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# DELAY-DIFFERENTIAL EQUATIONS WITH INFINITELY MANY STATE-DEPENDENT IMPULSES

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**Abstract**

The paper deals with existence and uniqueness of the solution to a system of delay-differential equations with infinitely many state-dependent impulses. A simple transformation allows us to show that the problem can be treated as a system of delay-differential equations without impulse. The fixed point approach is then applied for an appropriate norm. Ordinary differential equations with impulses can be seen as a special case.

## 1 Introduction

The object of this paper is to present existence and uniqueness results for the solution of a system of delay-differential equations with infinitely many state-dependent impulses. This type of problem is characterized by jumps in the solution of the system. It was brought to our attention by aerospace engineers and appear for example in impulsive control problems.

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The system is described as follows: let $0 < T < +\infty$,

$$(S) \quad \begin{cases} \dot{x}(t) = f(t, x_t) + \sum\limits_{j \in I\!N} \alpha_j(x(\widetilde{\tau}_j^-))\delta(t - \tau_j), & t \in [0, T], \\ (x(0), x_0) = (\phi^0, \phi^1) = \Phi \in M^p = I\!R^n \times L^p(-h, 0; I\!R^n), \end{cases}$$

where $I\!N = \{1, 2, 3, ...\}$ is the set of strictly positive integers and $h$, $0 < h \leq +\infty$, is the memory of the system. Also, $\delta(.)$ is the Dirac delta function at 0, $\{\widetilde{\tau}_j\}_{j \in I\!N}$ and $\{\tau_j\}_{j \in I\!N}$ are two sequences of strictly increasing real numbers in $[0, T]$ such that $0 < \widetilde{\tau}_j \leq \tau_j$, and $\alpha_j : I\!R^n \longrightarrow I\!R^n$ is a given map for each $j \in I\!N$.

For any $t$, $0 \leq t \leq T$, the notation $I(-h, t) = [-h, t] \cap I\!R$ allows us to simultaneously deal with $[-h, t]$ if the memory $h$ is finite ($h < +\infty$) and $(-\infty, t]$ if the memory $h$ is infinite ($h = +\infty$). Also $I^<(-h, t) = \{\theta \in I(-h, t) | \theta < t\}$.

The function $x : [0, T] \longrightarrow I\!R^n$ is a vector valued function, and for any $t$ such that $0 \leq t \leq T$, $x_t$ is defined on $I(-h, 0)$ by

$$x_t(\theta) = \begin{cases} x(t + \theta) & \text{for} \quad -t \leq \theta \leq 0, \\ \phi^1(t + \theta) & \text{for} \quad \theta < -t. \end{cases}$$

Finally $f : [0, T] \times K(-h, 0; I\!R^n) \longrightarrow I\!R^n$ is a given map.

To describe $K(-h, t; I\!R^n)$ we need the following function spaces. Let $C(-h, t; I\!R^n)$ be the space of vector valued continuous functions defined on the interval $I(-h, t)$,

$$C_c(-h, t; I\!R^n) = \left\{ \phi \in C(-h, t; I\!R^n) \;\middle|\; \begin{array}{l} \exists K_0 \text{ compact } \subset I(-h, t) \quad \text{such that} \\ \phi(\theta) = 0 \quad \forall \theta \in I(-h, t) \backslash K_0 \end{array} \right\},$$

and

$$C_0(-h, t; I\!R^n) = \left\{ \phi \in C(-h, t; I\!R^n) \;\middle|\; \begin{array}{l} \forall \epsilon > 0 \quad \exists K_\epsilon \text{ compact } \subset I(-h, t) \\ \text{such that } |\phi(\theta)| < \epsilon, \ \forall \theta \in I(-h, t) \backslash K_\epsilon \end{array} \right\}.$$

Then

$$K(-h, t; I\!R^n) = \begin{cases} C(-h, t; I\!R^n) & \text{if } h < +\infty, \\ C_0(-h, t; I\!R^n) & \text{if } h = +\infty, \end{cases}$$

and $L^p(-h, t; I\!R^n) = L^p(I(-h, t); I\!R^n)$ for $1 \leq p \leq +\infty$.

The case of a system of ordinary differential equations has been studied recently by Dubeau *et al* [3]. The principal result of this paper is a complement to the existing literature (see for example [1], [2], [4], [5]) and generalizes the result given in [3].

