of discrete semigroups of operators $\{T^n : n \geq 0\}$, we obtain criteria for asymptotic almost periodicity of solutions of Eq.(*).

1. Introduction.

Consider the difference equation

$$Bx_{n+1} = Ax_n, n = 0, 1, 2, ..., \tag{1}$$

where A and B are closed, densely defined, linear, generally unbounded operators defined on a Banach space X, with values in a Banach space Y. A

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sequence $\{x_n\}_{n=0}^{\infty}$ is called a solution to (1) if $x_n \in D(A) \cap D(B)$ for all $n \geq 0$ and (1) is satisfied.

Because of the presence of B, Eq.(1) is called degenerate, and it is not, in general, well posed even in the finite dimensional case. The study of the solvability and the asymptotic behavior of Eq.(1) is a natural problem which may have important applications, since many equations in physics and engineering have this form. On the other hand, to our knowledge, this problem has not been sufficiently considered in the literature.

In this paper, we study the asymptotic behavior of solutions of Eq.(1). Under a rather mild condition of the uniqueness of exponentially bounded (respectively, bounded) solutions, we associate with Eq.(1) a bounded (resp., a power bounded) operator T on a subspace L of exponentially bounded (resp., bounded) solutions, such that $BT = A$. Using a criterion of almost periodicity of discrete semigroups, we obtain a similar result on asymptotic almost periodicity and stability of a bounded solution to (1).

For the case A, B are bounded operators such that the spectrum of the operator pencil (A, B), $\sigma(A, B)$, is bounded, we show that there is a naturally defined projection operator P on X, and a bounded operator T on X such that $TP = PT$, and that Eq.(1) is equivalent to the non-degenerate equation $x_{n+1} = Tx_n$ on the subspace $E := PX$. Moreover, E coincides with the subspace L of exponentially bounded solutions. This gives a natural splitting of the phase space of Eq.(1) into "degenerate"and 'non-degenerate"subspaces.

2. Generalized spectrum and resolvent.

In this preliminary section, we introduce the resolvent and the spectrum for a pair (A, B) of closed operators and establish their basic properties. For similar results for bounded operators A and B, let us refer the reader to **[5]** and the literature cited therein.

Let A and B be closed, linear operators with dense domains, defined on a Banach space X and with values in a Banach space Y. Assume that $\mathcal{D} := \mathcal{D}(A) \cap \mathcal{D}(B)$ is dense.

Definition 2.1 For a complex number λ such that $(\lambda B - A)$ is one to one on $\mathcal{D}$, define the operator $C_{A,B}(\lambda) : \mathcal{D}(C_{A,B}(\lambda)) \subset X \to X$ as follows:

$$\begin{aligned} \mathcal{D}(C_{A,B}(\lambda)) := \{x \in \mathcal{D} : \ & \textit{there exists a unique} \quad y \in \mathcal{D} \\ & \textit{such that} \ \ Bx = \lambda By - Ay\}, \end{aligned} \tag{2}$$

and

$$C_{A,B}(\lambda)x := y.$$

Furthermore, we introduce the set $\rho(A,B)$ by

$$\rho(A,B) := \{\lambda \in C : C_{A,B}(\lambda) \textit{ is densely defined and is bounded}\}.$$

The set $\rho(A,B)$ is called the *resolvent set* of (A,B) and its complement in C, $\sigma(A,B) := C \setminus \rho(A,B)$, is called the *spectrum* of (A,B).

If $\lambda \in \rho(A,B)$, then we define

$$R_{A,B}(\lambda) := \overline{C_{A,B}(\lambda)} \in B(X), \quad \text{the extension of } C_{A,B}(\lambda) \text{ by continuity}$$

and call it the *generalized resolvent* of (A,B). Note that if A and B are bounded, then $\lambda \in \rho(A,B)$ if and only if $(\lambda B - A)$ is invertible, and $R_{A,B}(\lambda) = (\lambda B - A)^{-1}B$, so that our definition coincides with the well known definition of the resolvent for linear operator pencils (see e.g. **[5]**).

The following lemmas (Lemma 2.2-2.4) are extensions of the well known facts of the operator theory. The proofs are adaptations of the classical proofs to the new situation.

Lemma 2.2 *(i) Assume that $\lambda_0 \in \rho(A,B)$ and μ is a complex number satisfying $|\mu - \lambda_0| < \|C_{A,B}(\lambda_0)\|^{-1}$. Then $\mu \in \rho(A,B)$.*
(ii) $\rho(A,B)$ is an open set.

Proof (i) We show that $(\mu B - A)$ is one-to-one on $\mathcal{D}$. For this, suppose that $y \in \mathcal{D}$ and $\mu By - Ay = 0$. Then $\lambda_0 By - Ay = (\lambda_0 - \mu)By = B[(\lambda_0 - \mu)y]$, hence $(\lambda_0 - \mu)y \in \mathcal{D}(C_{\lambda_0})$ and $(C_{A,B}(\lambda_0)(\lambda_0 - \mu)y = y$, by the definition of $C_{A,B}(\lambda_0)$. Therefore,

$$\|y\| = \|C_{A,B}(\lambda_0)(\lambda_0 - \mu)y\| \le |\lambda_0 - \mu| \|C_{A,B}(\lambda_0)\| \|y\|,$$

which implies $y = 0$. Thus, $(\mu B - A)$ is one-to-one on $\mathcal{D}$.

