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STRUCTURE OF ANN-CATEGORIES AND MAC LANE-SHUKLA COHOMOLOGY

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Abstract

In this paper we study the structure of a class of categories having two operations which satisfy axioms analoguos to that of rings. Such categories are called "Ann-categories". We obtain the classification theorems for regular Ann-categories and Ann-functors by using Mac Lane-Shukla cohomology of rings. These results give new interpretations of the cohomology groups $H^3(R, M)$ and $H^2(R, M)$ of the rings R.

1 Introduction and Preliminaries

Monoidal categories and symetric monoidal categories were studied first by S. Mac Lane [8], J. Bénabou [1] and G. M. Kelly [3]. They are, respectively, categories $\mathcal{A}$ together with a bifunctor $\otimes$: $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a system of natural equivalences of associativity-unitivity, or a system of natural equivalences of associativity-unitivity-commutativity. A. Solian [14], H. X. Sinh [2] and K. H. Ulbrich [15] , investigated $\otimes$-categories from the point of view of algebraic structure. They examined the monoidal categories whose all objects are invertible.

The problem of coherence always plays a fundamental role in the study of any class of $\otimes$-categories. From initial conditions, we have to prove that the morphisms generated by a given ones depend only on its source butt. The consideration of structures arose later in the papers of H. X. Sinh [2] and B. Mitchell [9]. Here we obtained deep results on the classification by the cohomology of groups.

By the other direction, M. Laplaza [4] considered the coherence of natural

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equivalences of distributivity in a category having two operations $\oplus$ and $\otimes$. In the papers of Laplaza, the distribution of *monomorphisms* together with the natural isomorphisms of the two symmetrical monoidal structures must satisfy 24 commutative diagrams, that form natural relations between them.
In this paper, we consider a class of Pic-categories (see H. X. Sinh [2]) in which the second operation and natural equivalences of distributivity are defined so that the analogous axioms of rings are verified. Such categories are called Ann-categories. Coherence for Ann-categories was shown in [11].

Throughout we define invariants of Ann-category basing on construction of reduced Ann-categories and pre-sticked of the type (R, M). From this we obtain classification theorems for the regular Ann-categories and Ann-functors by using cohomology groups $H^3(R, M)$, $H^2(R, M)$ of the ring R. These theorems give a relation between the notion of Ann-category with the theory of cohomology of rings and the problem of extention of rings.

For convinience, the tensor product of two objects A and B is denoted by AB instead of $A \otimes B$, but for the morphisms we still write $f \otimes g$ to avoid confusion with the composition of morphisms.

The notions and results on monoidal categories are supposed to be familier to the readers (see [3, 5, 8] for example).

Recall that *a Pic-category* is a symmetric monoidal category $\mathcal{A}$ (or a $\otimes$ACU-category $\mathcal{A}$) in which every object is invertible and every morphism is an isomorphism (see [2]).

Definition 1.1. *An Ann-category is a category $\mathcal{A}$ together with*

(i) Two bifunctors $\oplus$, $\otimes$: $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

(ii) A fixed object $0 \in \mathcal{A}$ with natural isomorphisms a^+, c, g, d such that $\big(\mathcal{A}, \oplus, a^+, c, (0, g, d)\big)$ is a Pic-category.

(iii) A fixed object $1 \in \mathcal{A}$ with natural isomorphisms a, l, r such that $\big(\mathcal{A}, \otimes, a, (1, l, r)\big)$ is a monoidal category (i. e. a $\otimes AU$-category).

(iv) Two natural isomorphisms $\mathfrak{L}$, $\mathfrak{R}$

$$\mathfrak{L}_{A,X,Y} : A(X \oplus Y) \to AX \oplus AY$$

$$\mathfrak{R}_{X,Y,A} : (X \oplus Y)A \to XA \oplus YA$$

satisfying the following conditions

(Ann-1) For every object $A \in \mathcal{A}$, the pair of $\oplus$-functors $(L^A, \breve{L}^A)$, $(R^A, \breve{R}^A)$ defined by

$$\begin{cases} L^A : X \to AX \\ \breve{L}^A_{X,Y} = \mathfrak{L}_{A,X,Y} \end{cases} \quad \begin{cases} R^A : X \to AX \\ \breve{R}^A_{X,Y} = \mathfrak{R}_{X,Y,A} \end{cases}$$

are $\oplus AC$-functors.

(Ann-2) For any $A, B, X, Y \in \mathcal{A}$ the following diagrams are commutative

$$\begin{tikzcd}
A(B(X\oplus Y)) \arrow[r, "id\otimes\mathfrak{L}"] \arrow[d, "a"'] & A(BX\oplus BY) \arrow[r, "\mathfrak{L}"] & A(BX)\oplus A(BY) \arrow[d, "a\oplus a"] \\
(AB)(X\oplus Y) \arrow[rr, "\mathfrak{L}"] & & (AB)X\oplus(AB)Y
\end{tikzcd}$$

$$\begin{tikzcd}
(X\oplus Y)(AB) \arrow[rr, "\mathfrak{R}"] \arrow[d, "a"'] & & X(AB)\oplus Y(AB) \arrow[d, "a\oplus a"] \\
((X\oplus Y)A)B \arrow[r, "\mathfrak{R}\otimes id"] & (XA\oplus YA)B \arrow[r, "\mathfrak{R}"] & (XA)B\oplus(YA)B
\end{tikzcd}$$