## 2 Preliminary Results

A solution to the system $(S)$ is a function of bounded variation which can be written in the form

$$\begin{cases} x(t) = \phi^0 + \int_0^t f(\tau, x_\tau)d\tau + \sum\limits_{j \in IN} \alpha_j(x(\widetilde{\tau}_j^-)\chi_{[\tau_j,+\infty)}(t), & 0 \le t \le T, \\ x(t) = \phi^1(t), \qquad t \in I^<(-h,0). \end{cases} \tag{1}$$

We will further see that under certain assumptions, the expression (1) makes sense not only for functions $x_t$ in $K(-h,0;IR^n)$ but also in $L^p(-h,0;IR^n)$.

Suppose $\Phi = (\phi^0, \phi^1) \in M^p = IR^n \times L^p(-h,0;IR^n)$, and consider the set of functions

$$W_T(\Phi) = \left\{ y \in L^p(-h,T;IR^n) \,\middle|\, \begin{array}{l} y(t) = \phi^1(t) \quad \text{for} \quad t \in I^<(-h,0), \\ y(0) = \phi^0, \\ y\big|_{[0,T]} \in C(0,T;IR^n) \end{array} \right\}.$$

Using the sequences $\{\widetilde{\tau}_j\}_{j \in IN}$ and $\{\tau_j\}_{j \in IN}$, we consider the following partition of the set $IN$. Let $\tau_0 = 0$,

$$IN_k = \{ j \in IN \quad | \quad \tau_{k-1} < \widetilde{\tau}_j \le \tau_k \} \quad (k = 1, 2, 3, \ldots).$$

We introduce the family of functions $\{\theta_j\}_{j \in IN}$ defined on $W_T(\Phi)$ with values in $IR^n$ by

$$\theta_j(y) = y(\widetilde{\tau}_j^-) + \sum_{i=1}^{k-1} \alpha_i \circ \theta_i(y), \quad j \in IN_k, \quad k \in IN,$$

(where we use the notation $\sum\limits_{i=n}^{m} \gamma_i = 0$ if $m < n$ ), and for $y \in W_T(\Phi)$ we define on $[0,T]$ the following functions

$$\begin{aligned} \mathcal{J}y(t) &= \sum_{j \in IN} \alpha_j \circ \theta_j(y) \chi_{[\tau_j,+\infty)}(t), \\ \mathcal{F}y(t) &= y(t) + \mathcal{J}y(t), \\ \mathcal{R}y(t) &= \phi^0 + \int_0^t f(\tau, [\mathcal{F}y]_\tau)d\tau. \end{aligned}$$

In the following proposition it is shown that the existence of a solution to $(S)$ is equivalent to the existence of a fixed point for the operator $\mathcal{R}$.

**Proposition 1** *For each solution $x \in BV(0,T;IR^n)$ of the system (S), there exists a function $y \in W_T(\Phi)$ such that*

$$x = \mathcal{F}y \quad \textit{and } y = \mathcal{R}y.$$

*Conversely if $y \in W_T(\Phi)$ is such $y = \mathcal{R}y$, then the solution $x \in BV(0,T;IR^n)$ of the system (S) is $x = \mathcal{F}y$.*

**Proof** If $x$ is a solution of the original problem we have

$$\begin{cases} x(t) = \phi^0 + \int_0^t f(\tau, x_\tau)d\tau + \sum\limits_{j\in IN} \alpha_j(x(\widetilde{\tau}_j^-))\chi_{[\tau_j,+\infty)}(t), & 0 \le t \le T \\ x(t) = \phi^1(t), \qquad t \in I^<(-h,0). \end{cases}$$

Define $y \in W_T(\Phi)$ by

$$y(t) = \begin{cases} \phi^1(t), & t \in I^<(-h,0), \\ \phi^0 + \int_0^t f(\tau, x_\tau)d\tau, & t \in [0,T]\,. \end{cases}$$

Thus by recurrence we have,

$$\begin{aligned} x(\widetilde{\tau}_j^-) &= y(\widetilde{\tau}_j) + \sum_{i\in IN} \alpha_i(x(\widetilde{\tau}_i^-))\chi_{[\tau_i,+\infty)}(\widetilde{\tau}_j) \\ &= \theta_j(y) \end{aligned}$$

for all $j \in IN$. Then

$$\begin{aligned} x(t) &= \phi^0 + \int_0^t f(\tau, x_\tau)d\tau + \sum_{j\in IN} \alpha_j(x(\widetilde{\tau}_j^-))\chi_{[\tau_j,+\infty)}(t) \\ &= y(t) + \sum_{j\in IN} \alpha_j \circ \theta_j(y)\chi_{[\tau_j,+\infty)}(t) \\ &= \mathcal{F}y(t) \end{aligned}$$

and

$$y(t) = \phi^0 + \int_0^t f(\tau, [\mathcal{F}y]_\tau)d\tau = \mathcal{R}y(t).$$

