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ON (m, n)-PURITY OF MODULES

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Abstract

Let R be a ring. Given two positive integers m and n, an R-module V is said to be (m, n)-presented if there is an exact sequence of R-modules $0 \to K \to R^m \to V \to 0$ with K n-generated. A submodule U' of a right R-module U is said to be (m, n)-pure in U if for every (m, n)-presented left R-module V, the canonical map $U' \otimes_R V \to U \otimes_R V$ is a monomorphism. A right R-module A is said to be absolutely (m, n)-pure if A is (m, n)-pure in every module which contains A as a submodule. In this paper, several characterizations of (m, n)-purity are given and some properties of (m, n)-purity are investigated, various results of purity are developed, many extending known results. It is shown that a right R-module A is absolutely (m, n)-pure if and only if it is (n, m)-injective.

0. Introduction

Throughout R is an associative ring with identity and all modules are unitary. m and n will be two fixed positive integers (unless specified otherwise). $R^{m\times n}$ will denote the set of all $m \times n$ matrices over R. For an R-module M, M^m (M_m) denotes the set of all formal $1 \times m$ ($m \times 1$) matrices whose entries are elements of M and M^+ denotes the character module $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M. For any $x \in M^m$ and $A \in R^{m\times n}$, under the usual multiplication of matrices, xA is a well-defined element in M^n. We write M_R ($_RM$) to indicate a right

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(left) R-module. For convenience, "n-generated" means "having a generating set of cardinality at most n".

Following [2], A_R is called (m,n)*-injective* if every right R-homomorphism from an n-generated submodule of R^m to A extends to one from R^m to A. $(1,n)$-injective modules are also called n*-injective* modules in [10]. Cohn [4] called a submodule U' of U_R *pure* in U in case for every left R-module V, the canonical map $U' \otimes_R V \to U \otimes_R V$ is a monomorphism. We also recall that A_R is said to be *absolutely pure* [7] if it is pure in every module containing it as a submodule. A_R is said to be FP*-injective* [6] if $Ext^1_R(V,A) = 0$ for every finitely presented module V_R. A_R is said to be F*-injective* [5] if every right R-homomorphism from a finitely generated right ideal to A extends to one from R_R to A. Clearly, A_R is F-injective if and only if A is n-injective for every positive integer n.

In section 1, we introduce the concept of (m,n)-pure submodules. Various results are developed, many extending known results. In section 2, absolutely (m,n)-pure modules are investigated. In particular, it is shown that a right R-module A is absolutely (m,n)-pure if and only if it is (n,m)-injective. In the last section, we consider the relation between (m,n)-purity and (m,n)-flatness.

1. (m,n)-pure submodules

We start with the following definition.

Definition 1.1 A left R-module V is said to be (m,n)*-presented*, if there is an exact sequence of left R-modules $0 \to K \to R^m \to V \to 0$ with K n-generated.

Remark 1.2 It is easy to see that a left R-module A is (m,n)-injective if and only if $Ext^1_R(V,A) = 0$ for all (m,n)-presented left R-module V, so A is FP-injective if and only if A is (m,n)-injective for all positive integers m and n.

Definition 1.3 Given a right R-module U with submodule U', U' is called (m,n)*-pure* in U if the canonical map $U' \otimes_R V \to U \otimes_R V$ is a monomorphism for every (m,n)-presented left R-module V. U' is said to be $(m,\aleph_0)$*-pure* (resp., $(\aleph_0,n)$*-pure*) in U in case U' is (m,n)-pure in U for all positive integers n (resp., m).

Remark 1.4 (1) It is easy to see that U' is pure in U if and only if U' is (m,n)-pure in U for all positive integers m and n.

(2) Suppose $U' \leq U$ and every finitely generated submodule of U' is (m,n)-pure in U. As we all know, U' is the direct limit of its finitely generated submodules and $\otimes$ is commutative with $\varinjlim$. Then U' is (m,n)-pure in U.