$$\begin{tikzcd}
A((X\oplus Y)B) \arrow[r, "id\otimes\mathfrak{R}"] \arrow[d, "a"'] & A(XB\oplus YB) \arrow[r, "\mathfrak{L}"] & A(XB)\oplus A(YB) \arrow[d, "a\oplus a"] \\
(A(X\oplus Y))B \arrow[r, "\mathfrak{L}\otimes id"] & (AX\oplus AY)B \arrow[r, "\mathfrak{R}"] & (AX)B\oplus(AY)B
\end{tikzcd}$$

$$\begin{tikzcd}
(A\oplus B)X\oplus(A\oplus B)Y \arrow[d, "\mathfrak{R}\oplus\mathfrak{R}"] & (A\oplus B)(X\oplus Y) \arrow[l, "\mathfrak{L}"'] \arrow[r, "\mathfrak{R}"] & A(X\oplus Y)\oplus B(X\oplus Y) \arrow[d, "\mathfrak{L}\oplus\mathfrak{L}"] \\
(AX\oplus BX)\oplus(AY\oplus BY) \arrow[rr, "v"] & & (AX\oplus AY)\oplus(BX\oplus BY)
\end{tikzcd}$$

where $v = v_{A,B,C,D}$: $(A\oplus B)\oplus(C\oplus D) \to (A\oplus C)\oplus(B\oplus D)$ is the unique functorial morphism constructed from the morphisms a^+, c and id in the Pic-category $(\mathcal{A}, \oplus)$.

(Ann-3) The following diagrams are commutative

$$\begin{tikzcd}
1(X\oplus Y) \arrow[r, "\mathfrak{L}"] \arrow[d, "\ell"'] & 1X\oplus 1Y \arrow[d, "\ell\oplus\ell"] \\
X\oplus Y \arrow[r, equal] & X\oplus Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(X\oplus Y)1 \arrow[r, "\mathfrak{R}"] \arrow[d, "r"'] & X1\oplus Y1 \arrow[d, "r\oplus r"] \\
X\oplus Y \arrow[r, equal] & X\oplus Y
\end{tikzcd}$$

Definition 1.2. *Let $\mathcal{A}$ and $\mathcal{A}'$ be Ann-categories . An Ann-functor from $\mathcal{A}$ to $\mathcal{A}'$ is a functor $F\colon \mathcal{A}\to\mathcal{A}'$ together with natural isomorphisms $\breve{F}$, $\widetilde{F}$ such that: $(F,\breve{F})$ is a $\oplus AC$-functor, $(F,\widetilde{F})$ is a $\otimes A$-functor and $\breve{F}$, $\widetilde{F}$ are compatible with natural equivalences of distributivity in the sense that the following two* *diagrams are commutative*

$$\begin{tikzcd}
F(A(B\oplus C)) \arrow[r, "\widetilde{F}"] \arrow[d, "F(\mathfrak{L})"'] & FAF(B\oplus C) \arrow[r, "id\otimes \breve{L}"] & FA(FB\oplus FC) \arrow[d, "\mathfrak{L}'"] \\
F(AB\oplus AC) \arrow[r, "\breve{F}"] & F(AB)\oplus F(AC) \arrow[r, "\widetilde{F}\oplus\widetilde{F}"] & FAFB\oplus FAFC
\end{tikzcd}$$

$$\begin{tikzcd}
F((A\oplus B)C) \arrow[r, "\widetilde{F}"] \arrow[d, "F(\mathfrak{R})"'] & F(A\oplus B)FC \arrow[r, "\breve{L}\otimes id"] & (FA\oplus FB)FC \arrow[d, "\mathfrak{R}'"] \\
F(AC\oplus BC) \arrow[r, "\breve{F}"] & F(AC)\oplus F(BC) \arrow[r, "\widetilde{F}\oplus\widetilde{F}"] & FAFC\oplus FBFC
\end{tikzcd}$$

If F is an equivalence, then $(F, \breve{F}, \widetilde{F})$ is called an Ann-equivalence.

Proposition 1.3. *Let $\mathcal{A}$ be an Ann-category and $A \in \mathcal{A}$. Then there exist unique isomorphisms $\widehat{L}^A\colon A\otimes 0 \to 0$, $\widehat{R}^A\colon 0\otimes A \to 0$ so that $(L^A, \breve{L}^A, \widehat{L}^A)$ and $(R^A, \breve{R}^A, \widehat{R}^A)$ are symmetrical monoidal functors ($\oplus$ACU-functor)*

Proof. Since $(\mathcal{A}, \oplus)$ is a Pic-category, each $\oplus$AC-functor is also a $\oplus$ACU-functor. □

Proposition 1.4. *In any Ann-category $\mathcal{A}$, the isomorphisms $\widehat{L}^A$, $\widehat{R}^A$ have the following properties:*

(i) The family $\widehat{L}^- = \widehat{L}$ (resp. the family $\widehat{R}^- = \widehat{R}$) is a $\oplus$-morphism from the functor $(R^0, \breve{R}^0)$ (resp. $(L^0, \breve{L}^0)$) to the functor $(\theta : A \mapsto 0, \breve{\theta} = g_0^{-1})$ i. e. the following diagrams are commutative:

$$\begin{tikzcd}
A0 \arrow[r, "f\otimes id"] \arrow[d, "\widehat{L}^A"'] & B0 \arrow[d, "\widehat{L}^B"] \\
0 \arrow[r, equal] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
(X\oplus Y)0 \arrow[r, "\breve{R}^0"] \arrow[d, "\widehat{L}^{X\oplus Y}"] & X0\oplus Y0 \arrow[d, "\widehat{L}^X\oplus\widehat{L}^Y"] \\
0 & 0\oplus 0 \arrow[l, "g_0"']
\end{tikzcd}$$

(resp. $\widehat{R}^B(id\otimes f) = \widehat{R}^A$ and $\widehat{R}^{X\oplus Y} = g_0(\widehat{R}^X \oplus \widehat{R}^Y)\breve{L}^0$).