Conversely, if $y \in W_T(\Phi)$ and $y = \mathcal{R}y$, let us define $x(t) = \mathcal{F}y(t)$. Then

$$\begin{aligned} x(t) &= y(t) + \mathcal{J}y(t) \\ &= \phi^0 + \int_0^t f(\tau, [\mathcal{F}y]_\tau)d\tau + \sum_{j\in IN} \alpha_j \circ \theta_j(y)\chi_{[\tau_j,+\infty)}(t). \end{aligned}$$

Thus, by recurrence, we have $x(\widetilde{\tau}_j^-) = \theta_j(y)$ for all $j \in IN$ and finally

$$x(t) = \phi^0 + \int_0^t f(\tau, x_\tau)d\tau + \sum_{j\in IN} \alpha_j(x(\widetilde{\tau}_j^-))\chi_{[\tau_j,+\infty)}(t)$$

is a solution of the original problem. □

According to this result, one is brought back to find a fixed point $y = \mathcal{R}y$ or, which is equivalent, to find a continuous solution $y$ on $[0, T]$ of the following delay-differential equation

$$(R) \quad \begin{cases} \dot{y}(t) = F(t, y), \quad t \in [0, T], \\ (y(0), y_0) = (\phi^0, \phi^1) = \Phi \in M^p = I\!R^n \times L^p(-h, 0; I\!R^n), \end{cases}$$

with

$$\begin{aligned} F(t, y) &= f(t, [\mathcal{F}y]_t) \\ &= f(t, [y + \sum_{j \in I\!N} \alpha_j \circ \theta_j(y) \chi_{[\tau_j, +\infty)}]_t). \end{aligned}$$

## 3 The Main Result

The main result of this paper is the following theorem.

**Theorem 2** *Let $\phi^0 \in I\!R^n$ and $\phi^1 \in L^p(-h, 0; I\!R^n)$ where $1 \leq p < +\infty$ and $0 < h \leq +\infty$. Suppose that*
*(a) the map $f : [0, T] \times K(-h, 0; I\!R^n) \longrightarrow I\!R^n$ satisfies the following conditions:*
*$(h_1)$ for all $\phi \in K(-h, 0; I\!R^n)$, the function*

$$t \longrightarrow f(t, \phi) : [0, T] \longrightarrow I\!R^n$$

*is Lebesgue mesurable,*
*$(h_2)$ there exists a nonnegative function $n(.) \in L^1(0, T; I\!R)$ such that for any $\phi_1, \phi_2 \in K(-h, 0; I\!R^n)$*

$$|f(t, \phi_1) - f(t, \phi_2)| \leq n(t) \, \|\phi_1 - \phi_2\|_{C(-h,0)} \, ,$$

*$(h_3)$ the map*

$$t \longrightarrow f(t, 0) : [0, T] \longrightarrow I\!R^n$$

*is integrable,*
*$(h_4)$ there exists a nonnegative increasing function $m(t)$ such that for any $t \geq 0$, and any $z_1, z_2 \in C_c(-h, t; I\!R^n)$*

$$\int_0^t |f(s, (z_2)_s) - f(s, (z_1)_s)| \, ds \leq m(t) (\int_{-h}^t |z_2(s) - z_1(s)|^p \, ds)^{1/p};$$

*(b) the maps $\alpha_j : I\!R^n \longrightarrow I\!R^n$ $(j \in I\!N)$ satisfy the following conditions*
*$(h_5)$ $\sum_{j \in I\!N} |\alpha_j(0)| < +\infty$,*

*($h_6$) there exists a sequence of nonnegative real numbers $\{\lambda_j\}_{j\in IN}$ such that $\Lambda = \sum\limits_{j\in IN} \lambda_j < +\infty$ and for any $j \in IN$, and any $x_1, x_2 \in IR^n$*

$$|\alpha_j(x_1) - \alpha_j(x_2)| \leq \lambda_j \, |x_1 - x_2| \, ;$$

*(c) the sequences $\{\tau_j\}_{j\in IN}$ and $\{\widetilde{\tau}_j\}_{j\in IN}$ are strictly increasing sequences in $[0, T]$ and*

*($h_7$) $0 < \widetilde{\tau}_j \leq \tau_j$ for any $j \in IN$.*

*Therefore*

*(1) the solution $y$ of $(R)$ exists and is unique in $W^{1,1}(0, T; IR^n)$; moreover if $y(.; \phi_i^0, \phi_i^1)$ is the solution of $(R)$ for the initial condition $\Phi_i = (\phi_i^0, \phi_i^1)$ , $i = 1, 2$, then there exists a constant $C_1(T)$ such that*

$$\left\| y(.; \phi_1^0, \phi_1^1) - y(.; \phi_2^0, \phi_2^1) \right\|_{W^{1,1}(0,T)} \leq C_1(T) \, \|\Phi_1 - \Phi_2\|$$