Theorem 1.5 *Let $U'_R \leq U_R$, then the following statements are equivalent*:

(1) *U' is (m,n)-pure in U,*

(1)$'$ *For all $C \in R^{n\times m}$, the canonical map $U' \otimes_R (R^m/R^nC) \to U \otimes_R (R^m/R^nC)$ is a monomorphism,*

(2) *For every (m,n)-presented left R-module V, the canonical map $Tor_1^R(U,V) \to Tor_1^R(U/U',V)$ is surjective,*

(3) *For all $C \in R^{n\times m}$, $(U')^m \cap U^nC = (U')^nC$,*

(4) *For every n-generated submodule I of ${}_RR^m$, $(U')^m \cap UI = U'I$,*

(5) *For every (n,m)-presented right R-module V, the canonical map $Hom_R(V,U) \to Hom_R(V,U/U')$ is surjective,*

(5)$'$ *For all $C \in R^{n\times m}$, the canonical map*

$$Hom_R(R_n/CR_m, U) \to Hom_R(R_n/CR_m, U/U')$$

is surjective,

(6) *For every commutative diagram*

$$\begin{tikzcd} & R^n \arrow[r, "g"] \arrow[d, "f"'] & R^m \arrow[d] \\ 0 \arrow[r] & U' \arrow[r] & U \end{tikzcd}$$

there exists $h:\ R^m \to U'$ with $f = hg$,

(7) *For every (n,m)-presented right R-module V, the canonical map $Ext_R^1(V,U') \to Ext_R^1(V,U)$ is a monomorphism.*

Proof (1)$\Leftrightarrow$(1)$'$ and (5)$\Leftrightarrow$(5)$'$ are obvious.

(1)$\Leftrightarrow$(2) follows from the exact sequence

$$Tor_1^R(U,V) \to Tor_1^R(U/U',V) \to U' \otimes V \to U \otimes V.$$

(1)$\Rightarrow$(3) Suppose that $C = (c_{ij})_{n\times m}$ and $x \in (U')^m \cap U^nC$, then there exist $a_1, a_2, \cdots, a_m \in U'$, $u_1, u_2, \cdots, u_n \in U$ such that $x = (a_1, a_2, \cdots, a_m)$ and $a_i = \sum_{j=1}^n u_jc_{ji}$, $i = 1, 2, \cdots, m$. Let V be an (m,n)-presented left R-module with generators $v_1, v_2, \cdots, v_m$ and relations $\sum_{i=1}^m c_{ji}v_i = 0$, $j = 1, 2, \cdots, n$, then we have $\sum_{i=1}^m a_i \otimes v_i = 0$ in $U \otimes V$. Since U' is (m,n)-pure in U, $\sum_{i=1}^m a_i \otimes v_i = 0$ in $U' \otimes V$. It follows that $a_i = \sum_{j=1}^n u'_jc_{ji}$ for some $u'_1, u'_2, \cdots, u'_n \in U'$, $i = 1, 2, \cdots, m$, thus $x \in (U')^nC$. But $(U')^nC \subseteq (U')^m \cap U^nC$, so $(U')^m \cap U^nC = (U')^nC$.

(3)$\Rightarrow$(4) Suppose $I = Rb_1 + \cdots + Rb_n$, where $b_j = (c_{1j}, c_{2j}, \cdots, c_{mj}) \in R^m$, $j = 1, 2, \cdots, n$. If $x = (a_1, \cdots, a_m) = \sum_{j=1}^n u_jb_j \in (U')^m \cap UI$ with each $a_i \in U'$ and each $u_j \in U$, then $x = (u_1, u_2, \cdots, u_n)C \in U^nC \cap (U')^m$, where C is an $n \times m$ matrix with row vectors $b_1, \cdots, b_n$. By (3), $x = (u'_1, u'_2, \cdots, u'_n)C$ for some $u'_1, u'_2, \cdots, u'_n \in U'$. This implies that $x \in U'I$, and hence $(U')^m \cap UI = U'I$.