(ii) For any $A, B \in \mathcal{A}$, the following diagrams are commutative:

$$\begin{tikzcd}
X(0Y) \arrow[rr, "a"] \arrow[d, "id\otimes\widehat{R}^X"'] & & (X0)Y \arrow[d, "\widehat{L}^X\otimes id"] \\
X0 \arrow[r, "\widehat{L}^X"] & 0 & 0Y \arrow[l, "\widehat{R}^Y"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X(Y0) \arrow[r, "id\otimes\widehat{L}^Y"] \arrow[d, "a"'] & X0 \arrow[d, "\widehat{L}^X"] \\
(XY)0 \arrow[r, "\widehat{L}^{XY}"] & 0
\end{tikzcd}$$

and $\widehat{R}^{XY} = \widehat{R}^{Y}(\widehat{R}^{X} \otimes id)a_{0,X,Y}$.

(iii) $L^1 = l_0$, $R^1 = r_0$.

2 The first two invariants of an Ann-category

Let $\mathcal{A}$ be an Ann-category. Then the set $\Pi_0(\mathcal{A})$ of the isomorphic classes of objects of $\mathcal{A}$ is a ring with the operations induced by the ones $\oplus$, $\otimes$ in $\mathcal{A}$, and $\Pi_1(\mathcal{A}) = Aut(0)$ is an abelian group with operation denoted by $+$.

The following two Theorems on the structure of the Ann-categories can be found in [12].

Theorem 2.1. *$\Pi_1(\mathcal{A})$ is an $\Pi_0(\mathcal{A})$-bimodule where the left and right operations of the ring $\Pi_0(\mathcal{A})$ on the abeian group $\Pi_1(\mathcal{A})$ are defined respectively by*

$$su = \lambda_X(u), \quad us = \rho_X(u), \quad X \in s \in \Pi_0(\mathcal{A}),\ u \in \Pi_1(\mathcal{A})$$

in which λ_X, ρ_X are the two maps $Aut(0) \to Aut(0)$ given by the following commutative diagrams:

$$\begin{array}{ccc} X0 & \xrightarrow{\widehat{L}^X} & 0 \\ {\scriptstyle id\otimes u}\downarrow & & \downarrow{\scriptstyle \lambda_X(u)} \\ X0 & \xrightarrow{\widehat{L}^X} & 0 \end{array} \qquad \begin{array}{ccc} 0X & \xrightarrow{\widehat{R}^X} & 0 \\ {\scriptstyle u\otimes id}\downarrow & & \downarrow{\scriptstyle \rho_X(u)} \\ 0X & \xrightarrow{\widehat{R}^X} & 0 \end{array}$$

the following theorem shows the invariableness of $\Pi_0(\mathcal{A})$-bimodule $\Pi_1(\mathcal{A})$.

Theorem 2.2. *Given two Ann-categories $\mathcal{A}$, $\mathcal{A}'$. Then any Ann-functor $(F, \breve{F}, \widetilde{F})$: $\mathcal{A} \to \mathcal{A}'$ yields a ring homomorphism*

$$\begin{aligned} F_0 : \quad \Pi_0(\mathcal{A}) &\to \Pi_0(\mathcal{A}') \\ clX &\mapsto clFX \end{aligned}$$

and a group homomorphism

$$\begin{aligned} F_1 : \quad \Pi_1(\mathcal{A}) &\to \Pi_1(\mathcal{A}') \\ u &\mapsto \gamma_{F0}^{-1}(Fu) \end{aligned}$$

having the properties

$$F_1(su) = F_1(s)F_0(u) \qquad F_1(us) = F_0(u)F_1(s)$$

where γ_A: $Aut(0) \to Aut(0)$ is defined by $\gamma_A(u) = g_A(u \otimes id_A)g_A^{-1}$. Moreover, F is an Ann-equivalence if and only if F_0, F_1 are isomorphisms.

Hence $\Pi_0(\mathcal{A})$ and $\Pi_1(\mathcal{A})$ are the first two invariants of an Ann-category.

3 Reduced Ann-categories

In preparing to define the third invariant of Ann-categories, we construct reduced Ann-categories. Let $\mathcal{A}$ be an Ann-category. The reduced category $\mathcal{S}$ in constructed from $\Pi_0(\mathcal{A})$ and $\Pi_1(\mathcal{A})$ as follows: its objects are the elements of $\Pi_0(\mathcal{A})$, its morphisms are the automorphisms of the form (s, u) with $s \in \Pi_0(\mathcal{A}),\ u \in \Pi_1(\mathcal{A})$ i. e.

$$Aut(s) = \{s\} \times \Pi_1(\mathcal{A})$$

The composition law of morphisms is reduced by addition in $\Pi_1(\mathcal{A})$. We shall use the transmission of structures (see [10]) to change $\mathcal{S}$ into an Ann-category which is equivalent to $\mathcal{A}$. Choose for every $s \in \Pi_0(\mathcal{A})$ a representant $X_s \in \mathcal{A}$ such that $X_0 = 0$, $X_1 = 1$ and then, for every pair $s,\ t \in \Pi_0(\mathcal{A})$, two families of isomorphisms

$$\varphi_{s,t} :\ X_s \oplus X_t \to X_{s+t}, \qquad \psi_{s,t} :\ X_s X_t \to X_{st}$$

such that

$$\varphi_{0,t} = g_{X_t}, \qquad \varphi_{s,0} = d_{X_s}$$

$$\psi_{1,t} = 1_{X_t}, \qquad \psi_{s,1} = r_{X_s}, \qquad \psi_{0,t} = \widehat{R}^{X_t}, \qquad \psi_{0,s} = \widehat{L}^{X_s}$$