Next, we show that $C_{A,B}(\mu)$ is densely defined and bounded. For this, consider the set

$$\mathcal{E} := \{\tilde{x} = [I - (\lambda_0 - \mu)R_{A,B}(\lambda_0)]x : x \in \mathcal{D}(C_{A,B}(\lambda_0))\}.$$

Since $[I - (\lambda_0 - \mu)R_{A,B}(\lambda_0)]$ is invertible, it follows that see that $\mathcal{E}$ is a dense linear space. Moreover, $\tilde{x} = x - (\lambda_0 - \mu)y$, where $y = R_{A,B}(\lambda_0)x = C_{A,B}(\lambda_0)x$, so that

$$B\tilde{x} = B[x - (\lambda_0 - \mu)y] = Bx - \lambda_0 By + \mu By = \mu By - Ay.$$

Thus $\tilde{x} \in \mathcal{D}(C_{A,B}(\mu))$, so that $C_{A,B}(\mu)$ is densely defined.

To show that $C_{A,B}(\mu)$ is bounded, let $x \in \mathcal{D}(C_{A,B}(\mu))$ and $y = C_{A,B}(\mu)x$. Then $y \in \mathcal{D}(A) \cap \mathcal{D}(B)$ and

$$\lambda_0 By - Ay = B[x - (\mu - \lambda_0)y],$$

which implies that $[x-(\mu-\lambda_0)y] \in \mathcal{D}(C_{A,B}(\lambda_0))$ and $C_{A,B}(\lambda_0)[x-(\mu-\lambda_0)y] = y$. From this, it follows that

$$R_{A,B}(\lambda_0)x - (\mu - \lambda_0)R_{A,B}(\lambda_0)y = y,$$

or

$$R_{A,B}(\lambda_0)x = y + (\mu - \lambda_0)R_{\lambda_0}y.$$

Since $[I + (\mu - \lambda_0)R_{A,B}(\lambda_0)]$ is invertible, we have

$$y = [I + (\mu - \lambda_0)R_{A,B}(\lambda_0)]^{-1}R_{A,B}(\lambda_0)x,$$

so

$$\|y\| \leq \|[I + (\mu - \lambda_0)R_{A,B}(\lambda_0)]^{-1}R_{A,B}(\lambda_0)\|\|x\|,$$

which implies that $C_{A,B}(\mu)$ is bounded and

$$\|C_{A,B}(\mu)\| \leq \|[I + (\mu - \lambda_0)R_{A,B}(\lambda_0)]^{-1}R_{A,B}(\lambda_0)\|.$$

Statement (ii) follows immediately from (i). □

Lemma 2.3 *The resolvent identity holds:*

$$R_{A,B}(\lambda) - R_{A,B}(\mu) = (\mu - \lambda)R_{A,B}(\lambda)R_{A,B}(\mu), \text{ for all } \lambda, \mu \in \rho(A, B).$$

Proof As in the above, we have $C_{A,B}(\mu)x = [I + (\mu - \lambda)R_{A,B}(\lambda)]^{-1}R_{A,B}(\lambda)x$, for all $x \in \mathcal{D}(C_{A,B}(\mu))$, $\lambda, \mu \in \rho(A, B)$. Therefore

$$[I + (\mu - \lambda)R_{A,B}(\lambda)]C_{A,B}(\mu)x = R_{A,B}(\lambda)x, \text{ for all } x \in \mathcal{D}(C_{A,B}(\mu)),$$

which implies

$$[I + (\mu - \lambda)R_{A,B}(\lambda)]R\mu = R_{A,B}(\lambda),$$

or

$$R_{A,B}(\lambda) - R_{A,B}(\mu) = (\mu - \lambda)R_{A,B}(\lambda)R_{A,B}(\mu).$$

□

Lemma 2.4 $R_{A,B}(\lambda)$ *is an analytic function on* $\rho(A, B)$.

Proof For μ sufficiently close to λ we have

$$[I - (\lambda - \mu)R_{A,B}(\lambda)]^{-1} = \sum_{k=0}^{\infty}(\lambda - \mu)^k R_{A,B}(\lambda)^k,$$

or

$$[I-(\lambda-\mu)R_{A,B}(\lambda)]^{-1}-I=\sum_{k=1}^{\infty}(\lambda-\mu)^k R_{A,B}(\lambda)^k,$$

which converges to 0 as $\mu \to \lambda$. By Lemma 2.3 we have

$$R_{A,B}(\mu)=R_{A,B}(\lambda)[I-(\lambda-\mu)R_{A,B}(\lambda)]^{-1},$$

hence

$$R_{A,B}(\mu)-R_{A,B}(\lambda)=R_{A,B}(\lambda)[I-(\lambda-\mu)R_{A,B}(\lambda)]^{-1}-R_{A,B}(\lambda)\to 0 \text{ as } \mu\to\lambda,$$

so that $\lambda \mapsto R_{A,B}(\lambda)$ is a continuous function. Applying the resolvent identity (Lemma 2.3) again, we obtain that $R_{A,B}(\lambda)$ has derivative, hence is an analytic function on $\rho(A,B)$.

Lemmas 2.3 and 2.4 mean that the generalized resolvent $R_{A,B}(\lambda)$ of (A,B) is in fact *a pseudo-resolvent.* As such, $Ker(R_{A,B}(\lambda))$ and $Image(R_{A,B}(\lambda))$ are independent of λ. If $Ker(R_{A,B}(\lambda))=\{0\}$, then there exists a closed operator T such that $R_{A,B}(\lambda)$ is the resolvent of T, i.e. $R_{A,B}(\lambda)=(\lambda-T)^{-1}$. We refer the reader to ([**6**], p. 428) for related facts about pseudo-resolvents.