(4)$\Rightarrow$(5) Consider the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i_K"] & R^n \arrow[r, "\pi_2"] & V \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & U' \arrow[r, "i_{U'}"] & U \arrow[r, "\pi_1"] & U/U' \arrow[r] & 0
\end{tikzcd}$$

where $f \in Hom_R(V, U/U')$ and K is an m-generated submodule of R^n, with generators $y_i = (c_{i1}, c_{i2}, \cdots, c_{in})$, $i = 1, 2, \cdots, m$. Since R^n is projective, there exist $g \in Hom_R(R^n, U)$ and $h \in Hom_R(K, U')$ such that the diagram commutes. Now let $b_j = (c_{1j}, c_{2j}, \cdots, c_{mj}) \in R^m$, $j = 1, 2, \cdots, n$, $I = Rb_1 + \cdots + Rb_n$ and $u_i = \sum_{j=1}^n g(e_j)c_{ij}$, where $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ (with 1 in the jth position and 0's in all other positions), $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. Then $u_i = g(\sum_{j=1}^n e_j c_{ij}) = g(y_i) = h(y_i) \in U'$, $i = 1, 2, \cdots, m$. Note that $(u_1, u_2, \cdots, u_m) = \sum_{j=1}^n g(e_j)b_j \in UI$, by (4), $(u_1, u_2, \cdots, u_m) = \sum_{j=1}^n u'_j b_j$ for some $u'_1, u'_2, \cdots, u'_n \in U'$. Therefore, $u_i = \sum_{j=1}^n u'_j c_{ij}$, $i = 1, 2, \cdots, m$. Define $\sigma \in Hom_R(R^n, U')$ such that $\sigma(e_j) = u'_j$, $j = 1, 2, \cdots, n$. Then $\sigma i_K = h$. Finally, we define $\tau : V \to U$ by $\tau(z + K) = g(z) - \sigma(z)$, then τ is a well-defined right R-homomorphism and $\pi_1 \tau = f$. Whence $Hom_R(V, U) \to Hom_R(V, U/U')$ is surjective.

(5)⇒(3) Suppose that $C = (c_{ij})_{n \times m} \in R^{n \times m}$ and $x \in (U')^m \cap U^n C$. Then $x = (a_1, a_2, \cdots, a_m) = (u_1, u_2, \cdots, u_n)C$ for some $a_1, a_2, \cdots, a_m \in U'$ and $u_1, u_2, \cdots, u_n \in U$. Take $y_i = (c_{1i}, c_{2i}, \cdots, c_{ni})$ $(i = 1, 2, \cdots, m)$, $K = y_1R + y_2R + \cdots + y_mR$ and $V = R^n/K$. Then we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i_K"] \arrow[d, "f_1"] & R^n \arrow[r, "\pi_2"] \arrow[d, "f_2"] & V \arrow[r] & 0 \\
0 \arrow[r] & U' \arrow[r, "i_{U'}"] & U \arrow[r, "\pi_1"] & U/U' \arrow[r] & 0
\end{tikzcd}$$

where f_2 is defined by $f_2(e_j) = u_j$, $j = i, 2, \cdots, n$ and $f_1 = f_2|_K$. Define $f_3 : V \to U/U'$ by $f_3(z + K) = \pi_1 f_2(z)$. It is easy to see that f_3 is well defined and $f_3\pi_2 = \pi_1 f_2$. By hypothesis, $f_3 = \pi_1\tau$ for some $\tau \in Hom_R(V, U)$. Now we define $\sigma : R^n \to U'$ by $\sigma(z) = f_2(z) - \tau\pi_2(z)$. Then $\sigma \in Hom_R(R^n, U')$ and $i_{U'}\sigma = f_2$. Hence $a_i = f_2(y_i) = \sigma(y_i) = \sum_{j=1}^n \sigma(e_j)c_{ji}$, $i = 1, 2, \cdots, m$, and $x = (\sigma(e_1), \sigma(e_2), \cdots, \sigma(e_n))C \in (U')^n C$. Therefore $(U')^m \cap U^n C = (U')^n C$.