*where*

$$\|\Phi\| = \left\| (\phi^0, \phi^1) \right\| = \left| \phi^0 \right| + \left\| \phi^1 \right\|_{L^p(-h,0;IR^n)}$$

*(2) the solution $x$ of $(S)$ exists, is unique in $BV(0, T; IR^n)$ and is given by*

$$x(t) = y(t) + \sum_{j\in IN} \alpha_j \circ \theta_j(y) \chi_{[\tau_j, +\infty)}(t);$$

*moreover if $x(.; \phi_i^0, \phi_i^1)$ is the solution of $(S)$ for the initial condition $\Phi_i = (\phi_i^0, \phi_i^1)$, $i = 1, 2$, then there exists a constant $C_2(T) > 0$ such that*

$$\left\| x(.; \phi_1^0, \phi_1^1) - x(.; \phi_2^0, \phi_2^1) \right\|_{L^\infty(0,T)} \leq C_2(T) \, \|\Phi_1 - \Phi_2\| \, .$$

The proof of this theorem is based on some standard lemmas and technical results that are presented in the next section.

# 4 Technical Results

Let us start this section with two technical lemmas whose proofs are similar to those given in the case without delay by Dubeau *et al* [3]. Let us recall that $\Lambda = \sum\limits_{j\in IN} \lambda_j < +\infty$.

**Lemma 3** *Let $\Phi_i = (\phi_i^0, \phi_i^1)$ , $i = 1, 2$. For any $y_i \in W_T(\Phi_i)$, $i = 1, 2$, and $j \in IN$, we have*

$$\|\alpha_j \circ \theta_j(y_1) - \alpha_j \circ \theta_j(y_2)\| \leq \lambda_j e^{\Lambda} \, \|y_1 - y_2\|_{C(0,\widetilde{\tau}_j)} \, .$$

**Lemma 4** *The function* $y = 0$ *is in* $W_T(0)$ *and we have*

$$\sum_{j \in I\!N} |\alpha_j \circ \theta_j(0)| \leq e^{\Lambda} \sum_{j \in I\!N} |\alpha_j(0)| \,.$$

The next two lemmas will also be used in the proof of the main result. For the proofs see [1].

**Lemma 5** *Let* $1 \leq p < +\infty$, $0 < T < +\infty$ *and* $0 < h \leq +\infty$. *Suppose that hypotheses* $(h_1)$, $(h_2)$, $(h_3)$, *and* $(h_4)$ *are satisfied, and for any* $z \in C_c(-h, T; I\!R^n)$ *define the map* $f_z \in L^1(0, T; I\!R^n)$ *by*

$$f_z(t) = f(t, z_t), \qquad 0 \leq t \leq T.$$

*Then the map*

$$z \longrightarrow f_z : C_c(-h, T; I\!R^n) \longrightarrow L^1(0, T; I\!R^n)$$

*has a continuous extension to a map from* $L^p(-h, T; I\!R^n)$ *to* $L^1(0, T; I\!R^n)$, *and for any* $t \in [0, T]$ *and all* $z_1, z_2 \in L^p(-h, T; I\!R^n)$, *we have:*

$$\int_0^t |f(s, (z_2)_s) - f(s, (z_1)_s)| \, ds \leq m(t) \, \|z_2 - z_1\|_{L^p(-h,t)} \, .\Box$$

**Lemma 6** *Let* $0 < \alpha < 1, 1 \leq p < +\infty$ *and* $c > 0$ *be three given constants, then*

*(i) the map*

$$g_\alpha(t) = \exp\left[\left(\frac{c}{\alpha}\right)^p \frac{t}{p}\right], \qquad t \geq 0$$

*is increasing and* $\geq 1$;

*(ii) for any* $t \geq 0$,

$$c \, \|g_\alpha\|_{L^p(0,t)} \leq \alpha g_\alpha(t);$$

*(iii) for any* $T > 0$ *and any* $z \in C(0, T; I\!R^n)$,

$$\|x\|_{C_\alpha(0,T)} = \sup\left\{|x(t)/g_\alpha(t)| : t \in [0, T]\right\}$$

*is a norm equivalent to the usual one*

$$\|x\|_{C(0,T)} = \sup\left\{|x(t)| : t \in [0, T]\right\}$$

*and we have*

$$\frac{1}{g_\alpha(T)} \|x\|_{C(0,T)} \leq \|x\|_{C_\alpha(0,T)} \leq \|x\|_{C(0,T)} \, ;$$