3. Solution spaces and almost periodicity.

We now proceed to the study of the asymptotic behavior of Eq.(1). Our approach is to associate with each Eq.(1) a single linear operator T on a generally non-closed linear subspace L, on which solutions of (1) coincide with orbits of T. We introduce in L a new, generally stronger, norm, in which T is bounded and $\sigma(T)\subset\sigma(A,B)$. We then deduct results on the asymptotic behavior of solutiuons of Eq.(1) from the corresponding well known results for asymptotic behavior of orbits of single operators.

To formulate and prove the main results, we introduce some conditions of uniqueness of solutions and give some simple spectral conditions of uniqueness of solutions.

Definition 3.1 (cf. [**1**]). (i) We say that Eq.(1) satisfies the uniqueness condition if for any solution $\{x_n\}$ with $x_0=0$, we have $x_n=0$ for all n.

(ii) We say that Eq.(1) satisfies the uniqueness condition for bounded solutions if for any solution $\{x_n\}$ such that $\sup_{n\geq 0}\|x_n\|<\infty$ and $x_0=0$, we have $x_n=0$ for all n.

A solution $\{x_n\}, n\geq 0,$ is called *exponentially bounded* if there exist $M>0$, and $\omega>0$ such that $\|x_n\|\leq M\omega^n$. Such a solution is called $O(\omega)$-*solution.*

(iii) We say that Eq.(1) satisfies the uniqueness condition for exponentially bounded solutions if for any exponentially bounded solution $\{x_n\}$ such that $x_0 = 0$, we have $x_n = 0$ for all n.

We introduce the following linear subspaces

$$\mathcal{L} := \{x \in X : \text{there exists a solution } \{x_n\}_{n\geq 0} \text{ such that } x_0 = x\},$$

$$L(\omega) := \{x \in X : \text{there exists a } O(\omega) \text{ solution } \{x_n\}_{n\geq 0} \text{ such that } x_0 = x\},$$

and

$$L := \cup_{\omega\in R} L(\omega) = \{x \in X : \text{there exists an exponentially bounded solution } \{x_n\}_{n\geq 0} \text{ such that } x_0 = x\}.$$

The proof of the following proposition is straightforward.

Proposition 3.2 *If Eq.(1) satisfies the uniqueness condition (resp., the uniqueness condition for $O(\omega)$-bounded solutions,...), then $\mathcal{L}$ (resp., $L(\omega)$,...) is a linear subspace of X.*

Proposition 3.3 *Assume either (a) A is a bounded operator, or (b) there exists $\mu \in \rho(B)$ such that $(\mu - B)^{-1}A$ is closed.*

(i) If there is a set Λ such that for every $\omega > 0$, $\Lambda \cap \{z \in \mathbf{C} : |z| > \omega\}$ is not a zero set of an analytic function, and $(\lambda B - A)$ is injective for all $\lambda \in \Lambda$, then Eq.(1) satisfies the uniqueness condition for exponentially bounded solutions.

(ii) If there is a set Λ such that $\Lambda \cap \{z \in \mathbf{C} : |z| > 1\}$ is not a zero set of an analytic function, and $(\lambda B - A)$ is injective for all $\lambda \in \Lambda$, then Eq.(1) satisfies the uniqueness condition for bounded solutions.

Proof (i) Suppose that $\{x_n\}$ is an exponentially bounded solution of Eq.(1), i.e. $\|x_n\| \leq M\lambda_0^n$ for some $\lambda_0, M > 0$, with $x_0 = 0$. We must show that $x_n = 0$ for all $n \geq 0$.

Let $|\lambda| > \lambda_0$ and consider the vector-valued function

$$z(\lambda) = \sum_{k=0}^{\infty} \lambda^{-k} x_k.$$

Assume that (a) holds, i.e. A is a bounded operator. Since the operator B is closed, we have

$$\begin{aligned} Az(\lambda) &= \textstyle\sum_{k=0}^{\infty} \lambda^{-k} Ax_k &&= \textstyle\sum_{k=0}^{\infty} \lambda^{-k} Bx_{k+1} \\ &= \lambda \textstyle\sum_{k=0}^{\infty} \lambda^{-(k+1)} Bx_{k+1} &&= \lambda \textstyle\sum_{k=0}^{\infty} \lambda^{-k} Bx_k - \lambda Bx_0 \\ &= \lambda Bz(\lambda), \end{aligned}$$

hence $(\lambda B - A)z(\lambda) = 0$. This implies, by the injectivity of $(\lambda B - A)$, that $z(\lambda) = 0$ for all $\lambda \in \Lambda \cap \{z : |z| > |\lambda_0|\}$, or $z(\lambda) \equiv 0$, so that $x_n = 0$ for all n.

Now assume that (b) holds, i.e. there exists $\mu \in \rho(B)$ such that $(\mu - B)^{-1}A$ is closed. We have, for every $n \geq 1$,

$$\begin{aligned}
\sum_{k=0}^{n} \lambda^{-k} A x_k &= \sum_{k=0}^{n} \lambda^{-k} B x_{k+1} = \lambda \sum_{k=1}^{n+1} \lambda^{-k} B x_k \\
&= \lambda \sum_{k=0}^{n+1} \lambda^{-k} B x_k - \lambda B x_0 = \lambda B \sum_{k=0}^{n+1} \lambda^{-k} x_k - \lambda B x_0 \\
&= \lambda(B-\mu) \sum_{k=0}^{n+1} \lambda^{-k} x_k + \lambda\mu \sum_{k=0}^{n+1} \lambda^{-k} x_k - \lambda(B-\mu)x_0 - \lambda\mu x_0.
\end{aligned} \tag{3}$$

