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# ON J-RINGS AND PERIODIC RINGS

**Howard E. Bell**

*Department of Mathematics, Brock University,*
*St. Catharines, Ontario, Canada L2S 3A1*
*e-mail : hbell@spartan.ac.brocku.ca*

**Abstract**

We characterize infinite J-rings by using conditions on infinite subsets and infinite subrings, and we give a sufficient conditon for a ring to be periodic.

## 1 Introduction

Let $R$ be a ring, $N$ its set of nilpotent elements, $T$ its set of torsion elements, and $P(R)$ its prime radical. For each $x \in R$, denote by $A(x)$and $< x >$ respectively the two-sided annihilator of $x$ and the subring generated by $x$, and denote by $S$ the set of zero divisors $a \in R$ for which $A(a) \neq \{0\}$. An element $x \in R$ is called periodic if there exist distinct positive integers $m$, $n$ such that $x^m = x^n$; and $x$ is called potent if there exists an integer $n > 1$ such that $x^n = x$. The ring $R$ is defined to be periodic if every element of R is periodic; and periodic rings in which every element is potent are called J-rings, in honor of Jacobson's famous theorem asserting commutativity of such rings. Finally, $R$ is called reduced if $N = \{0\}$.

A number of theorems in the recent literature deduce certain elementwise conditions on infinite rings from corresponding partial conditions on infinite subsets - e.g. [2, Theorem 1.1] or [6, Theorem 4.1]. Our purpose is to prove results of this kind which characterize infinite J-rings, and to give sufficient conditions for certain rings to be periodic.

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## 2 Preliminaries

We mention some well-known and easily-established properties of potent elements and J-rings:

(i) If $x$ and $y$ are both potent, there exists a single $n$ such that $x^n = x$ and $y^n = y$. Moreover, if $x^n = x$, then $x^{j+k(n-1)} = x^j$ for all positive integers $j$, $k$; and thus $x^{n-1}$ is idempotent.

(ii) If $R$ is a J-ring, then $R$ is reduced and $R = T$.

(iii) If $R$ is a reduced ring, then every periodic element is potent. Thus every reduced periodic ring is a J-ring.

We now state some known theorems which we shall require. The first two deal with FZS - rings – that is, rings in which every zero subring is finite.

**Lemma 1** ([5], Lemma 1.6) *If $R$ is an FZS-ring and $x \in N$, then $A(x)$ is of finite index in $R$.*

**Lemma 2** ([8],Theorem 4) *If $R$ is a semiprime FZS-ring, then $R = B \oplus C$, where $B$ is reduced and $C$ is a direct sum of finitely many total matrix rings over finite fields.*

**Lemma 3** ([3], Theorem 4) *If $R$ is an infinite ring which is not a domain, then $S$ is infinite and $S$ contains an infinite subring of $R$.*

Our final lemma in this section is due to Chacron [7]; a proof is also found in [4].

**Lemma 4** *Let $R$ be a ring with the property that for each $x \in R$, there exists a positive integer $m$ and a polynomial $p(X) \in \mathbf{Z}[X]$ such that $x^m = x^{m+1}p(x)$. Then $R$ is periodic.*

## 3 Results on rings with infinite-subset conditions

Our first result, which is trivial but useful, is a characterization of periodic rings.

**Theorem 5** *If $R$ is a ring such that every infinite subset of $R$ contains a periodic element, then $R$ is periodic.*

**Proof** Suppose $R$ is not periodic, and $x$ is a nonperiodic element of $R$. Then $\{x, x^2, x^3, ...\}$ is an infinite subset containing no periodic element. □

**Theorem 6** *Let $R$ be an infinite ring which is not a domain. If every infinite subset of $S$ contains a potent element, then $R$ is a J-ring.*

**Proof** Note first that $R$ is an FZS-ring. We shall show that $R$ is reduced. Suppose, on the contrary, that $u \in N \setminus \{0\}$ and $u^k = 0 \neq u^{k-1}$. By Lemma 1, $A(u)$ is infinite; and since $A(u) \subseteq S$, every infinite subset of $A(u)$ contains a potent element. Therefore the set $\hat{A}(u)$ of all potent elements in $A(u)$ is infinite and hence $u + \hat{A}(u)$ is infinite; moreover $u + \hat{A}(u) \subseteq S$, since $u + \hat{A}(u) \subseteq A(u^{k-1})$. Thus there exists $a \in A(u)$ such that both $a$ and $a + u$ are potent; and there exists $n > 1$ such that $a^n = a$, $(a+u)^n = a + u$, and $u^n = 0$. But then $a + u = (a+u)^n = a^n = a$, so $u = 0 -a$ contradiction; therefore $R$ is reduced as claimed.

Since $R$ is not a domain, $S$ is infinite by Lemma 3; hence $S$ contains a nonzero potent element and hence a nonzero idempotent $e$, which must be central because $R$ is reduced. Thus $R = eR \oplus A(e)$, with both summands contained in $S$. If one of these summands is finite, it is a J-ring by (iii); and each infinite summand, of which there must be at least one, is a reduced periodic ring by Theorem 5 and hence a J-ring. But a direct sum of two J-rings is a J-ring, so we are finished. □

Our final theorem in this section is stated in [1], but the proof given there is not correct.

**Theorem 7** *If $R$ is an infinite ring in which every infinite subset contains a potent element, then $R$ is a J-ring.*

**Proof** If $R$ is not a domain, it is a J-ring by Theorem 6. If $R$ is a domain, it is clearly reduced; and it is periodic by Theorem 5, hence is a J-ring. □

# 4 Rings with conditions on infinite subrings

In this section we study a weaker condition - namely, that every infinite subring contains a nonzero potent element. A ring satisfying this condition will be called a PJS-ring (partially Jacobson subring ring).