*(iv) for any $T > 0$ and any $z \in L^\infty(0, T; IR^n)$*

$$\|x\|_{L^\infty_\alpha(0,T)} = ess\sup\{|x(t)/g_\alpha(t)| : t \in [0, T]\}$$

*is a norm equivalent to the usual one*

$$\|x\|_{L^\infty(0,T)} = ess\sup\{|x(t)| : t \in [0, T]\},$$

*and we have*

$$\frac{1}{g_\alpha(T)} \|x\|_{L^\infty(0,T)} \le \|x\|_{L^\infty_\alpha(0,T)} \le \|x\|_{L^\infty(0,T)}.$$

# 5 Proof of the Main Result

**Proof of (1).** Let $\Phi = (\phi^0, \phi^1)$ be given. For any $y \in W_T(\Phi)$ the map

$$t \longrightarrow y(t) + \sum_{j \in IN} \alpha_j \circ \theta_j(y) \chi_{[\tau_j, +\infty)}(t)$$

is $L^p$ integrable on $I(-h, T)$, being the sum of a function in $L^p(-h, T; IR^n)$ and a function of bounded variation. In fact, by Lemma 3, we have for any $y \in W_T(\Phi)$, $j \in IN$ and any $t \in I(-h, T)$

$$\begin{aligned}
\left|\alpha_j \circ \theta_j(y) \chi_{\tau_j, +\infty)}(t)\right| &\le |\alpha_j \circ \theta_j(y) - \alpha_j \circ \theta_j(0)| + |\alpha_j \circ \theta_j(0)| \\
&\le \lambda_j e^\Lambda \|y\|_{C(0,T)} + |\alpha_j \circ \theta_j(0)|.
\end{aligned}$$

Then

$$\sum_{j \in IN} \left|\alpha_j \circ \theta_j(y) \chi_{\tau_j, +\infty)}(t)\right| \le \Lambda e^\Lambda \|y\|_{C(0,T)} + \sum_{j \in IN} |\alpha_j \circ \theta_j(0)|,$$

and by Lemma 4

$$\sum_{j \in IN} \left|\alpha_j \circ \theta_j(y) \chi_{\tau_j, +\infty)}(t)\right| \le \Lambda e^\Lambda \|y\|_{C(0,T)} + e^\Lambda \sum_{j \in IN} |\alpha_j(0)|.$$

Using hypothesis $(h_1)$, $(h_2)$, $(h_3)$, $(h_4)$, and Lemma 5, the map

$$t \longrightarrow f(t, (y + \sum_{j \in IN} \alpha_j \circ \theta_j(y) \chi_{[\tau_j, +\infty)})_t)$$

is $L^1$ integrable on $[0, T]$ and $\int_0^t f(\tau, (y + \sum_{j \in IN} \alpha_j \circ \theta_j(y) \chi_{[\tau_j, +\infty)})_\tau) d\tau$ is well defined.

Let $\Phi_i = (\phi_i^0, \phi_i^1)$ and $y_i \in W_T(\Phi_i)$ for $i = 1, 2$, and consider

$$\mathcal{R}y_i(t) = \begin{cases} \phi_i^1(t), & t \in I^<(-h, 0), \\ \phi_i^0 + \int_0^t F(\tau, y_i)d\tau, & t \in [0, T]. \end{cases}$$

Then, for any $t \in [0, T]$ we have

$$\begin{aligned} (\mathcal{R}y_1 - \mathcal{R}y_2)(t) &= (\phi_1^0 - \phi_2^0) + \int_0^t [F(\tau, y_1) - F(\tau, y_2)]d\tau, \\ |\mathcal{R}y_1 - \mathcal{R}y_2|(t) &\le \left|\phi_1^0 - \phi_2^0\right| + \int_0^t |f(\tau, [\mathcal{F}y_1]_\tau) - f(\tau, [\mathcal{F}y_2]_\tau)|\, d\tau, \end{aligned}$$

and by Lemma 5

$$|\mathcal{R}y_1 - \mathcal{R}y_2|(t) \le \left|\phi_1^0 - \phi_2^0\right| + m(t) \left\|\mathcal{F}y_1 - \mathcal{F}y_2\right\|_{L^p(-h,t)}.$$