(3)⇒(1) Suppose that ${}_RV$ is (m, n)-presented, with generators $v_1, v_2, \cdots, v_m$ and relations $\sum_{j=1}^m c_{ij}v_j = 0$, $i = 1, \cdots, n$. If $\sum_{k=1}^s a_k \otimes b_k = 0$ in $U \otimes V$, where $a_k \in U'$, $b_k = \sum_{j=1}^m \alpha_{kj}v_j \in V$, then $\sum_{j=1}^m (\sum_{k=1}^s a_k\alpha_{kj}) \otimes v_j = 0$ in $U \otimes V$. So $\sum_{k=1}^s a_k\alpha_{kj} = \sum_{i=1}^n u_i c_{ij}$ for some $u_i \in U$. By (3), there exist $u'_1, u'_2, \cdots, u'_n \in U'$ such that $\sum_{k=1}^s a_k\alpha_{kj} = \sum_{i=1}^n u'_i c_{ij}$, $j = 1, \cdots, m$. Thus $\sum_{k=1}^s a_k \otimes b_k = \sum_{i=1}^n u'_i \otimes (\sum_{j=1}^m c_{ij}v_j) = 0$ in $U' \otimes V$.

(5)⇔(6) By Diagram Lemma (see [11, page 53]).

(5)⇔(7) It follows from the exact sequence
$Hom_R(V, U) \to Hom_R(V, U/U') \to Ext^1_R(V, U') \to Ext^1_R(V, U)$. □

Corollary 1.6 *Let $U'_R \leq U_R$, then U' is $(1, \aleph_0)$-pure in U if and only if $UI \cap U' = U'I$ for all finitely generated left ideals I.* $\square$

Proposition 1.7 *Let $U'_R \leq U_R$, then*

(1) *If U is n-generated, then U' is (m, n)-pure in U if and only if U' is $(m, \aleph_0)$-pure in U.*

(2) *If each finitely generated left ideal of R is n-generated, then U' is $(1, n)$-pure in U if and only if U' is $(1, \aleph_0)$-pure in U.*

(3) *If each finitely generated right ideal of R is m-generated, then U' is $(m, 1)$-pure in U if and only if U' is $(\aleph_0, 1)$-pure in U.*

Proof (2) can be proved by Theorem 1.5(4), and (3) can be proved by Theorem 1.5(5). Now we prove only the necessity of (1).

Let $u_1, u_2, \cdots, u_n$ be a generating set of U. For every positive integer l and each $C \in R^{l\times m}$, if $x \in (U')^m \cap U^l C$, then $x = (u_1, u_2, \cdots, u_n)AC$ for some $A \in R^{n\times l}$. Since U' is (m, n)-pure in U, by Theorem 1.5(3), $x = (u'_1, u'_2, \cdots, u'_n)AC$ for some $u'_1, u'_2, \cdots, u'_n \in U$. Thus $x \in (U')^m \cap (U')^l C$. Therefore U' is (m, l)-pure in U. $\square$

Remark 1.8 Suppose that every finitely generated left R-module is a direct limit of cyclic modules, then $(1, \aleph_0)$-pure submodules of any right R-module are pure. By Proposition 1.7(2), it follows that if R is a principal ideal domain, then $(1, 1)$-pure submodules of any R-module are pure.

Many properties of (m, n)-purity are similar to those of purity. For example, we have

Proposition 1.9 *Suppose E, F and G are right R-modules such that $E \subseteq F \subseteq G$.*

(1) *If E is (m, n)-pure in F and F is (m, n)-pure in G, then E is (m, n)-pure in G.*

(2) *If E is (m, n)-pure in G, then E is (m, n)-pure in F.*

(3) *If F is (m, n)-pure in G, then F/E is (m, n)-pure in G/E.*

(4) *If E is (m, n)-pure in G and F/E is (m, n)-pure in G/E, then F is (m, n)-pure in G.* $\square$

By Ramamurthi and Rangaswamy [8], a submodule A of an R-module B is called *strongly pure* if for each element $a \in A$ (equivalently, any finite set a_1, a_2, $\cdots$, a_n of elements of A) there exists a homomorphism $\alpha :\ B \to A$ such that $\alpha(a) = a$ ($\alpha(a_i) = a_i$, $i = 1, 2, \cdots, n$). Clearly, if A is strongly pure in B, then A is pure in B, but the converse is not true.

Proposition 1.10 *Suppose that P_R is a projective module and $K_R \leq P_R$, then the following statements are equivalent:*

(1) *K is pure in P,*
(2) *K is $(1,\aleph_0)$-pure in P,*
(3) *K is strongly pure in P.*

Proof (1)⇒(2) and (3)⇒(1) are obvious.