Defining the functor $H :\ \mathcal{S} \to \mathcal{A}$ by $H(s) = X_s,\ H(s,u) = \gamma_{X_s}(u)$ and putting $\breve{H} = \varphi^{-1},\ \widetilde{H} = \psi^{-1}$ we can use the theorem of transmission of structures (see [10]) to obtain $\mathcal{S}$ to be an Ann-category with the two operations in the explicit forms:

$$s \oplus t = s + t \qquad \text{(sum in ring } \Pi_0(\mathcal{A})) \tag{1}$$

$$(s,u) \oplus (t,v) = (s+t, u+v) \tag{2}$$

$$s \otimes t = st \qquad \text{(product in ring } \Pi_0(\mathcal{A})) \tag{3}$$

$$(s,u) \otimes (t,v) = (st, sv + ut) \tag{4}$$

and with the natural equivalences induced by that of $\mathcal{A}$. $\mathcal{S}$ is called the reduced Ann-category of $\mathcal{A}$. We now have:

Theorem 3.1. *In the reduced Ann-category $\mathcal{S}$ of $\mathcal{A}$, the natural equivalences of unitivity of the two operations $\oplus$, $\otimes$ are identities, and the natural equivalences $\xi,\ \eta,\ \alpha,\ \lambda,\ \rho$ induced from $a^+,\ c,\ a,\ \mathfrak{L},\ \mathfrak{R}$ by $(H, \breve{H}, \widetilde{H})$ are functions having the values in $\Pi_1(\mathcal{A})$ and satisfying the following relations*

1. $\xi(y,z,t) - \xi(x+y,z,t) + \xi(x,y+z,t) - \xi(x,y,z+t) + \xi(x,y,z) = 0$
2. $\xi(0,y,z) = \xi(x,0,t) = \xi(x,y,0) = 0$
3. $\xi(x,y,z) - \xi(x,z,y) + \xi(z,x,y) - \eta(x,z) + \eta(x+y,z) - \eta(y,z) = 0$
4. $\eta(x,y) + \eta(y,x) = 0$

5. $x\eta(y,z) - \eta(xy,xz) = \lambda(x,y,z) - \lambda(x,z,y)$

6. $\eta(x,y)z - \eta(xz,yz) = \rho(x,y,z) - \rho(y,x,z)$

7. $x\xi(y,z,t) - \xi(xy,xz,xt) =$
$\lambda(x,z,t) - \lambda(x,y+z,t) + \lambda(x,y,z+t) - \lambda(x,y,z)$

8. $\xi(x,y,z)t - \xi(xt,yt,zt) =$
$\rho(y,z,t) - \rho(x+y,z,t) + \rho(x,y+z,t) - \rho(x,y,t)$

9. $\rho(x,y,z+t) - \rho(x,y,z) - \rho(x,y,t) + \lambda(x,z,t)$
$+ \lambda(y,z,t) - \lambda(x+y,z,t) = -\xi(xz+xt,yz,yt)$
$+ \xi(xz,xt,yz) - \eta(xt,yz) + \xi(xz+yz,xt,yt) - \xi(xz,yz,xt)$

10. $\alpha(x,y,z+t) - \alpha(x,y,z) - \alpha(x,y,t) =$
$x\lambda(y,z,t) + \lambda(x,yz,yt) - \lambda(xy,z,t)$

11. $\alpha(x,y+z,t) - \alpha(x,y,t) - \alpha(x,z,t) =$
$x\rho(y,z,t) - \rho(xy,xz,t) + \lambda(x,yt,zt) - \lambda(x,y,z)t$

12. $\alpha(x+y,z,t) - \alpha(x,z,t) - \alpha(y,z,t) =$
$-\rho(x,y,z)t - \rho(xz,yz,t) + \rho(x,y,zt)$

13. $x\alpha(y,z,t) - \alpha(xy,z,t) + \alpha(x,yz,t)$
$- \alpha(x,y,zt) + \alpha(x,y,z)t = 0$

14. $\alpha(1,y,z) = \alpha(x,1,z) = \alpha(x,y,1) = 0$

15. $\alpha(0,y,z) = \alpha(x,0,t) = \alpha(x,y,0) = 0$

16. $\lambda(1,y,z) = \lambda(0,y,,z) = \lambda(x,0,z) = \lambda(x,y,0) = 0$

17. $\rho(x,y,1) = \rho(0,y,z) = \rho(x,0,z) = \rho(x,y,0) = 0$

for $x,y,z,t \in \Pi_0(A)$.

For the two choices of different representants (X_s, φ, ψ), we can prove the followings:

Proposition 3.2. *If* $\mathcal{S}$ *with* (X_s, φ, ψ) *and* $\mathcal{S}'$ *with* (X_s', φ', ψ') *are two reduced Ann-categories of* $\mathcal{A}$, *then there exists an Ann-equivalence* $(F, \breve{F}, \widetilde{F})$: $\mathcal{S} \to \mathcal{S}'$, *with* $F = id$.

