

ON (m, n) -PURITY OF MODULES

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Abstract

Let R be a ring. Given two positive integers m and n , an R -module V is said to be (m, n) -presented if there is an exact sequence of R -modules $0 \rightarrow K \rightarrow R^m \rightarrow V \rightarrow 0$ with K n -generated. A submodule U' of a right R -module U is said to be (m, n) -pure in U if for every (m, n) -presented left R -module V , the canonical map $U' \otimes_R V \rightarrow U \otimes_R V$ is a monomorphism. A right R -module A is said to be absolutely (m, n) -pure if A is (m, n) -pure in every module which contains A as a submodule. In this paper, several characterizations of (m, n) -purity are given and some properties of (m, n) -purity are investigated, various results of purity are developed, many extending known results. It is shown that a right R -module A is absolutely (m, n) -pure if and only if it is (n, m) -injective.

0. Introduction

Throughout R is an associative ring with identity and all modules are unitary. m and n will be two fixed positive integers (unless specified otherwise). $R^{m \times n}$ will denote the set of all $m \times n$ matrices over R . For an R -module M , M^m (M_m) denotes the set of all formal $1 \times m$ ($m \times 1$) matrices whose entries are elements of M and M^+ denotes the character module $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M . For any $x \in M^m$ and $A \in R^{m \times n}$, under the usual multiplication of matrices, xA is a well-defined element in M^n . We write M_R (${}_R M$) to indicate a right

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(left) R -module. For convenience, “ n -generated” means “having a generating set of cardinality at most n ”.

Following [2], A_R is called (m, n) -*injective* if every right R -homomorphism from an n -generated submodule of R^m to A extends to one from R^m to A . $(1, n)$ -injective modules are also called n -*injective* modules in [10]. Cohn [4] called a submodule U' of U_R *pure* in U in case for every left R -module V , the canonical map $U' \otimes_R V \rightarrow U \otimes_R V$ is a monomorphism. We also recall that A_R is said to be *absolutely pure* [7] if it is pure in every module containing it as a submodule. A_R is said to be *FP-injective* [6] if $\text{Ext}_R^1(V, A) = 0$ for every finitely presented module V_R . A_R is said to be *F-injective* [5] if every right R -homomorphism from a finitely generated right ideal to A extends to one from R_R to A . Clearly, A_R is *F-injective* if and only if A is n -injective for every positive integer n .

In section 1, we introduce the concept of (m, n) -pure submodules. Various results are developed, many extending known results. In section 2, absolutely (m, n) -pure modules are investigated. In particular, it is shown that a right R -module A is absolutely (m, n) -pure if and only if it is (n, m) -injective. In the last section, we consider the relation between (m, n) -purity and (m, n) -flatness.

1. (m, n) -pure submodules

We start with the following definition.

Definition 1.1 A left R -module V is said to be (m, n) -*presented*, if there is an exact sequence of left R -modules $0 \rightarrow K \rightarrow R^m \rightarrow V \rightarrow 0$ with K n -generated.

Remark 1.2 It is easy to see that a left R -module A is (m, n) -injective if and only if $\text{Ext}_R^1(V, A) = 0$ for all (m, n) -presented left R -module V , so A is *FP-injective* if and only if A is (m, n) -injective for all positive integers m and n .

Definition 1.3 Given a right R -module U with submodule U' , U' is called (m, n) -*pure* in U if the canonical map $U' \otimes_R V \rightarrow U \otimes_R V$ is a monomorphism for every (m, n) -presented left R -module V . U' is said to be (m, \aleph_0) -*pure* (resp., (\aleph_0, n) -*pure*) in U in case U' is (m, n) -pure in U for all positive integers n (resp., m).

Remark 1.4 (1) It is easy to see that U' is pure in U if and only if U' is (m, n) -pure in U for all positive integers m and n .

(2) Suppose $U' \leq U$ and every finitely generated submodule of U' is (m, n) -pure in U . As we all know, U' is the direct limit of its finitely generated submodules and \otimes is commutative with \varinjlim . Then U' is (m, n) -pure in U .