Applying $(B-\mu)^{-1}$ to both parts of (3) and taking into account that $x_0 = 0$, we have

$$\begin{aligned}
(B-\mu)^{-1} &\left[\sum_{k=0}^{n} \lambda^{-k} A x_k\right] = \\
&= \lambda \left[\sum_{k=0}^{n+1} \lambda^{-k} x_k\right] + \lambda\mu(B-\mu)^{-1} \left[\sum_{k=0}^{n+1} \lambda^{-k} x_k\right] - \lambda x_0 - \lambda\mu(B-\mu)^{-1} x_0 \\
&= \lambda \left[\sum_{k=0}^{n+1} \lambda^{-k} x_k\right] + \lambda\mu(B-\mu)^{-1} \left[\sum_{k=0}^{n+1} \lambda^{-k} x_k\right]
\end{aligned}$$

Since the right hand side converges and the operator $(B-\mu)^{-1}A$ is closed, it follows that $z(\lambda) \in \mathcal{D}((B-\mu)^{-1}A)$ and

$$(B-\mu)^{-1} A z(\lambda) = \lambda z(\lambda) + \lambda\mu(B-\mu)^{-1} z(\lambda),$$

hence $z(\lambda) \in \mathcal{D}(B)$ and

$$Az(\lambda) = \lambda B z(\lambda),$$

or $(\lambda B - A)z(\lambda) = 0$. This implies, by the injectivity of $(\lambda B - A)$, that $z(\lambda) \equiv 0$, or $x_n = 0$ for all n.

The proof of (ii) is analogous.

The results in the next corollary are immediate from Proposition 3.3 and from the fact that $\rho(A, B)$ is an open set.

Corollary 3.4 *Assume that condition (a) or (b) in Proposition 3.3 holds.*

(i) If $\sigma(A, B)$ is bounded, then Eq.(1) satisfies the uniqueness condition for exponentially bounded solutions.

(ii) If $\sigma(A, B) \subset \{z : |z| \leq 1\}$ or $\rho(A, B) \cap \{z : |z| = 1\} \neq \emptyset$, then Eq.(1) satisfies the uniqueness condition for bounded solutions.

Definition 3.5 A complex number λ is said to be in the point spectrum $\sigma_p(A, B)$ if there exists $x \in \mathcal{D}(A) \cap D(B), x \neq 0$, such that $(\lambda B - A)x = 0$.

A complex number λ is said to be in the approximate point spectrum $\sigma_a(A,B)$ if there exist $\{x_n\}_{n=1}^{\infty}$, $0<\delta\le\|x_n\|\le\alpha$ for all n, $x_n\in\mathcal{D}$, and $\{y_n\}_{n=1}^{\infty}$, such that $y_n\in\mathcal{D}$, $\|y_n\|\to 0$ as $n\to\infty$, and $(\lambda Bx_n-Ax_n)=By_n$. Equivalently, $\lambda\in\sigma_a(A,B)$ if for every $\varepsilon>0$ there exist $x,y\in\mathcal{D}$ such that $\|x\|=1$, $\|y\|<\varepsilon$ and $\lambda Bx-Ax=By$.

Lemma 3.6 $\sigma_a(A,B)\subset\sigma(A,B)$.

Proof Assuming the contrary, there exists $\lambda\in\sigma_a(A,B)$ such that $\lambda\notin\sigma(A,B)$. Then, there exist $x_n,y_n,n=1,2,...$, such that $(\lambda Bx_n-Ax_n)=By_n$, $y_n\to 0$, and $0<\delta\le\|x_n\|\le\alpha<\infty$. By the definition, $y_n\in\mathcal{D}(C_{A,B}(\lambda))$ and $R_{A,B}(\lambda)y_n=C_{A,B}(\lambda)y_n=x_n$. This implies that $\|x_n\|\le\|R_{A,B}(\lambda)\|\|y_n\|\to 0$, a contradiction. □

Assume that Eq.(1) satisfies the uniqueness condition for exponentially bounded solutions. We introduce a construction, which associates with Eq.(1) a linear subspace with a new norm on which solutions of (1) are orbits of a bounded linear operator.

Recall that $L(\omega)$ consists off x such that there exists a solution $\{x_n\}$ of Eq.(1) with $x_0=x$ and $\|x_n\|/\omega^n$ are uniformly bounded.

Define, for every $\omega>0$, a new norm $|||\cdot|||$ in $L(\omega)$ by:

$$|||x|||:=\sup_{n\ge 0}\{\|x_n\|/\omega^n : n\ge 0\},\ x\in L(\omega).$$

Then $L(\omega)$ is normed space with the new norm $|||\cdot|||$. Note that $\|x\|\le|||x|||$, so that the embedding $L(\omega)\to X$ is continuous, but in general the two norms are not equivalent.

Note that if A and B are bounded, then, as can be easily seen, $(L(\omega),|||\cdot|||)$ is a complete normed space. But in general $(L(\omega),|||\cdot|||)$ is not complete.

Definition 3.7 Define a linear operator T_ω on $L(\omega)$ by

$$T_\omega x=x_1,$$

where $\{x_n\}$ is the solution of Eq.(1) in $L(\omega)$ with $x_0=x$.

Note that T_ω is a bounded linear operator on $(L(\omega),|||\cdot|||)$ and it is easy to verify that $|||T_\omega|||\le\omega$ and such that $BT_\omega z=Az$ for all $z\in L(\omega)$. We summarize these facts in the following theorem.

Theorem 3.8 *T_ω is a bounded linear operator in the norm space $(L(\omega),|||\cdot|||)$ with $|||T_\omega|||\le\omega$ and satisfies $BT_\omega z=Az$ for all $z\in L(\omega)$.*

Below we denote by $\widehat{L}_\omega$ the completion of $(L(\omega),|||\cdot|||)$ and by $\widehat{T}_\omega$ the extension of T_ω from $L(\omega)$ to $\widehat{L}(\omega)$ by continuity.