But we also have

$$\begin{aligned} \|\mathcal{F}y_1 - \mathcal{F}y_2\|_{L^p(-h,t)} &\le \|y_1 - y_2\|_{L^p(-h,t)} + \|\mathcal{J}y_1 - \mathcal{J}y_2\|_{L^p(0,t)} \\ &\le \left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)} + \|y_1 - y_2\|_{L^p(0,t)} \\ &\quad + \|\mathcal{J}y_1 - \mathcal{J}y_2\|_{L^p(0,t)}. \end{aligned}$$

For any $\alpha$, $0 < \alpha < 1$, let us introduce the map $g_\alpha(t) = \exp\left[\left(\frac{m(T)}{\alpha}\right)^p \frac{t}{p}\right]$. Hence we have

$$\begin{aligned} m(t) \|y_1 - y_2\|_{L^p(0,t)} &= m(t)(\int_0^t g_\alpha^p(\tau) |\frac{y_1 - y_2}{g_\alpha(\tau)}|^p d\tau)^{1/p} \\ &\le m(t) \|y_1 - y_2\|_{C_\alpha(0,t)} \|g_\alpha\|_{L^p(0,t)}, \end{aligned}$$

and by Lemma 6, with $c = m(T)$, we have

$$m(t) \|y_1 - y_2\|_{L^p(0,t)} \le \alpha g_\alpha(t) \|y_1 - y_2\|_{C_\alpha(0,t)}.$$

Also

$$m(t) \|\mathcal{J}y_1 - \mathcal{J}y_2\|_{L^p(0,t)} \le \alpha g_\alpha(t) \|\mathcal{J}y_1 - \mathcal{J}y_2\|_{L^\infty_\alpha(0,t)}.$$

But

$$\begin{aligned} |\mathcal{J}y_1 - \mathcal{J}y_2|(\tau) &\le \sum_{j \in I\!N} |\alpha_j \circ \theta_j(y_1) - \alpha_j \circ \theta_j(y_2)|\, \chi_{\tau_j, +\infty)}(\tau) \\ &\le \sum_{j \in I\!N} \lambda_j e^\Lambda \|y_1 - y_2\|_{C(0,\tau)} \\ &\le \Lambda e^\Lambda \|y_1 - y_2\|_{C(0,\tau)}. \end{aligned}$$

Then

$$\begin{aligned}\frac{|\mathcal{J}y_1 - \mathcal{J}y_2|(\tau)}{g_\alpha(\tau)} &\leq \Lambda e^\Lambda \frac{\|y_1 - y_2\|_{C(0,\tau)}}{g_\alpha(\tau)} \\ &\leq \Lambda e^\Lambda \|y_1 - y_2\|_{C_\alpha(0,\tau)}.\end{aligned}$$

Consequently

$$\|\mathcal{J}y_1 - \mathcal{J}y_2\|_{L^\infty_\alpha(0,t)} \leq \Lambda e^\Lambda \|y_1 - y_2\|_{C_\alpha(0,t)}. \tag{2}$$

Therefore

$$\begin{aligned}|\mathcal{R}y_1 - \mathcal{R}y_2|(t) &\leq \left|\phi_1^0 - \phi_2^0\right| + m(t)\left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)} \\ &\quad + \alpha g_\alpha(t)(1 + \Lambda e^\Lambda)\|y_1 - y_2\|_{C_\alpha(0,t)}.\end{aligned}$$

If $\Phi_1 = \Phi_2$ we have

$$|\mathcal{R}y_1 - \mathcal{R}y_2|(t) \leq \alpha g_\alpha(t)(1 + \Lambda e^\Lambda)\|y_1 - y_2\|_{C_\alpha(0,t)}.$$

Then

$$\|\mathcal{R}y_1 - \mathcal{R}y_2\|_{C_\alpha(0,T)} \leq \alpha(1 + \Lambda e^\Lambda)\|y_1 - y_2\|_{C_\alpha(0,T)},$$

and we have a contraction for $\alpha$ small enough. The map $\mathcal{R}$ has a fixed point which is the unique solution of $(R)$.

Moreover, as $g_\alpha(t) \geq 1$ and $m(t)$ is an increasing function, for $\Phi_i = (\phi_i^0, \phi_i^1), i = 1, 2$, we have

$$\begin{aligned}\frac{|\mathcal{R}y_1 - \mathcal{R}y_2|(t)}{g_\alpha(t)} &\leq \left|\phi_1^0 - \phi_2^0\right| + m(T)\left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)} \\ &\quad + \alpha(1 + \Lambda e^\Lambda)\|y_1 - y_2\|_{C_\alpha(0,t)},\end{aligned}$$

then

$$\begin{aligned}\|\mathcal{R}y_1 - \mathcal{R}y_2\|_{C_\alpha(0,T)} &\leq \left|\phi_1^0 - \phi_2^0\right| + m(T)\left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)} \\ &\quad + \alpha(1 + \Lambda e^\Lambda)\|y_1 - y_2\|_{C_\alpha(0,t)}.\end{aligned}$$