(2)⇒(3) Let F_R be a free module with basis $X=\{x_i \,|\, i\in I\}$ and $F=P\oplus P'$, then K is $(1,\aleph_0)$-pure in F by (2). Let $y=\sum_{i=1}^n x_ir_i\in K$ and $I=Rr_1+Rr_2+\cdots+Rr_n$, then $y\in K\cap FI$. Since K is $(1,n)$-pure in F, by Theorem 1.5(4), $y\in KI$. Assume $y=\sum_{i=1}^n k_ir_i$ for some $k_i\in K$, define $\theta: F\to K$ by $\theta(x_i)=k_i$, $1\le i\le n$, and $\theta(x_i)=0$ for other x_i's, then $\theta(y)=y$, and hence K is strongly pure in P. □

2. Absolutely (m,n)-pure modules

Definition 2.1 A_R is said to be *absolutely (m,n)-pure* if A is (m,n)-pure in every module which contains A as a submodule.

Theorem 2.2 *For a right R-module A, the following statements are equivalent:*
(1) *A is absolutely (m,n)-pure,*
(2) *A is (m,n)-pure in its injective envelope $E(A)$,*
(3) *A is (n,m)-injective .*

Proof (1)⇒(2) is obvious.

(2)⇒(1) Suppose $A\le B$, then $A\le E(A)\le E(B)$. Since A is (m,n)-pure in $E(A)$ and $E(A)$ is pure in $E(B)$, A is (m,n)-pure in $E(B)$ by Proposition 1.9(1). Note that $A\le B\le E(B)$, by Proposition 1.9(2), A is (m,n)-pure is B.

(2)⇔(3) follows from the the exact sequence

$$Hom_R(V,E(A))\to Hom_R(V,E(A)/A)\to Ext_R^1(V,A)\to 0$$

and Remark 1.2 and Theorem 1.5(5). □

Proposition 2.3 *If A_R is an (m,n)-pure submodule of an absolutely (m,n)-pure module B_R, then A is absolutely (m,n)-pure.*

Proof For every (m,n)-presented module ${}_RV$, since A is (m,n)-pure in B and B is (m,n)-pure in $E(B)$, $A\otimes V\to B\otimes V$ and $B\otimes V\to E(B)\otimes V$ are monomorphisms. Thus the following commutative diagram

$$\begin{tikzcd} A\otimes V \arrow[r] \arrow[d] & B\otimes V \arrow[d] \\ E(A)\otimes V \arrow[r] & E(B)\otimes V \end{tikzcd}$$

gives that the map $A\otimes V\to E(A)\otimes V$ is a monomorphism. □

The following theorem is immediate.

Theorem 2.4 *Suppose that $A_R \leq B_R$ and B_R is (m,n)-injective, then A is (m,n)-injective if and only if A is (n,m)-pure in B.* □

Corollary 2.5 *Let $A_R \leq B_R$.*

(1) *If B is FP-injective, then A is FP-injective if and only if A is pure in B.*

(2) *If B is F-injective, then A is F-injective if and only if A is $(\aleph_0, 1)$-pure in B.*

(3) *If B is n-injective, then A is n-injective if and only if A is $(n,1)$-pure in B. In particular, if B is P-injective, then A is P-injective if and only if A is $(1,1)$-pure in B.* □

Corollary 2.6 *Let A be a right R-module, then*

(1) *A is FP-injective if and only if A is absolutely pure.*

(2) *A is F-injective if and only if for every finitely generated free left R-module F and every cyclic submodule K of F, $A \otimes F/K \to E(A) \otimes F/K$ is a monomorphism.*

(3) *A is n-injective if and only if for every finitely generated free left R-module F and every cyclic submodule K of the left R-module ${}_RR^n$, $A \otimes R^n/K \to E(A) \otimes R^n/K$ is a monomorphism. In particular, A is P-injective if and only if for each $a \in R$, $A \otimes R/Ra \to E(A) \otimes R/Ra$ is a monomorphism.* □

Remark 2.7 Following Jain [6], ring R is said to be right IF in case every injective right R-module is flat. By Corollary 2.6, it is easy to see that if R is a right IF ring, then A_R is n-injective if and only if $Tor_1^R(E(A)/A, R^n/I) = 0$ for all cyclic submodules I of ${}_RR^n$.