Theorem 1.5 Let $U'_R \leq U_R$, then the following statements are equivalent:

- (1) U' is (m, n) -pure in U ,
(1)' For all $C \in R^{n \times m}$, the canonical map $U' \otimes_R (R^m/R^n C) \rightarrow U \otimes_R (R^m/R^n C)$ is a monomorphism,
(2) For every (m, n) -presented left R -module V , the canonical map $Tor_1^R(U, V) \rightarrow Tor_1^R(U/U', V)$ is surjective,
(3) For all $C \in R^{n \times m}$, $(U')^m \cap U^n C = (U')^n C$,
(4) For every n -generated submodule I of ${}_R R^m$, $(U')^m \cap UI = U'I$,
(5) For every (n, m) -presented right R -module V , the canonical map $Hom_R(V, U) \rightarrow Hom_R(V, U/U')$ is surjective,
(5)' For all $C \in R^{n \times m}$, the canonical map

$$Hom_R(R_n/CR_m, U) \rightarrow Hom_R(R_n/CR_m, U/U')$$

is surjective,

- (6) For every commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{g} & R^m \\ f \downarrow & & \downarrow \\ 0 & \longrightarrow & U' \longrightarrow U \end{array}$$

there exists $h : R^m \rightarrow U'$ with $f = hg$,

- (7) For every (n, m) -presented right R -module V , the canonical map $Ext_R^1(V, U') \rightarrow Ext_R^1(V, U)$ is a monomorphism.

Proof (1) \Leftrightarrow (1)' and (5) \Leftrightarrow (5)' are obvious.

(1) \Leftrightarrow (2) follows from the exact sequence

$$Tor_1^R(U, V) \rightarrow Tor_1^R(U/U', V) \rightarrow U' \otimes V \rightarrow U \otimes V.$$

(1) \Rightarrow (3) Suppose that $C = (c_{ij})_{n \times m}$ and $x \in (U')^m \cap U^n C$, then there exist $a_1, a_2, \dots, a_m \in U'$, $u_1, u_2, \dots, u_n \in U$ such that $x = (a_1, a_2, \dots, a_m)$ and $a_i = \sum_{j=1}^n u_j c_{ji}$, $i = 1, 2, \dots, m$. Let V be an (m, n) -presented left R -module with generators v_1, v_2, \dots, v_m and relations $\sum_{i=1}^m c_{ji} v_i = 0$, $j = 1, 2, \dots, n$, then we have $\sum_{i=1}^m a_i \otimes v_i = 0$ in $U \otimes V$. Since U' is (m, n) -pure in U , $\sum_{i=1}^m a_i \otimes v_i = 0$ in $U' \otimes V$. It follows that $a_i = \sum_{j=1}^n u'_j c_{ji}$ for some $u'_1, u'_2, \dots, u'_n \in U'$, $i = 1, 2, \dots, m$, thus $x \in (U')^n C$. But $(U')^n C \subseteq (U')^m \cap U^n C$, so $(U')^m \cap U^n C = (U')^n C$.

(3) \Rightarrow (4) Suppose $I = Rb_1 + \dots + Rb_n$, where $b_j = (c_{1j}, c_{2j}, \dots, c_{mj}) \in R^m$, $j = 1, 2, \dots, n$. If $x = (a_1, \dots, a_m) = \sum_{j=1}^n u_j b_j \in (U')^m \cap UI$ with each $a_i \in U'$ and each $u_j \in U$, then $x = (u_1, u_2, \dots, u_n)C \in U^n C \cap (U')^m$, where C is an $n \times m$ matrix with row vectors b_1, \dots, b_n . By (3), $x = (u'_1, u'_2, \dots, u'_n)C$ for some $u'_1, u'_2, \dots, u'_n \in U'$. This implies that $x \in U'I$, and hence $(U')^m \cap UI = U'I$.