For $y_i$ solution of $(R)$, we have $y_i = \mathcal{R}y_i$. Then

$$\begin{aligned}\|y_1 - y_2\|_{C_\alpha(0,T)}(1 - \alpha(1 + \Lambda e^\Lambda)) &\leq \left|\phi_1^0 - \phi_2^0\right| \\ &\quad + m(T)\left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)}\end{aligned} \tag{3}$$

and by Lemma 6

$$\begin{aligned}\|y_1 - y_2\|_{C(0,T)} &\leq \frac{g_\alpha(T)}{(1 - \alpha(1 + \Lambda e^\Lambda))}[\left|\phi_1^0 - \phi_2^0\right| \\ &\quad + m(T)\left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)}].\end{aligned} \tag{4}$$

Therefore

$$\begin{aligned}\|y_1 - y_2\|_{L^1(0,T)} &\le \frac{Tg_\alpha(T)}{(1-\alpha(1+\Lambda e^\Lambda))}[\left|\phi_1^0 - \phi_2^0\right| \\ &+m(T)\left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)}]. \qquad (5)\end{aligned}$$

For the derivative we have

$$\begin{aligned}\left\|\dot{y}_1 - \dot{y}_2\right\|_{L^1(0,T)} &= \int_0^T \left|\dot{y}_1 - \dot{y}_2\right|(t)dt \\ &= \int_0^T |F(t,y_1) - F(t,y_2)|\,dt \\ &\le m(T)\left\|\mathcal{F}y_1 - \mathcal{F}y_2\right\|_{L^p(-h,T)} \\ &\le m(T)[\left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)} + \|y_1 - y_2\|_{L^p(0,T)} + \\ &\quad \|\mathcal{J}y_1 - \mathcal{J}y_2\|_{L^p(0,T)}] \\ &\le m(T)[\left\|\phi_1^1 - \phi_2^1\right\|_{L^p(-h,0)} + \alpha g_\alpha(T)[\|y_1 - y_2\|_{C_\alpha(0,T)} \\ &\quad + \|\mathcal{J}y_1 - \mathcal{J}y_2\|_{L_\alpha^\infty(0,T)}]. \qquad (6)\end{aligned}$$

The result is obtained by combining relations (2), (3), (5), and (6) .

**Proof of (2)**. The solution $x$ is given by

$$x(t) = \mathcal{F}y(t) = y(t) + \mathcal{J}y(t)$$

with $y = \mathcal{R}y$. It is a function of bounded variation since

$$Var(x;[0,T]) \le Var(y;[0,T]) + Var(\mathcal{J}y;[0,T]).$$

But the two terms on the righthand side are bounded since

$$\begin{aligned}Var(y;[0,T]) &\le \int_0^T |F(\tau,y)|\,d\tau \\ &\le \int_0^T |f(\tau,[\mathcal{F}y]_\tau) - f(\tau,[0]_\tau)|\,d\tau + \int_0^T |f(\tau,[0]_\tau)|\,d\tau \\ &\le m(T)\|\mathcal{F}y\|_{L^p(-h,T)} + \int_0^T |f(\tau,[0]_\tau)|\,d\tau \\ &\le m(T)[\|y\|_{L^p(-h,T)} + \|\mathcal{J}y\|_{L^p(-h,T)}] + \int_0^T |f(\tau,0)|\,d\tau.\end{aligned}$$

But,

$$\|\mathcal{J}y\|_{L^p(-h,T)} \le T^{1/p}\|\mathcal{J}y\|_{L^\infty(0,T)},$$

and

$$\begin{aligned}\|\mathcal{J}y\|_{L^\infty(0,T)} &\le \sum_{j\in IN} |\alpha_j\circ\theta_j(y) - \alpha_j\circ\theta_j(0)| + \sum_{j\in IN} |\alpha_j\circ\theta_j(0)| \\ &\le \Lambda e^{\Lambda}\|y\|_{C(0,T)} + e^{\Lambda}\sum_{j\in IN}|\alpha_j(0)|,\end{aligned}$$

Then the result follows.