Definition 2.8 A right R-module A is said to be $(m, \aleph_0)$-*injective* (resp., $(\aleph_0, n)$-*injective*) if A is (m,n)-injective, for every positive integer n (resp., m). Clearly, A is F-injective if and only if A is $(1, \aleph_0)$-injective.

Proposition 2.9 *Suppose that A_R is (n,m)-injective and $E(A)$ is n-generated, then A is $(\aleph_0, m)$-injective.*

Proof As A is (n,m)-injective, so A is (m,n)-pure in $E(A)$ by Theorem 2.2. But $E(A)$ is n-generated, by proposition 1.8(1), A is $(m, \aleph_0)$-pure in $E(A)$, and hence A is $(\aleph_0, m)$-injective. □

Remark 2.10 Rutter [9] has an example of a left P-injective ring which is not left 2-injective. Similarly, we can give an example of a right P-injective ring which is not right 2-injective. So, in general, an $(1,n)$-pure submodules of a right R-module need not to be its $(2,n)$-pure submodules. From Theorem

2.2, we see also that if for every positive integer m, $(m,1)$-pure submodules of each injective R-module are its $(m,\aleph_0)$-pure submodules, then F-injective R-modules are FP-injective modules.

3. (m,n)-purity and (m,n)-flatness

In this section, we consider the relation between (m,n)-purity and (m,n)-flatness. Firstly, we give a definition as below.

Definition 3.1 A right R-module V is said to be (m,n)-*flat*, if for every n-generated submodule I of ${}_RR^m$, the canonical map $V\otimes I\to V\otimes R^m$ is monic. A right R-module V is called $(m,\aleph_0)$-*flat* (resp., $(\aleph_0,n)$-*flat*), if for every positive integer n (resp., m), V is (m,n)-flat.

Note that $(1,n)$-flat module is n-*flat* in sense of [10] and the next proposition is easy to verify.

Proposition 3.2 *For a right R-module V, the following statements are equivalent*:

(1) *V is (m,n)-flat,*

(2) *$Tor_1^R(V,M)=0$ for all (m,n)-presented left R-module M,*

(3) *V^+ is (m,n)-injective ,*

(4) *For every n-generated submodule I of ${}_RR^m$, the map $\mu_I: V\otimes I\to VI$; $\sum v_i\otimes x_i\mapsto\sum v_ix_i$ is a monomorphism,*

(5) *For all $X\in V^n$, $A\in R^{n\times m}$, if $XA=0$, then exist a positive integer l and $Y\in V^l$, $C\in R^{l\times n}$, such that $CA=0$ and $X=YC$.* □

Remark 3.3 From Proposition 3.2, the (m,n)-flatness of V_R can be characterized by the (m,n)-injectivity of V^+. On the other hand, by [3, Lemma 2.7(1)], the sequence $Tor_1^R(V^+,M)\to Ext_R^1(M,V)^+\to 0$ is exact for all finitely presented left R-module M, so if V^+ is (m,n)-flat, then V is (m,n)-injective.

Proposition 3.4 *$(n,\aleph_0)$-pure submodules of (m,n)-flat modules are (m,n)-flat.*

Proof Suppose that V_R is (m,n)-flat, K is $(n,\aleph_0)$-pure in V. Let $X\in K^n$, $A\in R^{n\times m}$ satisfy $XA=0$, then by the (m,n)-flatness of V, there exist positive integer l, $U\in V^l$ and $C\in R^{l\times n}$ such that $CA=0$ and $X=UC$. Since K is $(n,\aleph_0)$-pure in V and hence (n,l)-pure, by Theorem 1.5(3), $X=YC$ for some $Y\in K^l$. And so K is (m,n)-flat. □

Corollary 3.5 *Pure submodules of a flat module are flat.* □

Theorem 3.6 *Let $U_R'\leq U_R$.*

(1) *If U/U' is (m,n)-flat, then U' is (m,n)-pure in U.*
(2) *If U' is (m,n)-pure in U and U is (m,n)-flat, then U/U' is (m,n)-flat.*