If we substitute $\Pi_0(\mathcal{A})$ by a ring R and $\Pi_1(\mathcal{A})$ by an R-bimodule M, we can construct an Ann-category $\mathcal{I}$ with the operations $\oplus$, $\otimes$ defined by the relations (3.1)-(3.4) and the natural equivalences

$$a^+ = \xi, c = \eta, a = \alpha, \mathfrak{L} = \lambda, \mathfrak{R} = \rho$$

satisfying the relations in the theorem 3.1. This Ann-category $\mathcal{I}$ is called an *Ann-category of type* (R, M).

If the function η satisfies the *regular condition* $\eta(x,x) = 0$, the family $(\xi, \eta, \alpha, \lambda, \rho)$ is a 3-cocycle of the ring R with coefficients in the R-bimodule M in the Mac Lane-Shukla sense (see theorem 4.3). In particular, when $\lambda = 0, \rho = 0, \xi = 0$ we have $\eta = 0$ and hence α becomes a normal 3-cocycle of the $\mathbb{Z}$-algebra R in the Hochshild sense (see [10]).

Any ring R with the unit $1 \neq 0$ may be considered as an Ann-category of the type $(R, 0)$. Hence we have proved the following theorem:

Theorem 3.3. *Any Ann-category is an Ann-equivalence to an Ann-category of the type* (R, M).

4 Cohomology classification of the regular Ann-categories

According to theorem 3.3 we have only to consider the classification of the Ann-categories having the first two common invariants.

Definition 4.1. *Let R be a ring with unit, M be an $R-$bimodule considered as a ring with the null multiplication. An Ann-category $\mathcal{A}$ is called having pre-stick of the type (R, M) if there exists a pair of ring isomorphisms (ϵ_0, ϵ_1)*

$$\epsilon_0 : R \longrightarrow \Pi_0(\mathcal{A}), \quad \epsilon_1 : M \longrightarrow \Pi_1(\mathcal{A})$$

satisfying the conditions:

$$\epsilon_1(su) = \epsilon_0(s)\epsilon_1(u), \quad \epsilon_1(us) = \epsilon_1(u)\epsilon_0(s), \quad s \in R, u \in M.$$

A morphism between two Ann-categories $\mathcal{A}, \mathcal{A}'$ having the same pre-stick of the type (R, M) is an Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \longrightarrow \mathcal{A}'$ such that the following diagrams are commutative

$$\begin{array}{ccc} \Pi_0(\mathcal{A}) & \xrightarrow{F_0} & \Pi_0(\mathcal{A}') \\ {\scriptstyle \epsilon_0}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_0} \\ R & =\!= & R \end{array} \qquad \begin{array}{ccc} \Pi_1(\mathcal{A}) & \xrightarrow{F_1} & \Pi_1(\mathcal{A}') \\ {\scriptstyle \epsilon'_1}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon'_1} \\ M & =\!= & M \end{array}$$

in which F_0, F_1 are two ring morphisms induced from $(F, \breve{F}, \widetilde{F})$. It follows directly from the definition that F is an equivalence.

The two Ann-categories $\mathcal{A}, \mathcal{A}'$ are called congruences *if there exists a morphism $(F, \breve{F}, \widetilde{F})$betweenthem.*

Definition 4.2. *An Ann-category $\mathcal{A}$ having a natural equivalence c of commutativity so that $c_{X,X} = id$ is called a* regular *Ann-category.*

For the regular Ann-categories we can define its third invariant, that is an element of Mac Lane - Shukla cohomology group $H^3(R, M)$ of the ring R.

Recall that the cohomology of an algebra Λ with coefficients in an $\Lambda-$bimodule coincides with the Mac Lane cohomology of the ring $R = \Lambda$, considered as a $\mathbb{Z}-$algebra. We have

$$H^*(R, M) = H^*(\sum_{n\geq 0} Hom_{\mathbb{Z}}(U^n, M))$$

where U is a graded differential algebra and a free resolution over $\mathbb{Z}$ of R. The differential δ over graded module $\sum Hom_{\mathbb{Z}}(U^n, M)$ is defined by the relation $\delta f = g + h$, where

$$g(u_1, ..., u_n) = -\sum_{i=1}^{n}(-1)^{e_{i-1}} f(u_1, ..., du_i, ..., u_n),$$

$$h(u_1, ..., u_n) = u_1 f(u_2, ..., u_{n+1}) + \\ \sum_{i=1}^{n}(-1)^{e_i} f(u_1, \ldots, u_i u_{i+1}, \ldots, u_{n+1}) + (-1)^{e_{n+1}} f(u_1, \ldots, u_n)u_{n+1},$$

$e_0 = 0, e_i = i + degu_1 + \cdots + degu_i$ (see [13]).

Theorem 4.3. *A 3-cochain $f =< \zeta, \eta, \alpha, \lambda, \rho >$ of the ring R with coefficients in the $R-$bimodule M is a 3-cocycle if and only if $(\zeta, \eta, \alpha, -\lambda, \rho)$ is a family of natural equivalences of a regular Ann-category of the type (R, M).*

Proof. The essence of the proof is to compute the group $\mathbb{Z}^3(R, M)$ by choosing a convenient resolution of the ring R (as a $\mathbb{Z}$-algebra), different from the two resolutions of Shukla and Mac Lane. For the additional structure of R, we consider the complex of abelian groups:

$$0 \longrightarrow B_4 \xrightarrow{d_4} B_3 \xrightarrow{d_3} B_2 \xrightarrow{d_2} B_1 \xrightarrow{d_1} B_0 \xrightarrow{\nu} R \longrightarrow 0$$

in which

$$B_0 = \mathbb{Z}(\dot{R}), \quad B_1 = \mathbb{Z}(\dot{R} \times \dot{R}), \quad B_2 = \mathbb{Z}(\dot{R} \times \dot{R} \times \dot{R}) \oplus \mathbb{Z}(\dot{R} \times \dot{R})$$

$$B_3 = \mathbb{Z}(\dot{R} \times \dot{R} \times \dot{R} \times \dot{R}) \oplus \mathbb{Z}(\dot{R} \times \dot{R} \times \dot{R}) \oplus \mathbb{Z}(\dot{R} \times \dot{R}) \oplus \mathbb{Z}(\dot{R})$$

$$B_4 = Kerd_3, \quad \dot{R} = R \backslash \{0\}$$

($\mathbb{Z}(\dot{R}^i)$, $i = 1,2,3,4$ are the free abelian groups generated by the set $\dot{R}^i$).