Let $x_i(.) = x(.;\phi_i^0,\phi_i^1)$ be the solution of $(S)$ with the initial condition $\Phi_i = (\phi_i^0,\phi_i^1)$ $(i=1,2)$, then

$$\|x_1 - x_2\|_{L^\infty(0,T)} \le \|y_1 - y_2\|_{C(0,T)} + \|\mathcal{J}y_1 - \mathcal{J}y_2\|_{L^\infty(0,T)}.$$

The result is then obtained by combining (2), (3), (4) and using inequalities given in Lemma 6(ii). □

# 6 A Particular Case: Systems Without Delay

The following system of ordinary differential equations with impulses, considered in [3],

$$(S')\qquad \begin{cases} \dot{x}(t) = g(t,x(t)) + \sum\limits_{j\in IN}\alpha_j(x(\tau_j^-))\delta(t-\tau_j), & t\in[0,T], \\ x(0) = x^0, \end{cases}$$

with $g:[0,T]\times IR^n \longrightarrow IR^n$, is a particular case of system $(S)$. Indeed, let $h=0$, set $\widetilde{\tau_j} = \tau_j$, and define the map $f$ by $f(t,\phi) = g(t,\phi(0))$, then the system $(S)$ is exactely $(S')$ and the main result can be stated as follows:

**Theorem 7** *Let $x^0\in IR^n$ and suppose*
*(a) the map $g:[0,T]\times IR^n\longrightarrow IR^n$ verifies the following conditions:*
*($h_1'$) for any $x\in IR^n$, the map $t\longrightarrow g(t,x)$ is Lebesgue measurable,*
*($h_2'$) there exists a nonnegative function $n(.)\in L^1(0,T;IR)$ such that, for any $x_1,x_2\in IR^n$*

$$|g(t,x_1) - g(t,x_2)| \le n(t)\,|x_1 - x_2|,$$

*($h_3'$) the map*

$$t\longrightarrow g(t,0):[0,T]\longrightarrow IR^n$$

*is integrable;*
*(b) the maps $\alpha_j : IR^n\longrightarrow IR^n$ $(j\in IN)$ verify the following conditions:*
*($h_4'$) $\sum\limits_{j\in IN}|\alpha_j(0)| < +\infty$,*

*($h_5'$) there exists a sequence of nonnegative real numbers $\{\lambda_j\}_{j\in IN}$ such that $\Lambda = \sum_{j\in IN} \lambda_j < +\infty$ and for any $j \in IN$, and any $x_1, x_2 \in IR^n$*

$$|\alpha_j(x_1) - \alpha_j(x_2)| \leq \lambda_j |x_1 - x_2| .$$

*Then*
*(1) there exists a unique solution $y$ in $W^{1,1}(0,T; IR^n)$ to*

$$(R') \quad \begin{cases} \dot{y}(t) = G(t,y), & t \in [0,T], \\ y(0) = x^0, \end{cases}$$

*where*

$$G(t,y) = g(t, [\mathcal{F}y](t)) = g(t, y(t) + \sum_{j\in IN} \alpha_j \circ \theta_j(y)\chi_{[\tau_j,+\infty)}(t))$$

*Moreover if $y(.;x_i^0)$ is the solution of $(R')$ for the initial condition $x_i^0$ $(i = 1,2)$, then there exists a constant $C_1'(T)$ such that*

$$\left\| y(.;x_1^0) - y(.;x_2^0) \right\|_{W^{1,1}(0,T)} \leq C_1'(T) \left| x_1^0 - x_2^0 \right| ;$$

*(2) there exists a unique solution $x$ in $BV(0,T; IR^n)$ of $(S')$ and it is given by*

$$x(t) = y(t) + \sum_{j\in IN} \alpha_j \circ \theta_j(y)\chi_{[\tau_j,+\infty)}(t).$$

*Moreover if $x(.;x_i^0)$ is the solution of $(S')$ for the initial condition $x_i^0$, there exists a constant $C_2'(T) > 0$ such that*

$$\left\| x(.;x_1^0) - x(.;x_2^0) \right\|_{L^\infty(0,T)} \leq C_2'(T) \left| x_1^0 - x_2^0 \right| .$$

**Proof** For $f$ defined by $f(t,\phi) = g(t,\phi(0))$, hypotheses $(h_1')$, $(h_2')$, $(h_3')$ become $(h_1)$, $(h_2)$, $(h_3)$, the hypothesis $(h_4)$ is directly obtained from $(h_2')$, $(h_4')$ and $(h_5')$ are equivalent to $(h_5)$ and $(h_6)$, and $(h_7)$ is always satisfied. □

**Remark 8** *If we replace $\Phi_i = (\phi_i^0, \phi_i^1)$ by $(x_i, 0)$ in the expressions of $C_i(T)$, $i = 1,2$, given in the main result, we obtain the expressions given in the paper of Dubeau et al [3].* □

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