Proof It follows from the exact sequence

$$Tor_1^R(U,M) \to Tor_1^R(U/U',M) \to U' \otimes M \to U \otimes M$$

and Proposition 3.2(2). □

Corollary 3.7 *Let $U'_R \leq U_R$.*
(1) *If U is flat, then U/U' is flat if and only if U' is pure in U.*
(2) *If U is n-flat, then U/U' is n-flat if and only if U' is $(1,n)$-pure in U.*
□

By Proposition 3.2(2), we see that V_R is flat if and only if V is (m,n)-flat for all positive integers m and n. Suppose V is an n-generated and n-flat module, there exists an exact sequence $0 \to K \to F \to V \to 0$ with F free and $rank(F) = n$. Then K is $(1,n)$-pure in F by Corollary 3.7 and hence $(1,\aleph_0)$-pure by Proposition 1.7, so V is $(1,\aleph_0)$-flat. It is well known that V_R is flat if and only if V is $(1,\aleph_0)$-flat. So we have the following corollary.

Corollary 3.8[10] *n-generated and n-flat modules are flat.* □

Proposition 3.9 *Let U_R be an n-generated flat module, $U'_R \leq U_R$, then U' is pure in U if and only if U' is $(1,n)$-pure. In particular, for every right ideal I of R, I_R is pure in R_R if and only if I_R is $(1,1)$-pure in R_R.*

Proof By Proposition 1.7. □

Corollary 3.10 *Suppose R_R is FP-injective (resp., (m,n)-injective, F-injective, n-injective), $I_R \leq R_R$, then I_R is FP-injective (resp., (m,n)-injective, F-injective, n-injective) if and only if I_R is P-injective.*

Proof It follows immediately from Theorem 2.4 and Proposition 3.9. □

Proposition 3.11 *Suppose that R is a right IF ring and $(1,n)$-flat right R-modules are (m,n)-flat, then $(n,1)$-injective right R-modules are (n,m)-injective.*

Proof If A_R is $(n,1)$-injective, then A is $(1,n)$-pure in $E(A)$. Since R is a right IF ring, $E(A)$ is flat. Hence $E(A)/A$ is $(1,n)$-flat. By hypothesis, $E(A)/A$ is (m,n)-flat and hence A is (m,n)-pure in $E(A)$, that is, A is (n,m)-injective. □

Corollary 3.12 *Suppose that R is a right IF ring and $(1,1)$-flat right R-modules are $(\aleph_0,1)$-flat. Then P-injective right R-modules are F-injective and $(1,1)$-flat left R-modules are flat.* □

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References

[1] F.W.Anderson and K.R.Fuller, "Rings and Categories of Modules", GTM 13, Springer-Verlag, New York, 1974.

[2] J.L.Chen, N.Q.Ding, Y.L.Li, Y.Q.Zhou, *On (m,n)-injectivity of modules*, Comm.Algebra, **29**(12)(2001), 5589-5603.

[3] J.L.Chen, N.Q.Ding. *On n-coherent rings*, Comm.Algebra, **24**(10)(1996), 3211-3216.

[4] P.M.Cohn, *On the free product of associative rings*, Math. Z., **71**(1959), 380-398.

[5] R.N.Gupta, *On f-injective modules and semihereditary rings*, Proc.Nat.Inst. Sci.India Part A **35**(1969),323-328.

[6] S.Jain, *Flat and FP-injective modules*, Proc. Amer. Math. Soc., **41**(2)(1973), 437-442.

[7] B.Maddox, *Absolutely pure modules*, Proc. Amer. Math. Soc., **18**(1967), 155-158.

[8] V.S.Ramamurthi, K.M.Rangaswamy, *On finitely injective modules*, J. Austral. Math. Soc., **16**(1973), 239-248.

[9] E.A.Rutter, *Rings with the principal extenaion property*, Comm. Algebra, **3**(3)(1975), 203-212.

[10] A.Shamsuddin, *n-injective and n-flat modules*, Comm. Algebra, **29**(5)(2001), 2039-2050.

[11] R.Wisbauer, "Foundations of Module and Ring Theory", Gordon and Breach Since Publishers, Philadelphia, Pennsylvania, 1991.