The following theorem shows that, if A and B are bounded operators, then $L(\omega)$ is invariant subspace for $R_{A,B}(\lambda)$ and on $L(\omega)$, $R_{A,B}(\lambda)$ coincides with the resolvent of the operator T_ω.

Theorem 3.9 *Assume that A and B are bounded. Then the following statements hold:*

(i) $L(\omega)$ is invariant subspace for the resolvent $R_{A,B}(\lambda)$. The operator $R_{A,B}(\lambda)$ is a bounded in the norm $|||\cdot|||$ and $|||R_{A,B}(\lambda)||| \leq \|R_{A,B}(\lambda)\|$, so that it can be extended by continuity to an operator $\widehat{R}_{A,B}(\lambda)$ on $\widehat{L}(\omega)$.

(ii) For every $x \in L(\omega)$ and $\lambda \in \rho(A,B)$, we have $R_{A,B}(\lambda)x = (\lambda - T_\omega)^{-1}x$.

(iii) $\widehat{R}_{A,B}(\lambda) = (\lambda I - T\omega)^{-1}$ for all $\lambda \in \rho(T_\omega) \cap \rho(A,B)$.

Proof (i) Assume that $x \in L(\omega)$, so that there exists in $L(\omega)$ a $O(\omega)$-solution $\{x_n\}_{n\geq 0}$ of Eq.(1) with $x_0 = x$. From the equality $Bx_{n+1} = Ax_n$ it follows that

$$\begin{aligned} BR_{A,B}(\lambda)x_{n+1} &= B(\lambda B - A)^{-1}Bx_{n+1} = B(\lambda B - A)^{-1}Ax_n \\ &= B\left[\lambda B - A)^{-1}(A - \lambda B)x_n\right] + \lambda B(\lambda B - A)^{-1}x_n \\ &= -Bx_n + \lambda B(\lambda B - A)^{-1}Bx_n = \left[-I + \lambda B(\lambda B - A)^{-1}\right]Bx_n \\ &= \left[-I + (\lambda B - A)(\lambda B - a)^{-1} + A(\lambda B - A)^{-1}\right]Bx_n \\ &= A(\lambda B - A)^{-1}Bx_n = AR(\lambda)x_n. \end{aligned}$$

So that $y_n := R_{A,B}(\lambda)x_n, n \geq 0$, is a solution of Eq.(1). Clearly, $\|y_n\| \leq \|R_{A,B}(\lambda)\|\|x_n\| \leq M\omega^n$, so that y_n belong to $L(\omega)$. In particular, $R_{A,B}(\lambda)x \in L(\omega)$.

Furthermore, we have, for every $x \in L(\omega)$,

$$\begin{aligned} |||R_{A,B}(\lambda)x||| &= \sup_{n\geq 0}\|R_{A,B}(\lambda)x_n\|/\omega^n \leq \|R_{A,B}(\lambda)\|\sup_{n\geq 0}\|x_n\|/\omega^n \\ &= \|R_{A,B}(\lambda)\| |||x|||, \end{aligned}$$

which implies that $R_{A,B}(\lambda)$ is a bounded in the norm $|||\cdot|||$ and $|||R_{A,B}(\lambda)||| \leq \|R_{A,B}(\lambda)\|$.

(ii) We have, for every λ such that $|\lambda| > \|T_\omega\|$ and $x \in L(\omega)$,

$$(\lambda I - T_\omega)^{-1}x = \lambda^{-1}\sum_{k=0}^{\infty}\lambda^{-k}T_\omega^k x = \lambda^{-1}\sum_{k=0}^{\infty}\lambda^{-k}x_k.$$

Therefore

$$A(\lambda I - T_\omega)^{-1}x = \lambda^{-1}\sum_{k=0}^{\infty}\lambda^{-k}Ax_k = \sum_{k=0}^{\infty}\lambda^{-(k+1)}Bx_{k+1} \tag{4}$$

and

$$\lambda B(\lambda I - T_\omega)^{-1}x = \sum_{k=0}^{\infty} \lambda^{-k} Bx_k \tag{5}$$

From (4) and (5) it follows that

$$(\lambda B - A)(\lambda I - T_\omega)^{-1}x = \sum_{k=0}^{\infty} \lambda^{-k} Bx_k - \sum_{k=0}^{\infty} \lambda^{-(k+1)} Bx_{k+1} = Bx_0 = Bx.$$

Hence

$$(\lambda I - T_\omega)^{-1}x = (\lambda B - A)^{-1}Bx = R_{A,B}(\lambda)x.$$

(iii) follows immediately from (ii). □

Proposition 3.10 *Assume that Eq.(1) satisfies the uniqueness condition for exponentially bounded solutions. Then* $\sigma_a(T_\omega) \subset \sigma_a(A,B) \cap \{z \in \mathbf{C} : |z| \leq \omega\}$.

Proof

Let $\lambda \in \sigma_a(T_\omega)$. Then, for every $\varepsilon > 0$ there exists $z \in L(\omega)$ such that $|||z||| = 1$ and $|||T_\omega z - \lambda z||| < \varepsilon$.

