ON J-RINGS AND PERIODIC RINGS

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Abstract

We characterize infinite J-rings by using conditions on infinite subsets and infinite subrings, and we give a sufficient conditon for a ring to be periodic.

1 Introduction

Let R be a ring, N its set of nilpotent elements, T its set of torsion elements, and P(R) its prime radical. For each $x \in R$, denote by A(x) and $\langle x \rangle$ respectively the two-sided annihilator of x and the subring generated by x, and denote by S the set of zero divisors $a \in R$ for which $A(a) \neq \{0\}$. An element $x \in R$ is called periodic if there exist distinct positive integers m, n such that $x^m = x^n$; and x is called potent if there exists an integer n > 1 such that $x^n = x$. The ring R is defined to be periodic if every element of R is periodic; and periodic rings in which every element is potent are called J-rings, in honor of Jacobson's famous theorem asserting commutativity of such rings. Finally, R is called reduced if $N = \{0\}$.

A number of theorems in the recent literature deduce certain elementwise conditions on infinite rings from corresponding partial conditions on infinite subsets - e.g. [2, Theorem 1.1] or [6, Theorem 4.1]. Our purpose is to prove results of this kind which characterize infinite J-rings, and to give sufficient conditions for certain rings to be periodic.

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2 Preliminaries

We mention some well-known and easily-established properties of potent elements and J-rings:

(i) If x and y are both potent, there exists a single n such that $x^n = x$ and $y^n = y$. Moreover, if $x^n = x$, then $x^{j+k(n-1)} = x^j$ for all positive integers j, k; and thus x^{n-1} is idempotent.

(ii) If R is a J-ring, then R is reduced and R = T.

(iii) If R is a reduced ring, then every periodic element is potent. Thus every reduced periodic ring is a J-ring.

We now state some known theorems which we shall require. The first two deal with FZS - rings – that is, rings in which every zero subring is finite.

Lemma 1 ([5], Lemma 1.6) If R is an FZS-ring and $x \in N$, then A(x) is of finite index in R.

Lemma 2 ([8], Theorem 4) If R is a semiprime FZS-ring, then $R = B \oplus C$, where B is reduced and C is a direct sum of finitely many total matrix rings over finite fields.

Lemma 3 ([3], Theorem 4) If R is an infinite ring which is not a domain, then S is infinite and S contains an infinite subring of R.

Our final lemma in this section is due to Chacron [7]; a proof is also found in [4].

Lemma 4 Let R be a ring with the property that for each $x \in R$, there exists a positive integer m and a polynomial $p(X) \in \mathbb{Z}[X]$ such that $x^m = x^{m+1}p(x)$. Then R is periodic.

3 Results on rings with infinite-subset conditions

Our first result, which is trivial but useful, is a characterization of periodic rings.

Theorem 5 If R is a ring such that every infinite subset of R contains a periodic element, then R is periodic.

Proof Suppose R is not periodic, and x is a nonperiodic element of R. Then $\{x, x^2, x^3, ...\}$ is an infinite subset containing no periodic element.

Theorem 6 Let R be an infinite ring which is not a domain. If every infinite subset of S contains a potent element, then R is a J-ring.

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Proof Note first that R is an FZS-ring. We shall show that R is reduced. Suppose, on the contrary, that $u \in N \setminus \{0\}$ and $u^k = 0 \neq u^{k-1}$. By Lemma 1, A(u) is infinite; and since $A(u) \subseteq S$, every infinite subset of A(u) contains a potent element. Therefore the set $\hat{A}(u)$ of all potent elements in A(u) is infinite and hence $u + \hat{A}(u)$ is infinite; moreover $u + \hat{A}(u) \subseteq S$, since $u + \hat{A}(u) \subseteq A(u^{k-1})$. Thus there exists $a \in A(u)$ such that both a and a + u are potent; and there exists n > 1 such that $a^n = a$, $(a + u)^n = a + u$, and $u^n = 0$. But then $a + u = (a + u)^n = a^n = a$, so u = 0 - a contradiction; therefore R is reduced as claimed.

Since R is not a domain, S is infinite by Lemma 3; hence S contains a nonzero potent element and hence a nonzero idempotent e, which must be central because R is reduced. Thus $R = eR \oplus A(e)$, with both summands contained in S. If one of these summands is finite, it is a J-ring by (iii); and each infinite summand, of which there must be at least one, is a reduced periodic ring by Theorem 5 and hence a J-ring. But a direct sum of two J-rings is a J-ring, so we are finished.

Our final theorem in this section is stated in [1], but the proof given there is not correct.

Theorem 7 If R is an infinite ring in which every infinite subset contains a potent element, then R is a J-ring.

Proof If R is not a domain, it is a J-ring by Theorem 6. If R is a domain, it is clearly reduced; and it is periodic by Theorem 5, hence is a J-ring.

4 Rings with conditions on infinite subrings

In this section we study a weaker condition - namely, that every infinite subring contains a nonzero potent element. A ring satisfying this condition will be called a PJS-ring (partially Jacobson subring ring).