The morphisms are given by:

$$
\begin{aligned}
\nu[x] &= x, \quad x \in \dot{R} \\
d_1[x,y] &= [y] - [x+y] + [x] \\
d_2[x,y,z] &= [x,z] - [x+y,z] + [x,y+z] - [x,y] \\
d_2[x,y] &= [x,y] - [y,x] \\
d_3[x,y,z,t] &= [y,z,t] - [x+y,z,t] + [x,y+z,t] - [x,y,z+t] + [x,y,z] \\
d_3[x,y,z] &= [x,y,z] - [x,z,y] + [z,x,y] + [x+y,z] - [x,z] - [y,z] \\
d_3[x,y] &= [x,y] + [y,x] \\
d_3[x] &= [x,x]
\end{aligned}
$$

$d_4 = i$ is the natural embedding.

We now define a distributive multiplication in $B = \sum B_i$ such that B becomes a graded differential algebra over $\mathbb{Z}$. A 3-cochain f is an element of a direct sum

$$
Hom_{\mathbb{Z}}(B_2, M) \oplus Hom_{\mathbb{Z}}(B_1 \otimes B_0, M) \oplus Hom_{\mathbb{Z}}(B_0 \otimes B_1, M) \oplus Hom_{\mathbb{Z}}(B_0 \otimes B_0 \otimes B_0, M)
$$

This implies that f is defined by a family of mappings

$$
\begin{aligned}
\zeta(x,y,z) &= f([x,y,z]) \\
\eta(x,y) &= f([x,y]) \\
\lambda(x,y,z) &= f([x] \otimes [y,z]) \\
\rho(x,y,z) &= f([x,y] \otimes [z]) \\
\alpha(x,y,z) &= f([x] \otimes [y] \otimes [z])
\end{aligned}
$$

From the formula of differentiation of the above resolution we complete the proof. □

Theorem 4.4 (Classification theorem). *There exists a bijection between the set of the congruence classes of pre-sticked regular Ann-categories of the type (R,M) and the cohomology group $H^3(R,M)$ of the ring R, with coefficients in the R-bimudule M.*

Proof. Consider the resolution that is shown in the proof of the theorem 4.3. If $f =< \zeta, \eta, \alpha, \lambda, \rho >$ is 3-coboundary, $f = \delta g$, with g is a pair of mappings

$$
\begin{aligned}
\mu &: B_1 \longrightarrow M \\
\nu &: B_0 \otimes B_0 \longrightarrow M
\end{aligned}
$$

we have the following relations

$$\begin{array}{rcl}
-\zeta(x,y,z) & = & \mu(y,z)-\mu(x+y,z)+\mu(x,y+z)-\mu(x,y)\\
-\eta(x,y) & = & \mu(x,y)-\mu(y,x)=ant\mu(x,y)\\
\alpha(x,y,z) & = & x\nu(y,z)-\nu(xy,z)+\nu(x,yz)-\nu(x,y)z\\
-\lambda(x,y,z) & = & \nu(x,y+z)-\nu(x,y)-\nu(x,z)+x\mu(y,z)-\mu(xy,xz)\\
\rho(x,y,z) & = & \nu(x+y,z)-\nu(x,z)-\nu(y,z)-\mu(x,y)z+\mu(xz,yz)
\end{array}$$

These relations imply what we have to prove. □

This theorem leads to the investigation of application of the Ann-category concept into the theory of ring extensions. The classtification theorem in the general case is still an open problem.

5 Ann-functors and low dimension cohomology groups of rings

In this section given problem is that of finding whether there is Ann-functor between two Ann-categories and, if so, how many. Since each Ann-category is Ann-equivalent to one Ann-category of the type (R, M) so the solution of problem for a class of Ann-categories of the type (R, M) is enough.

If $f =< \zeta, \eta, \alpha, \lambda, \rho >$ is a 3-cocycle in $\mathbb{Z}^3(R, M)$ the structure $(\zeta, \eta, \alpha, -\lambda, \rho)$ of Ann-category (R, M) is denoted by $\hat{f}$. Moreover, if

$$F = (F, \breve{F}, \widetilde{F}) : (R, M, \hat{f}) \longrightarrow (R', M', \hat{f}')$$

is an Ann-functor, this functor is a pair of ring homomorphisms (F_0, F_1) compatible with actions of bimodule. So sometimes F is denoted by (F_0, F_1). R'-bimodule M may be changed into R-bimodule by the homomorphism F_0,

$$m'r = mF(r), \quad rm' = F(r)m', \quad r \in R, m' \in M'.$$

Because $f \in \mathbb{Z}^3(R, M)$ and $f' \in \mathbb{Z}^3(R', M')$, F induces canonically 3-cocycles

$$f_*, f'^* \in \mathbb{Z}^3(R, M').$$

For axample

$$\zeta_*(x,y,z) = F(\zeta(x,y,z))$$
$$\zeta'^*(x,y,z) = \zeta'(Fx, Fy, Fz).$$

Isomorphisms $\breve{F}, \widetilde{F}$ are mappings $R \times R \longrightarrow M'$

$$\begin{array}{rcccccc}
\mu(x,y) & = & \breve{F}_{x,y} & : & F(x+y) & \longrightarrow & Fx+Fy\\
\nu(x,y) & = & \widetilde{F}_{x,y} & : & F(xy) & \longrightarrow & (Fx)(Fy)
\end{array}$$