(4) \Rightarrow (5) Consider the following diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \xrightarrow{i_K} & R^n & \xrightarrow{\pi_2} & V \longrightarrow 0 \\
& & & & & & \downarrow f \\
0 & \longrightarrow & U' & \xrightarrow{i_{U'}} & U & \xrightarrow{\pi_1} & U/U' \longrightarrow 0
\end{array}$$

where $f \in \text{Hom}_R(V, U/U')$ and K is an m -generated submodule of R^n , with generators $y_i = (c_{i1}, c_{i2}, \dots, c_{in})$, $i = 1, 2, \dots, m$. Since R^n is projective, there exist $g \in \text{Hom}_R(R^n, U)$ and $h \in \text{Hom}_R(K, U')$ such that the diagram commutes. Now let $b_j = (c_{1j}, c_{2j}, \dots, c_{mj}) \in R^m$, $j = 1, 2, \dots, n$, $I = Rb_1 + \dots + Rb_n$ and $u_i = \sum_{j=1}^n g(e_j)c_{ij}$, where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ (with 1 in the j th position and 0's in all other positions), $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. Then $u_i = g(\sum_{j=1}^n e_j c_{ij}) = g(y_i) = h(y_i) \in U'$, $i = 1, 2, \dots, m$. Note that $(u_1, u_2, \dots, u_m) = \sum_{j=1}^n g(e_j)b_j \in UI$, by (4), $(u_1, u_2, \dots, u_m) = \sum_{j=1}^n u'_j b_j$ for some $u'_1, u'_2, \dots, u'_n \in U'$. Therefore, $u_i = \sum_{j=1}^n u'_j c_{ij}$, $i = 1, 2, \dots, m$. Define $\sigma \in \text{Hom}_R(R^n, U')$ such that $\sigma(e_j) = u'_j$, $j = 1, 2, \dots, n$. Then $\sigma i_K = h$. Finally, we define $\tau : V \rightarrow U$ by $\tau(z + K) = g(z) - \sigma(z)$, then τ is a well-defined right R -homomorphism and $\pi_1 \tau = f$. Whence $\text{Hom}_R(V, U) \rightarrow \text{Hom}_R(V, U/U')$ is surjective.

(5) \Rightarrow (3) Suppose that $C = (c_{ij})_{n \times m} \in R^{n \times m}$ and $x \in (U')^m \cap U^n C$. Then $x = (a_1, a_2, \dots, a_m) = (u_1, u_2, \dots, u_n)C$ for some $a_1, a_2, \dots, a_m \in U'$ and $u_1, u_2, \dots, u_n \in U$. Take $y_i = (c_{1i}, c_{2i}, \dots, c_{ni})$ ($i = 1, 2, \dots, m$), $K = y_1 R + y_2 R + \dots + y_m R$ and $V = R^n/K$. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \xrightarrow{i_K} & R^n & \xrightarrow{\pi_2} & V \longrightarrow 0 \\
& & \downarrow f_1 & & \downarrow f_2 & & \\
0 & \longrightarrow & U' & \xrightarrow{i_{U'}} & U & \xrightarrow{\pi_1} & U/U' \longrightarrow 0
\end{array}$$

where f_2 is defined by $f_2(e_j) = u_j$, $j = 1, 2, \dots, n$ and $f_1 = f_2|_K$. Define $f_3 : V \rightarrow U/U'$ by $f_3(z + K) = \pi_1 f_2(z)$. It is easy to see that f_3 is well defined and $f_3 \pi_2 = \pi_1 f_2$. By hypothesis, $f_3 = \pi_1 \tau$ for some $\tau \in \text{Hom}_R(V, U)$. Now we define $\sigma : R^n \rightarrow U'$ by $\sigma(z) = f_2(z) - \tau \pi_2(z)$. Then $\sigma \in \text{Hom}_R(R^n, U')$ and $i_{U'} \sigma = f_2$. Hence $a_i = f_2(y_i) = \sigma(y_i) = \sum_{j=1}^n \sigma(e_j) c_{ji}$, $i = 1, 2, \dots, m$, and $x = (\sigma(e_1), \sigma(e_2), \dots, \sigma(e_n))C \in (U')^n C$. Therefore $(U')^m \cap U^n C = (U')^n C$.

(3) \Rightarrow (1) Suppose that ${}_R V$ is (m, n) -presented, with generators v_1, v_2, \dots, v_m and relations $\sum_{j=1}^m c_{ij} v_j = 0$, $i = 1, \dots, n$. If $\sum_{k=1}^s a_k \otimes b_k = 0$ in $U \otimes V$, where $a_k \in U'$, $b_k = \sum_{j=1}^m \alpha_{kj} v_j \in V$, then $\sum_{j=1}^m (\sum_{k=1}^s a_k \alpha_{kj}) \otimes v_j = 0$ in $U \otimes V$. So $\sum_{k=1}^s a_k \alpha_{kj} = \sum_{i=1}^n u_i c_{ij}$ for some $u_i \in U$. By (3), there exist $u'_1, u'_2, \dots, u'_n \in U'$ such that $\sum_{k=1}^s a_k \alpha_{kj} = \sum_{i=1}^n u'_i c_{ij}$, $j = 1, \dots, m$. Thus $\sum_{k=1}^s a_k \otimes b_k = \sum_{i=1}^n u'_i \otimes (\sum_{j=1}^m c_{ij} v_j) = 0$ in $U' \otimes V$.