Let $\{z_n\}$ be solution of Eq.(1) in $L(\omega)$ with $z_0 = z$ and put $y_n = Tz_n - \lambda z_n = z_{n+1} - \lambda z_n$. Then, as can be easily seen, $\{y_n\}$ is a solution in $L(\omega)$ of Eq.(1) and $y_0 = z_1 - \lambda z_0 = T_\omega z - \lambda z$. Since

$$\sup_{n\geq 0} \|y_n\|/\omega^n \equiv |||y_0||| = |||T_\omega z - \lambda z||| < \varepsilon,$$

we have $\|y_n/\omega^n\| < \varepsilon$ for all $n \geq 0$. Since $|||z||| = \sup_{n\geq 0} \|z_n\|/\omega^n = 1$, there is n such that $1/2 < \|z_n/\omega^n\| \leq 1$. Let $u = z_n/\omega^n$ and $v = -y_n/\omega^n$. Then $1/2 < \|u\| \leq 1$ and $\|v\| < \varepsilon$. On the other hand,

$$(\lambda B - A)u = (\lambda Bz_n - Az_n)/\omega^n = (\lambda Bz_n - Bz_{n+1})/\omega^n = -B(y_n/\omega^n) = Bv.$$

Hence $\lambda \in \sigma_a(A,B)$. □

Remark 3.11: (i) Since $\|T_\omega\| \leq \omega$, we have $\sigma(T_\omega) \subset \{z \in \mathbf{C} : |z| \leq \omega\}$. On the other hand, $\partial\sigma(T_\omega) \subset \sigma_a(T_\omega)$. Therefore, Proposition 3.10 implies that $\sigma(T_\omega) \cap \{z : |z| = \omega\} \subset \sigma(A,B)$.

(ii) If $\lambda \in \sigma_p(T_\omega)$ and there exists an eigenvector z such that $z \in L(\omega)$ and $T_\omega z = \lambda z$, then $\lambda \in \sigma_p(A,B)$. In fact, by Theorem 3.8, $(\lambda B - A)z = \lambda Bz - BT_\omega z = B(\lambda z - T_\omega z) = 0$. But in general we cannot claim that $\sigma_p(T_\omega) \subset \sigma_p(A,B)$, since there might not exist an eigenvector in $L(\omega)$.

(iii) The converse statement to (ii) always holds in the following form: $\sigma_p(A,B) \cap \{z : |z| \leq \omega\} \subset \sigma_p(T_\omega)$. In fact, if $\lambda \leq \omega$ and $(\lambda B - A)z = 0$, then $B(\lambda^{n+1}z) - A(\lambda^n z) = \lambda^n(\lambda B - A)z = 0$, so that $z \in L(\omega)$ and $T_\omega z = \lambda z$.

From Proposition 3.10 we obtain the following criterion for non-emptyness of the spectrum $\sigma(A,B)$.

Corollary 3.12 *Assume that Eq.(1) satisfies the uniqueness condition for exponentially bounded solutions and that there exists at least one such nonzero solution. Then* $\sigma(A,B)\neq\emptyset$.

Theorem 3.13 *Assume that Eq.(1) satisfies the uniqueness condition for exponentially bounded solutions and* $\sigma(A,B)\subset\{z\in\mathbf{C}:|z|\leq\rho\}$, *and let* $\{x_n\}_{n\geq 0}$ *be an exponentially bounded solution of Eq.(1). Then for every* $\varepsilon>0$ *there exists* $M>0$ *such that* $\|x_n\|\leq M(\rho+\varepsilon)^n$. *In particular, if* $\sigma(A,B)\subset\{z\in\mathbf{C}:|z|<1\}$ *then every exponentially bounded solution of Eq.(1) converge to zero exponentially (as* $n\to\infty$*).*

Proof By Proposition 3.10 and Lemma 3.5, $\sigma_a(T_\rho)\subset\sigma(A,B)\subset\{z\in\mathbf{C}:|z|\leq\rho\}$, hence $\sigma(T_\rho)\subset\{z\in\mathbf{C}:|z|\leq\rho\}$. This implies that for every $\varepsilon>0$ there exists $M_0>0$ such that $\|T_\rho^n\|\leq M_0(\rho+\varepsilon)^n$. In particular, $\|x_n\|\leq|||x_n|||=|||T_\rho^n x_0|||\leq M_0(\rho+\varepsilon)^n|||x_0|||=M(\rho+\varepsilon)^n$, where $M=M_0|||x_0|||$. □

For power bounded linear operators on a Banach space E (i.e. operators satisfying $\sup_{n\geq 0}\{\|T^n\|\}<\infty$) there have been established well known results (see [9], [8]) on stability and almost periodicity of the orbits $T^n x$ for all $x\in E$. Recall that a sequence $\mathbf{x}=\{x_n\}_{n\geq 0}$ is called asymptotically almost periodic, if the family of translates $\mathbf{x}_i=\{x_{n+i}\}_{n\geq 0}$ is relatively compact in $l^\infty(\mathbf{Z}_+,X)$ (the space of X-valued sequences with sup-norm). An operator T is said to generate an almost periodic discrete semigroup, if every orbit $\{T^n x:n\geq 0\}$ is relatively compact, or, equivalently, every sequence $\mathbf{x}=\{T^n x\}_{n\geq 0}$ is asymptotically almost periodic. The main theorem in [8], when applied to the semigroup $\mathbf{Z}_+$ of non-negative integers, states that if T is a power bounded operator on a Banach space such that $\sigma(T)\cap\{z:|z|=1\}$ is countable, and if the discrete semigroup $\{T^n:n\geq 0\}$ if totally ergodic (i.e. the sequence $\frac{1}{n}\sum_{k=j}^{n-1+j}\lambda^{-k}T^k x$ converges uniformly in j), then T generates an almost periodic semigroup $\{T^n:n\geq 0\}$ (see also [9] for details). Using Theorem 3.8 and Proposition 3.10, we can extend these results to solutions of Eq.(1).