These mappings, according to definition, satisfy diagrams in definition 1.2. On the other hand, $< \mu, \nu >$ is a 2-cochain of ring cohomology. From a calculation of $H^3(R, M)$ we have

$$f_* - {f'}^* = \delta < \mu, \nu > \tag{5}$$

Theorem 5.1. *Let $\mathcal{I} = (R, M, \hat{f})$, $\mathcal{I}' = (R', M', \hat{f}')$ be two regular Ann-categories and*

$$F = (F_0, F_1) : \mathcal{I} \longrightarrow \mathcal{I}'$$

be a functor that satisfies the condition (5.1). Then F is an Ann-functor if and only if $H_*(f) - H^*(f') = 0$ in $H^3(R, M')$. In this case, we can say that Ann-functor $(F, \breve{F}, \widetilde{F})$ is induced by the functor F.

Proof. If $(F, \breve{F}, \widetilde{F})$ is an Ann-functor with $\breve{F} = \mu, \widetilde{F} = \nu$, the condition (5.1) gives equation

$$H_*(f) - H^*(f') = 0$$

Conversely, the equation $H_*(f) - H^*(f') = 0$ automatically implies $f_* - {f'}^* = \delta g$, there $g =< \mu, \nu >$ is a 2-cochain. Let $\breve{F} = \mu, \widetilde{F} = \nu$, we have an Ann-functor $(F, \breve{F}, \widetilde{F})$. □

Definition 5.2. *An Ann-functor $F : (R, M, f) \longrightarrow (R', M', f')$ is called regular if F satisfies condition $f_* = {f'}^*$.*

In case there exists a regular Ann-functor F, we have the following theorem

Theorem 5.3. *(i) There exists a bijection between the set of the congruence classes of regular Ann-functors induced by a pair (F_0, F_1) and the cohomology group $H^2(R, M')$ of the ring R with coefficients in the R-bimodule M'.*

(ii) If $F : (R, M, f) \longrightarrow (R', M', f')$ is an Ann-functor, there exists a bijection

$$Aut(F) \longrightarrow \mathbb{Z}^1(R, M')$$

between the group of automorphisms of Ann-functor F and the group $\mathbb{Z}^1(R', M')$.

Proof. (i) Let $(F, \breve{F}, \widetilde{F})$ be a regular Ann-functor

$$(F, \breve{F}, \widetilde{F}) : (R, M, f) \longrightarrow (R', M', f)$$

Then

$$f_* - {f'}^* = \delta < \mu, \nu >= 0$$

where $\breve{F} = \mu, \widetilde{F} = \nu$. It means $< \mu, \nu >$ is 2-cocycle.

Suppore that $(G, \breve{G}, \widetilde{G})$ is another regular Ann-functor

$$(G, \breve{G}, \widetilde{G}) : (R, M, f) \longrightarrow (R', M', f)$$

and $\alpha : F \longrightarrow G$ is an Ann-morphism. Then, by to definition, the following diagrams are commutative

$$\begin{array}{ccc} F(x+y) & \xrightarrow{\breve{F}} & Fx+Fy \\ {\scriptstyle \alpha_{x+y}}\downarrow & & \downarrow{\scriptstyle \alpha_x+\alpha_y} \\ G(x+y) & \xrightarrow{\breve{G}} & Gx+Gy \end{array}$$

$$\begin{array}{ccc} F(xy) & \xrightarrow{\widetilde{F}} & (Fx)(Fy) \\ {\scriptstyle \alpha_{xy}}\downarrow & & \downarrow{\scriptstyle \alpha_x\otimes\alpha_y} \\ G(xy) & \xrightarrow{\widetilde{G}} & (Gx)(Gy) \end{array}$$

where $x, y \in R$. Also from the definition we have

$$\alpha_x \otimes \alpha_y = (Fx)\alpha_y + \alpha_x(Fy) = x\alpha_y + \alpha_x y$$

so

$$\begin{array}{rcl} \breve{G}_{x,y} - \breve{F}_{x,y} & = & \alpha_x - \alpha_{x+y} + \alpha_y \\ \widetilde{G}_{x,y} - \widetilde{F}_{x,y} & = & x\alpha_y - \alpha_{x+y} + \alpha_x y. \end{array}$$

Because $g =< \breve{F}, \widetilde{F} >$, $\quad g' =< \breve{G}, \widetilde{G} >$ are 2-cocycles and α is 1-cochain and by a calculation of $H^2(R, M)$ we have

$$g' - g = \delta\alpha \tag{6}$$

Equation (5.2) proves the existance of a correspondence from a class of regular Ann-functors cls$(F, \breve{F}, \widetilde{F})$ to a class of cohomologies $g + B^2(R, M')$, $g =< \breve{F}, \widetilde{F} >$. Moreover this correspondence is an injection. We now prove that it is a projection. In fact, let $g =< \mu, \nu >$ be any 2-cocycle. Then we can directly verify that (F, μ, ν) is a regular Ann-functor (R, M, f) to (R', M', f) corresponding to 2-cocycle g, proving (i).