(5) \Leftrightarrow (6) By Diagram Lemma (see [11, page 53]).

(5) \Leftrightarrow (7) It follows from the exact sequence

$$\text{Hom}_R(V, U) \rightarrow \text{Hom}_R(V, U/U') \rightarrow \text{Ext}_R^1(V, U') \rightarrow \text{Ext}_R^1(V, U). \quad \square$$

Corollary 1.6 *Let $U'_R \leq U_R$, then U' is $(1, \aleph_0)$ -pure in U if and only if $UI \cap U' = U'I$ for all finitely generated left ideals I . \square*

Proposition 1.7 *Let $U'_R \leq U_R$, then*

- (1) *If U is n -generated, then U' is (m, n) -pure in U if and only if U' is (m, \aleph_0) -pure in U .*
- (2) *If each finitely generated left ideal of R is n -generated, then U' is $(1, n)$ -pure in U if and only if U' is $(1, \aleph_0)$ -pure in U .*
- (3) *If each finitely generated right ideal of R is m -generated, then U' is $(m, 1)$ -pure in U if and only if U' is $(\aleph_0, 1)$ -pure in U .*

Proof (2) can be proved by Theorem 1.5(4), and (3) can be proved by Theorem 1.5(5). Now we prove only the necessity of (1).

Let u_1, u_2, \dots, u_n be a generating set of U . For every positive integer l and each $C \in R^{l \times m}$, if $x \in (U')^m \cap U^l C$, then $x = (u_1, u_2, \dots, u_n)AC$ for some $A \in R^{n \times l}$. Since U' is (m, n) -pure in U , by Theorem 1.5(3), $x = (u'_1, u'_2, \dots, u'_n)AC$ for some $u'_1, u'_2, \dots, u'_n \in U$. Thus $x \in (U')^m \cap (U')^l C$. Therefore U' is (m, l) -pure in U . \square

Remark 1.8 Suppose that every finitely generated left R -module is a direct limit of cyclic modules, then $(1, \aleph_0)$ -pure submodules of any right R -module are pure. By Proposition 1.7(2), it follows that if R is a principal ideal domain, then $(1, 1)$ -pure submodules of any R -module are pure.

Many properties of (m, n) -purity are similar to those of purity. For example, we have

Proposition 1.9 *Suppose E, F and G are right R -modules such that $E \subseteq F \subseteq G$.*

- (1) *If E is (m, n) -pure in F and F is (m, n) -pure in G , then E is (m, n) -pure in G .*
- (2) *If E is (m, n) -pure in G , then E is (m, n) -pure in F .*
- (3) *If F is (m, n) -pure in G , then F/E is (m, n) -pure in G/E .*
- (4) *If E is (m, n) -pure in G and F/E is (m, n) -pure in G/E , then F is (m, n) -pure in G . \square*

By Ramamurthi and Rangaswamy [8], a submodule A of an R -module B is called *strongly pure* if for each element $a \in A$ (equivalently, any finite set a_1, a_2, \dots, a_n of elements of A) there exists a homomorphism $\alpha : B \rightarrow A$ such that $\alpha(a) = a$ ($\alpha(a_i) = a_i, i = 1, 2, \dots, n$). Clearly, if A is strongly pure in B , then A is pure in B , but the converse is not true.

Proposition 1.10 *Suppose that P_R is a projective module and $K_R \leq P_R$, then the following statements are equivalent:*

- (1) K is pure in P ,
- (2) K is $(1, \aleph_0)$ -pure in P ,
- (3) K is strongly pure in P .

Proof (1) \Rightarrow (2) and (3) \Rightarrow (1) are obvious.