Theorem 3.14 *Assume* $\sigma(A,B)\cap\{|z|=1\}$ *is countable. Then every bounded solution* x_n *of Eq.(1) is asymptotically almost periodic if (and only if) the means* $\frac{1}{n}\sum_{k=j}^{n-1+j}\lambda^{-k}x_k$ *converge uniformly in* j, *for all* $\lambda\in\sigma(A,B)\cap\{|z|=1\}$. *In addition, if the above means converge to* 0, *for all* $\lambda\in\sigma(A,B)\cap\{|z|=1\}$, *then* $\lim_{n\to\infty}\|x_n\|=0$.

Proof Since $\{x_n\}$ is a bounded solution of (1), we have $x_n\in L(1)$, and $x_{n+1}=Tx_n, -\infty<n<\infty$. Let $T:=T_1$ be the contraction defined on $L(1)$ as in Theorem 3.8 (with $\omega=1$). By Lemma 3.6 and Proposition 3.10,

$\sigma(T_1) \cap \{z \in C : |z| = 1\}$ is countable. Assume that

$$y_n^{(j)} := \frac{1}{n} \sum_{k=j}^{n-1+j} \lambda^{-k} x_k$$

converge uniformly in j, in the norm of X. Then for every $\varepsilon > 0$ there exists N such that $\sup_{j \geq 1} \|y_n^{(j)} - y_m^{(j)}\| < \varepsilon$ whenever $n, m > N$. Then

$$|||y_n^{(j)} - y_m^{(j)}||| = \sup_{i \geq 0} \| \sum_{k=n+j}^{m-1+j} \lambda^{-k} x_{k+i} \| = \sup_{i \geq 0} \|\lambda^i \sum_{k=n+j+i}^{m-1+j+i} \lambda^{-k-i} x_{k+i}\| < \varepsilon,$$

uniformly in j.

Let M be a subspace of $L(1)$, which consists of such vectors z that

$$\frac{1}{n} \sum_{k=0}^{n-1} \lambda^{-k} T^k z$$

converges for all $\lambda \in \sigma(T) \cap \{z : |z| = 1\}$. Then M is a closed invariant subspace of T and $T|M$ is ergodic. Consequently, by the Almost Periodicity Theorem proved in [9] (see also [8], [4]), the operator $T|M$ generates an almost periodic (discrete) semigroup. In particular, since $x_0 \in M$, the sequence $\{x_n\}_{n \geq 0}$ is asymptotically almost periodic in $L(1)$. But then $\{x_n\}$ also is asymptotically almost periodic in the weaker original norm of X. □

Next, we present an extension of another well known result, due to Katznelson-Tzafriri (see [7]), which states that if T is a power-bounded operator and $\sigma(T) \cap \{z : |z| = 1\} \subset \{1\}$, then $\|T^{n+1} - T^n\| \to 0$ as $n \to \infty$.

Theorem 3.15 *Assume $\sigma(A, B) \cap \{|z| = 1\} \subset \{1\}$ and let $\{x_n\}_{n \geq 0}$ be a bounded solution of Eq.(1). Then $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.*

Proof By Theorem 3.8 and Proposition 3.10, T_1 is a contraction on $L(1)$ such that $\sigma(T_1) \cap \{z : |z| = 1\} \subset \{1\}$. Hence, by Katznelson-Tzafriri's Theorem, $|||T_1^{n+1} - T_1^n||| \to 0$ as $n \to \infty$. Since $x_0 \in L(1)$, we have $|||x_{n+1} - x_n||| = |||(T^{n+1} - T_1^n)x_0||| \to 0$, so that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. □.

4. Appendix: Eq.(1) with bounded operators A and B.

In this section, we present a special functional calculus for a pair (A, B), which is based on the resolvent identity (Lemma 2.3). This functional calculus naturally generates a projection operator and we show that the image of this projection operator coincides with the space $L(\omega)$, which has been studied in Section 3.

Assume that A and B are bounded linear operators from X to Y such that $\sigma(A,B)$ is a bounded nonempty set. Let Γ be a Cauchy contour around $\sigma(A,B)$ (see e.g. **[2]**). Assume that $\phi(\lambda)$ is an analytic function on a neighborhood of $\sigma(A,B)$. Then we can define a bounded operator $\phi(A,B)$ by

$$\phi(A,B) := \frac{1}{2\pi i}\int_\Gamma \phi(\lambda)R_{A,B}(\lambda)d\lambda.$$

Thanks to the resolvent identity, we can reproduce the standard arguments in the operator theory (see e.g. **[2]**, **[3]**) to show that the above defined functional calculus is a homomorphism from the algebra of analytic functions on $\sigma(A,B)$ to a subalgebra of bounded linear operators on X.

Proposition 4.1 *The mapping $\phi(\lambda) \mapsto \phi(A,B)$ is a homomorphism from the algebra Δ of analytic functions on $\sigma(A,B)$ into the algebra $L(X)$ of bounded linear operators on X:*

(i) If $\phi(\lambda) = \alpha_1\phi_1(\lambda) + \alpha_2\phi_2(\lambda)$, then $\phi(A,B) = \alpha_1\phi_1(A,B) + \alpha_2\phi_2(A,B)$.
(ii) If $\phi(\lambda) = \phi_1(\lambda)\phi_2(\lambda)$, then $\phi(A,B) = \alpha_1\phi_1(A,B)\alpha_2\phi_2(A,B)$.

From Proposition 4.1 it follows that the operator P defined by

$$P := \frac{1}{2\pi i}\int_\Gamma R_{A,B}(\lambda)d\lambda \tag{6}$$

satisfies $P^2 = P$, i.e. is a projection from X onto a subspace $E := P(X)$. Since the resolvents $R_{A,B}(\lambda)$ are commuting operators, they commute with the projection P as well. Let

$$T := \frac{1}{2\pi i}\int_\Gamma \lambda R_{A,B}(\lambda)d\lambda. \tag{7}$$

Then T is a bounded linear operator on X, such that T, P and the resolvent $R_{A,B}(\lambda)$ are pairwise commuting. In particular, E and its complement $F = (I-P)X$ are invariant subspaces of T, with $X = E \oplus F$.