It is clear that an infinite PJS-ring need not be a J-ring; one need only consider a ring $R_1 \oplus R_2$, where R_1 is an infinite J-ring and R_2 is a finite ring which is not a J-ring. However, it follows from Proposition 4 and Lemma 8 of [3] that every infinite PJS-ring contains an infinite J-subring.

Our next theorem provides a complete characterization of J-rings.

Theorem 8 A ring R is a J-ring if and only if R is a reduced PJS-ring.

Proof Our condition is obviously necessary, so we proceed to establish sufficiency. Let R be a reduced PJS-ring. Since finite reduced rings are J-rings, we may assume that R is infinite.

We show first that R = T. Suppose, on the contrary, that $a \in R \setminus T$. Then $\langle a \rangle$ is infinite, so there exists a nonzero potent element $n_1 a^{j_1} + \cdots + n_s a^{j_s}$; and it follows that there exist a relation

(4.1) $m_1 a^{k_1} + \dots + m_t a^{k_t} = 0,$

where $m_1, m_2, ..., m_t \in \mathbf{Z} \setminus \{0\}$ and $1 \leq k_1 < k_2 < ... < k_t$. We may assume without loss that k_1 is minimal among all such relations. If $k_1 > 1$, then ac = 0, where $c = m_1 a^{k_1-1} + \cdots + m_t a^{k_t-1}$. But this implies $c^2 = 0 = c$, contradicting the minimality of k_1 ; therefore $k_1 = 1$ and (4.1) takes the form

$$m_1 a = aq(a),$$

where $q(X) \in X\mathbf{Z}[X]$. Taking b = q(a), we have $b \notin T$ and $b^2 = m_1 b$; thus $\langle b \rangle$ is isomorphic to the subring of \mathbf{Z} generated by m_1 . Now if $m_1 \neq \pm 1$, $\langle b \rangle$ cannot contain a nonzero potent element; and if $m_1 = \pm 1$, $\langle 2b \rangle$ cannot contain a nonzero potent element. Therefore our assumption that $a \in R \setminus T$ must be false, and R = T.

For each prime p, let R_p denote the p-primary component of R; and note that, since R is reduced, $pR_p = \{0\}$. Let $0 \neq a \in R$. Then $a = a_1 + a_2 + \cdots + a_k$, where $0 \neq a_i \in R_{p_i}$ for primes p_1, p_2, \ldots, p_k . Let b be a typical a_i and q the corresponding p_i . If $\langle b \rangle$ is finite there exist positive integers m, n with m < n such that $b^m = b^n$. If $\langle b \rangle$ is infinite, then there exists a nonzero potent element $c = n_1 b^{k_1} + n_2 b^{k_2} + \cdots + n_s b^{k_s}$, where each $n_j \not\cong 0 \pmod{q}$ and $k_1 < k_2 < \cdots < k_s$; therefore, there exists $f(X) \in X\mathbf{Z}[X]$ with lowest degree term $n_1 X^{k_1}$, such that f(b) = 0. Taking $m_1 \in \mathbf{Z}$ such that $n_1 m_1 \equiv 1 \pmod{q}$, we have $m_1 f(b) = 0$, so $X\mathbf{Z}[X]$ contains a co-monic polynomial g(X) (i.e. a polynomial with lowest-degree coefficient equal to 1) such that g(b) = 0.

We have shown that for each a_i , there is co-monic $g_i(X) \in X\mathbb{Z}[X]$ for which $g_i(a_i) = 0$. Therefore $G(X) = \prod g_i(X)$ is co-monic and G(a) = 0. It follows by Lemma 4 that R is periodic; and since R is reduced, R is a J-ring.

Theorem 9 Let R be reduced ring. If R is a not a domain and every infinite subring of R which is contained in S contains a nonzero potent element, then R is a J-ring.

Proof If R is finite, there is nothing to prove, since R is reduced. Therefore, assume R is infinite, in which case Lemma 3 guarantees that S contains an infinite subring and hence a nonzero idempotent e. As in the proof of Theorem 6, e is central and $R = eR \oplus A(e)$, with each summand contained in S. Now each infinite summand is a J-ring by Theorem 8; and if either summand is finite, it is a J-ring. \Box

Our final theorem may be regarded as an extension of Theorem 8.

Theorem 10 If R is a ring such that every infinite subring contains a nonnilpotent periodic element, then R is periodic.

Proof Let $\overline{R} = \frac{R}{P(R)}$. It is easy to see that \overline{R} inherits the given hypothesis, which implies that \overline{R} is an FZS-ring. Moreover, \overline{R} is semiprime; hence by

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Lemma 2, $\overline{R} = B \oplus C$, where B is reduced and C is finite. Now B is a J-ring by Theorem 8, and C is periodic; hence \overline{R} is periodic. Since P(R) is a nil ideal, it follows by Lemma 4 that R is periodic.

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