It is clear that an infinite PJS-ring need not be a J-ring; one need only consider a ring $R_1 \oplus R_2$, where $R_1$ is an infinite J-ring and $R_2$ is a finite ring which is not a J-ring. However, it follows from Proposition 4 and Lemma 8 of [3] that every infinite PJS-ring contains an infinite J-subring.

Our next theorem provides a complete characterization of J-rings.

**Theorem 8** *A ring $R$ is a J-ring if and only if $R$ is a reduced PJS-ring.*

**Proof** Our condition is obviously necessary, so we proceed to establish sufficiency. Let $R$ be a reduced PJS-ring. Since finite reduced rings are J-rings, we may assume that $R$ is infinite.

We show first that $R = T$. Suppose, on the contrary, that $a \in R \setminus T$. Then $< a >$ is infinite, so there exists a nonzero potent element $n_1 a^{j_1} + \cdots + n_s a^{j_s}$; and it follows that there exist a relation

$$(4.1) \qquad m_1a^{k_1} + \cdots + m_ta^{k_t} = 0,$$

where $m_1, m_2, ..., m_t \in \mathbf{Z} \setminus \{0\}$ and $1 \leq k_1 < k_2 < ... < k_t$. We may assume without loss that $k_1$ is minimal among all such relations. If $k_1 > 1$, then $ac = 0$, where $c = m_1a^{k_1-1} + \cdots + m_ta^{k_t-1}$. But this implies $c^2 = 0 = c$, contradicting the minimality of $k_1$; therefore $k_1 = 1$ and (4.1) takes the form

$$m_1a = aq(a),$$

where $q(X) \in X\mathbf{Z}[X]$. Taking $b = q(a)$, we have $b \notin T$ and $b^2 = m_1b$; thus $< b >$ is isomorphic to the subring of $\mathbf{Z}$ generated by $m_1$. Now if $m_1 \neq \pm 1$, $< b >$ cannot contain a nonzero potent element; and if $m_1 = \pm 1$, $< 2b >$ cannot contain a nonzero potent element. Therefore our assumption that $a \in R \setminus T$ must be false, and $R = T$.

For each prime $p$, let $R_p$ denote the p-primary component of $R$; and note that, since $R$ is reduced, $pR_p = \{0\}$. Let $0 \neq a \in R$. Then $a = a_1 + a_2 + \cdots + a_k$, where $0 \neq a_i \in R_{p_i}$ for primes $p_1, p_2, ..., p_k$. Let $b$ be a typical $a_i$ and $q$ the corresponding $p_i$. If $< b >$ is finite there exist positive integers $m, n$ with $m < n$ such that $b^m = b^n$. If $< b >$ is infinite, then there exists a nonzero potent element $c = n_1b^{k_1} + n_2b^{k_2} + \cdots + n_sb^{k_s}$, where each $n_j \not\equiv 0 (\mathrm{mod}\, q)$ and $k_1 < k_2 < \cdots < k_s$; therefore, there exists $f(X) \in X\mathbf{Z}[X]$ with lowest degree term $n_1X^{k_1}$, such that $f(b) = 0$. Taking $m_1 \in \mathbf{Z}$ such that $n_1m_1 \equiv 1 (\mathrm{mod}\, q)$, we have $m_1f(b) = 0$, so $X\mathbf{Z}[X]$ contains a co-monic polynomial $g(X)$ (i.e. a polynomial with lowest-degree coefficient equal to 1) such that $g(b) = 0$.

We have shown that for each $a_i$, there is co-monic $g_i(X) \in X\mathbf{Z}[X]$ for which $g_i(a_i) = 0$. Therefore $G(X) = \Pi g_i(X)$ is co-monic and $G(a) = 0$. It follows by Lemma 4 that $R$ is periodic; and since $R$ is reduced, $R$ is a J-ring. □

**Theorem 9** *Let $R$ be reduced ring. If $R$ is a not a domain and every infinite subring of $R$ which is contained in $S$ contains a nonzero potent element, then $R$ is a J-ring.*

**Proof** If $R$ is finite, there is nothing to prove, since $R$ is reduced. Therefore, assume $R$ is infinite, in which case Lemma 3 guarantees that $S$ contains an infinite subring and hence a nonzero idempotent $e$. As in the proof of Theorem 6, $e$ is central and $R = eR \oplus A(e)$, with each summand contained in $S$. Now each infinite summand is a J-ring by Theorem 8; and if either summand is finite, it is a J-ring. Therefore $R$ is a J-ring. □

Our final theorem may be regarded as an extension of Theorem 8.

**Theorem 10** *If $R$ is a ring such that every infinite subring contains a non-nilpotent periodic element, then $R$ is periodic.*

**Proof** Let $\bar{R} = \frac{R}{P(R)}$. It is easy to see that $\bar{R}$ inherits the given hypothesis, which implies that $\bar{R}$ is an FZS-ring. Moreover, $\bar{R}$ is semiprime; hence by

Lemma 2, $\bar{R} = B \oplus C$, where $B$ is reduced and $C$ is finite. Now $B$ is a J-ring by Theorem 8, and $C$ is periodic; hence $\bar{R}$ is periodic. Since $P(R)$ is a nil ideal, it follows by Lemma 4 that $R$ is periodic. □

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