Lemma 4.2 *If $Tx = y$, then $By = APx$. In particular, if $x \in E$, then $Tx = y$ implies $By = Ax$.*

Lemma 4.2 implies that if $\{x_n\}_{n\geq 0}$ is an orbit of T in E, i.e. $x_n = T^n x_0$, then it is a solution of Eq.(1).

Proof First remark that $(\lambda B - A)R_{A,B}(\lambda)x = Bx$ for all $x \in X$. In fact, this is true for all $x \in \mathcal{D}(C_{A,B}(\lambda))$ by the definition of $C_{A,B}(\lambda)$, and hence for any $x \in X$ because A and B are bounded.

From $Tx = y$ we have

$$By = BTx = \frac{1}{2\pi i}\int_\Gamma \lambda BR_{A,B}(\lambda)xd\lambda = \frac{1}{2\pi i}\int_\Gamma (\lambda B - A + A)R_{A,B}(\lambda)xd\lambda$$
$$= \frac{1}{2\pi i}\int_\Gamma Bxd\lambda + A\frac{1}{2\pi i}\int_\Gamma R_{A,B}(\lambda)xd\lambda = 0 + APx = APx.$$

Lemma 4.2 immediately implies the following fact.

Lemma 4.3 *The equation $Bx_{n+1} = Ax_n$ have solutions in E for all $x_0 \in E$, and $x_n = T^n x_0$.*

Lemma 4.4 *If $x \in E$ and $By = Ax$, then $B(y - Tx) = 0$. In particular, if B is one-to-one, then $y = Tx$, and equations $y = Tx$ and $By = Ax$ are equivalent on E.*

Proof If $z = Tx$, then, as shown above, $Bz = APx = Ax$, hence $B(y - z) = By - Ax = 0$. □

Lemma 4.5 *The operator B is one-to-one on E, $A(E) \subset B(E)$, and $B^{-1}A|E = T|E$. In particular, if $P = I$, then B is one-to-one on X, $B^{-1}A$ is bounded, and $T = B^{-1}A$.*

Proof By the definition, if $Bx = 0$, then $x \in D(C_{A,B}(\lambda))$ and $C_{A,B}(\lambda)x = R_{A,B}(\lambda)x = 0$ for all $\lambda \in \rho(A,B)$, which implies $Px = 0$. Hence, B is one-to-one on E. By Lemma 4.4, from $y = Tx$ $(x \in E)$ it follows $By = Ax$, so that $Tx = B^{-1}Ax$. Hence $A(E) \subset B(E)$ and $T|E \subset B^{-1}A|E$. Since both operators are defined on E, they must be identical. This implies that $Ax \in \mathcal{D}(B^{-1})$ and $B^{-1}Ax = Tx$ for all $x \in E$.

We collect the facts from Lemmas 4.2-4.5 in the following theorem.

Theorem 5 *Suppose that A and B are bounded operators from X to Y such that $\sigma(A,B)$ is a bounded nonempty set. Let $P : X \to X$ and $T : X \to X$ be defined by (6) and (7). Then the following holds;*

(i) $TP = PT$;

(ii) The subspace $E = P(X)$ coincides with $L(\omega)$, where $\omega = \|T\|$, and $T = T_\omega$, $R_{A,B}(\lambda)|E = (\lambda I - T)^{-1}|E$;

(iii) Eq.(1) is equivalent to $x_{n+1} = Tx_n$ on $E := P(X)$;

(iv) B is injective on E;

(v) $P = I$ if and only if B is injective and $B^{-1}A = T$.

Proof It remains only to prove (ii). Suppose $x \in E$. Let $x_0 = x$, $x_k = T^k x_0$ for all $k \geq 1$. By Lemma 4.3, $\{x_n\}_{n\geq 0}$ is a solution of Eq.(1). Clearly, $\|x_n\| = \|T^n x\| \leq \|T^n\|\|x_0\|$, so that $x \in L(\omega)$, where $\omega = \|T\|$).

Conversely, let $x \in L(\omega)$, where $\omega = \|T\|$. By Theorem 3.9, $R_{A,B}(\lambda)x = (\lambda I - T_\omega)^{-1}x$. Therefore

$$\frac{1}{2\pi i}\int_\Gamma R_{A,B}(\lambda)x d\lambda = \frac{1}{2\pi i}\int_\Gamma (\lambda I - T_\omega)^{-1} x d\lambda,$$

where the convergence in the above integrals is in the norm $\|\cdot\|$. Since $|||\cdot|||$ is a strongger norm and in this norm we have

$$\frac{1}{2\pi i}\int_\Gamma (\lambda I - T_\omega)^{-1} x d\lambda = x,$$

it follows that

$$Px = \frac{1}{2\pi i}\int_\Gamma R_{A,B}(\lambda)x d\lambda = \frac{1}{2\pi i}\int_\Gamma (\lambda I - T_\omega)^{-1} x d\lambda = x,$$

so that $x \in E$. For $x \in L_\omega = E$, there exists a solution $\{x_n\}_{n\geq 0}$ of Eq.(1), with $x_0 = x$, and we have $T_\omega x = Tx = x_1$. Hence $T = T_\omega$, and from Theorem 3.9. it follows that $R_{A,B}(\lambda)|E = (\lambda I - T)^{-1}|E$. □

To our knowledge, Theorem 4.6 is new even for Eq.(1) on finite dimensional spaces.

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