(2) \Rightarrow (3) Let F_R be a free module with basis $X = \{x_i \mid i \in I\}$ and $F = P \oplus P'$, then K is $(1, \aleph_0)$ -pure in F by (2). Let $y = \sum_{i=1}^n x_i r_i \in K$ and $I = Rr_1 + Rr_2 + \cdots + Rr_n$, then $y \in K \cap FI$. Since K is $(1, n)$ -pure in F , by Theorem 1.5(4), $y \in KI$. Assume $y = \sum_{i=1}^n k_i r_i$ for some $k_i \in K$, define $\theta : F \rightarrow K$ by $\theta(x_i) = k_i$, $1 \leq i \leq n$, and $\theta(x_i) = 0$ for other x_i 's, then $\theta(y) = y$, and hence K is strongly pure in P . \square

2. Absolutely (m, n) -pure modules

Definition 2.1 A_R is said to be *absolutely (m, n) -pure* if A is (m, n) -pure in every module which contains A as a submodule.

Theorem 2.2 For a right R -module A , the following statements are equivalent:

- (1) A is absolutely (m, n) -pure,
- (2) A is (m, n) -pure in its injective envelope $E(A)$,
- (3) A is (n, m) -injective.

Proof (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (1) Suppose $A \leq B$, then $A \leq E(A) \leq E(B)$. Since A is (m, n) -pure in $E(A)$ and $E(A)$ is pure in $E(B)$, A is (m, n) -pure in $E(B)$ by Proposition 1.9(1). Note that $A \leq B \leq E(B)$, by Proposition 1.9(2), A is (m, n) -pure in B .

(2) \Leftrightarrow (3) follows from the exact sequence

$$\text{Hom}_R(V, E(A)) \rightarrow \text{Hom}_R(V, E(A)/A) \rightarrow \text{Ext}_R^1(V, A) \rightarrow 0$$

and Remark 1.2 and Theorem 1.5(5). \square

Proposition 2.3 If A_R is an (m, n) -pure submodule of an absolutely (m, n) -pure module B_R , then A is absolutely (m, n) -pure.

Proof For every (m, n) -presented module ${}_R V$, since A is (m, n) -pure in B and B is (m, n) -pure in $E(B)$, $A \otimes V \rightarrow B \otimes V$ and $B \otimes V \rightarrow E(B) \otimes V$ are monomorphisms. Thus the following commutative diagram

$$\begin{array}{ccc} A \otimes V & \longrightarrow & B \otimes V \\ \downarrow & & \downarrow \\ E(A) \otimes V & \longrightarrow & E(B) \otimes V \end{array}$$

gives that the map $A \otimes V \rightarrow E(A) \otimes V$ is a monomorphism. \square

The following theorem is immediate.

Theorem 2.4 *Suppose that $A_R \leq B_R$ and B_R is (m, n) -injective, then A is (m, n) -injective if and only if A is (n, m) -pure in B . \square*

Corollary 2.5 *Let $A_R \leq B_R$.*

- (1) *If B is FP-injective, then A is FP-injective if and only if A is pure in B .*
- (2) *If B is F -injective, then A is F -injective if and only if A is $(\aleph_0, 1)$ -pure in B .*
- (3) *If B is n -injective, then A is n -injective if and only if A is $(n, 1)$ -pure in B . In particular, if B is P -injective, then A is P -injective if and only if A is $(1, 1)$ -pure in B . \square*

Corollary 2.6 *Let A be a right R -module, then*

- (1) *A is FP-injective if and only if A is absolutely pure.*
- (2) *A is F -injective if and only if for every finitely generated free left R -module F and every cyclic submodule K of F , $A \otimes F/K \rightarrow E(A) \otimes F/K$ is a monomorphism.*
- (3) *A is n -injective if and only if for every finitely generated free left R -module F and every cyclic submodule K of the left R -module ${}_R R^n$, $A \otimes R^n/K \rightarrow E(A) \otimes R^n/K$ is a monomorphism. In particular, A is P -injective if and only if for each $a \in R$, $A \otimes R/Ra \rightarrow E(A) \otimes R/Ra$ is a monomorphism. \square*

Remark 2.7 Following Jain [6], ring R is said to be right IF in case every injective right R -module is flat. By Corollary 2.6, it is easy to see that if R is a right IF ring, then A_R is n -injective if and only if $Tor_1^R(E(A)/A, R^n/I) = 0$ for all cyclic submodules I of ${}_R R^n$.