(ii) Let

$$F = (F, \mu, \nu) : (R, M, f) \longrightarrow (R', M', f)$$

be an Ann-functor and $\alpha \in Aut(F)$. Then the equation (5.2) becomes $\delta(\alpha) = 0$, i.e. $\alpha \in \mathbb{Z}^1(R, M')$, proving (ii). □

6 Ann-category and theory of the extensions of rings

In this section, we establish a direct relation between theory of the extensions of rings and theory of Ann-categories. According to Mac Lane [7] we call a *bimultiplication* of a ring A a pair of mappings $a \longmapsto \sigma a, a \longmapsto a\sigma$ of A into itself which satisfy the rules

$$\begin{aligned} \sigma(a+b) = & \sigma a + \sigma b \quad , \quad (a+b)\sigma = a\sigma + b\sigma \\ \sigma(ab) = & \quad (\sigma a)b \quad , \quad (ab)\sigma = \quad a(b\sigma) \\ & a(\sigma b) \quad = \quad (a\sigma)b \end{aligned}$$

for all elements $a, b \in A$. The sum $\sigma + \nu$ and the product $\sigma\nu$ of two bimultiplications σ and ν are defined by the equations

$$\begin{aligned} (\sigma+\nu)a = & \sigma a + \nu a \quad , \quad a(\sigma+\nu) = a\sigma + a\nu \\ (\sigma\nu)a = & \quad \sigma(\nu a) \quad , \quad a(\sigma\nu) = \quad (a\sigma)\nu \end{aligned}$$

for all a in A.

The set of all bimultiplications of A is a ring denoted by M_A. For each element c of A, a bimultiplication μ_c is defined by

$$\mu_c a = ca, \quad a\mu_c = ac, \quad a \in A$$

We call μ_c an *inner bimultiplication*. Clearly $\mu : A \longrightarrow M_A$ is a ring homomorphism and the image μA of this homomorphism is a two-sided ideal in M_A. The quotient ring $P_A = M_A/\mu A$ is called the ring of *outer bimultiplications* of A and ring homomorphism $\theta : R \longrightarrow P_A$ is called *regular* if $\theta(1) = 1$ and two any elements of $\theta(R)$ are *permutable* (the bimultiplications σ and ν are called permutable if $\sigma(a\nu) = (\sigma a)\nu$ and $\nu(a\sigma) = (\nu a)\sigma$ for every a in A). Then

$$C_A = \{c \in A | ca = ac = 0, \forall a \in A\}$$

is called *bicenter* of A, and C_A is a R-bimodule under the operations

$$xc = (\theta x)c, \quad cx = c(\theta x), \quad c \in C_A, x \in A.$$

The "Extention problem" of rings requires finding the exact sequence of rings

$$0 \longrightarrow A \longrightarrow S \longrightarrow R \longrightarrow 1$$

induces homomorphism $\theta : R \longrightarrow P_A$.

Let $\sigma : R \longrightarrow M_A$ be a mapping such that $\sigma(x) \in \theta x, x \in R$ and $\sigma(0) = 0$, $\sigma(1) = 1$. Then we define two mappings

$$\begin{aligned} f : R \times R & \longrightarrow A \\ g : R \times R & \longrightarrow A \end{aligned}$$

such that

$$\begin{aligned} \mu f(x,y) &= \sigma(x+y) - \sigma(x) - \sigma(x) \\ \mu g(x,y) &= \sigma(xy) - \sigma(x)\sigma(x) \end{aligned}$$

for all $x, y \in R$. The ring structure of M_A implies mappings $\zeta, \alpha, \lambda, \rho : {M_A}^3 \longrightarrow C_A$ and $\eta : {M_A}^2 \longrightarrow C_A$

$$\begin{aligned}
\zeta(x,y,z) &= f(x,y) - f(x+y,z) + f(x,y+z) - f(x,y) \\
\eta(x,y) &= f(x,y) - f(y,x) \\
\alpha(x,y,z) &= xg(y,z) - g(x,y,z) + g(x,y,z) - g(x,y)z \\
\lambda(x,y,z) &= xf(y,z) - f(xy,xz) + g(x,y+z) - g(x,y) - g(x,z) \\
\rho(x,y,z) &= f(x,y)z - f(xz,yz) + g(x+y,z) - g(x,z) - g(y,z)
\end{aligned}$$

We call the family $(\zeta, \eta, \alpha, \lambda, \rho)$ of the above mappings an obstruction of the regular homomorphism θ. We can prove that if all these mappings are null, the homomorphism $\theta : R \longrightarrow P_A$ can be realized by a ring extention. It is the ring

$$S = \{(a, r) \quad | \quad a \in A, r \in R\}$$

with operations

$$\begin{aligned}
(a_1, r_1) + (a_2, r_2) &= (a_1 + a_2 + f(r_1, r_2), r_1 + r_2) \\
(a_1, r_1)(a_2, r_2) &= (r_1 a_2 + a_1 r_2 + g(r_1 r_2), r_1 r_2)
\end{aligned}$$

In the general case we have

Proposition 6.1. *If $(\zeta, \eta, \alpha, \lambda, \rho)$ is an* obstruction *of the regular homomorphism $\theta : R \longrightarrow P_A$, it is a family of natural equivalences of Ann-categories of the type* (R, C_A).

Proof. We can verify directly that $\zeta, \eta, \alpha, \lambda, \rho$ satisfy the relations in the proposition 3.1 . □

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