Definition 2.8 A right R -module A is said to be (m, \aleph_0) -injective (resp., (\aleph_0, n) -injective) if A is (m, n) -injective, for every positive integer n (resp., m). Clearly, A is F -injective if and only if A is $(1, \aleph_0)$ -injective.

Proposition 2.9 *Suppose that A_R is (n, m) -injective and $E(A)$ is n -generated, then A is (\aleph_0, m) -injective.*

Proof As A is (n, m) -injective, so A is (m, n) -pure in $E(A)$ by Theorem 2.2. But $E(A)$ is n -generated, by proposition 1.8(1), A is (m, \aleph_0) -pure in $E(A)$, and hence A is (\aleph_0, m) -injective. \square

Remark 2.10 Rutter [9] has an example of a left P -injective ring which is not left 2-injective. Similarly, we can give an example of a right P -injective ring which is not right 2-injective. So, in general, an $(1, n)$ -pure submodules of a right R -module need not to be its $(2, n)$ -pure submodules. From Theorem

2.2, we see also that if for every positive integer m , $(m, 1)$ -pure submodules of each injective R -module are its (m, \aleph_0) -pure submodules, then F -injective R -modules are FP -injective modules.

3. (m, n) -purity and (m, n) -flatness

In this section, we consider the relation between (m, n) -purity and (m, n) -flatness. Firstly, we give a definition as below.

Definition 3.1 A right R -module V is said to be (m, n) -flat, if for every n -generated submodule I of ${}_R R^m$, the canonical map $V \otimes I \rightarrow V \otimes R^m$ is monic. A right R -module V is called (m, \aleph_0) -flat (resp., (\aleph_0, n) -flat), if for every positive integer n (resp., m), V is (m, n) -flat.

Note that $(1, n)$ -flat module is n -flat in sense of [10] and the next proposition is easy to verify.

Proposition 3.2 For a right R -module V , the following statements are equivalent:

- (1) V is (m, n) -flat,
- (2) $Tor_1^R(V, M) = 0$ for all (m, n) -presented left R -module M ,
- (3) V^+ is (m, n) -injective ,
- (4) For every n -generated submodule I of ${}_R R^m$, the map $\mu_I : V \otimes I \rightarrow VI$; $\sum v_i \otimes x_i \mapsto \sum v_i x_i$ is a monomorphism,
- (5) For all $X \in V^n$, $A \in R^{n \times m}$, if $XA = 0$, then exist a positive integer l and $Y \in V^l$, $C \in R^{l \times n}$, such that $CA = 0$ and $X = YC$. \square

Remark 3.3 From Proposition 3.2, the (m, n) -flatness of V_R can be characterized by the (m, n) -injectivity of V^+ . On the other hand, by [3, Lemma 2.7(1)], the sequence $Tor_1^R(V^+, M) \rightarrow Ext_R^1(M, V)^+ \rightarrow 0$ is exact for all finitely presented left R -module M , so if V^+ is (m, n) -flat, then V is (m, n) -injective.

Proposition 3.4 (n, \aleph_0) -pure submodules of (m, n) -flat modules are (m, n) -flat.

Proof Suppose that V_R is (m, n) -flat, K is (n, \aleph_0) -pure in V . Let $X \in K^n$, $A \in R^{n \times m}$ satisfy $XA = 0$, then by the (m, n) -flatness of V , there exist positive integer l , $U \in V^l$ and $C \in R^{l \times n}$ such that $CA = 0$ and $X = UC$. Since K is (n, \aleph_0) -pure in V and hence (n, l) -pure, by Theorem 1.5(3), $X = YC$ for some $Y \in K^l$. And so K is (m, n) -flat. \square

Corollary 3.5 Pure submodules of a flat module are flat. \square

Theorem 3.6 Let $U'_R \leq U_R$.

- (1) If U/U' is (m, n) -flat, then U' is (m, n) -pure in U .
- (2) If U' is (m, n) -pure in U and U is (m, n) -flat, then U/U' is (m, n) -flat.

Proof It follows from the exact sequence

$$\mathrm{Tor}_1^R(U, M) \rightarrow \mathrm{Tor}_1^R(U/U', M) \rightarrow U' \otimes M \rightarrow U \otimes M$$

and Proposition 3.2(2). \square

Corollary 3.7 Let $U'_R \leq U_R$.

- (1) If U is flat, then U/U' is flat if and only if U' is pure in U .
- (2) If U is n -flat, then U/U' is n -flat if and only if U' is $(1, n)$ -pure in U .

\square

By Proposition 3.2(2), we see that V_R is flat if and only if V is (m, n) -flat for all positive integers m and n . Suppose V is an n -generated and n -flat module, there exists an exact sequence $0 \rightarrow K \rightarrow F \rightarrow V \rightarrow 0$ with F free and $\mathrm{rank}(F) = n$. Then K is $(1, n)$ -pure in F by Corollary 3.7 and hence $(1, \aleph_0)$ -pure by Proposition 1.7, so V is $(1, \aleph_0)$ -flat. It is well known that V_R is flat if and only if V is $(1, \aleph_0)$ -flat. So we have the following corollary.

Corollary 3.8^[10] n -generated and n -flat modules are flat. \square

Proposition 3.9 Let U_R be an n -generated flat module, $U'_R \leq U_R$, then U' is pure in U if and only if U' is $(1, n)$ -pure. In particular, for every right ideal I of R , I_R is pure in R_R if and only if I_R is $(1, 1)$ -pure in R_R .

Proof By Proposition 1.7. \square

Corollary 3.10 Suppose R_R is FP -injective (resp., (m, n) -injective, F -injective, n -injective), $I_R \leq R_R$, then I_R is FP -injective (resp., (m, n) -injective, F -injective, n -injective) if and only if I_R is P -injective.

Proof It follows immediately from Theorem 2.4 and Proposition 3.9. \square

Proposition 3.11 Suppose that R is a right IF ring and $(1, n)$ -flat right R -modules are (m, n) -flat, then $(n, 1)$ -injective right R -modules are (n, m) -injective.

Proof If A_R is $(n, 1)$ -injective, then A is $(1, n)$ -pure in $E(A)$. Since R is a right IF ring, $E(A)$ is flat. Hence $E(A)/A$ is $(1, n)$ -flat. By hypothesis, $E(A)/A$ is (m, n) -flat and hence A is (m, n) -pure in $E(A)$, that is, A is (n, m) -injective. \square

Corollary 3.12 Suppose that R is a right IF ring and $(1, 1)$ -flat right R -modules are $(\aleph_0, 1)$ -flat. Then P -injective right R -modules are F -injective and $(1, 1)$ -flat left R -modules are flat. \square

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References

- [1] F.W.Anderson and K.R.Fuller, “Rings and Categories of Modules”, GTM 13, Springer-Verlag, New York, 1974.
- [2] J.L.Chen, N.Q.Ding, Y.L.Li, Y.Q.Zhou, *On (m, n) -injectivity of modules*, Comm.Algebra, **29**(12)(2001), 5589-5603.
- [3] J.L.Chen, N.Q.Ding. *On n -coherent rings*, Comm.Algebra, **24**(10)(1996), 3211-3216.
- [4] P.M.Cohn, *On the free product of associative rings*, Math. Z., **71**(1959), 380-398.
- [5] R.N.Gupta, *On f -injective modules and semihereditary rings*, Proc.Nat.Inst. Sci.India Part A **35**(1969),323-328.
- [6] S.Jain, *Flat and FP -injective modules*, Proc. Amer. Math. Soc., **41**(2)(1973), 437-442.
- [7] B.Maddox, *Absolutely pure modules*, Proc. Amer. Math. Soc., **18**(1967), 155-158.
- [8] V.S.Ramamurthi, K.M.Rangaswamy, *On finitely injective modules*, J. Austral. Math. Soc., **16**(1973), 239-248.
- [9] E.A.Rutter, *Rings with the principal extenaion property*, Comm. Algebra, **3**(3)(1975), 203-212.
- [10] A.Shamsuddin, *n -injective and n -flat modules*, Comm. Algebra, **29**(5)(2001), 2039-2050.
- [11] R.Wisbauer, “Foundations of Module and Ring Theory”, Gordon and Breach Since Publishers, Philadelphia, Pennsylvania, 1991.