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ψ-DENSITY TOPOLOGIES

B.K. Lahiri* and **Pratulananda Das**†

B-1/146 Kalyani, West Bengal - 741235, India
e-mail : indrajit@cal2.vsnl.net.in

†*Department of Mathematics, Jadavpur University*
Kolkata - 700 032, India
e-mail : pratulananda@yahoo.co.in

Abstract

In this survey article we highlight some recent works on ψ-density point, ψ-density topology, ψ-approximately continuous functions etc. where ψ is a suitable real valued function of the real variable. Whenever possible, comparisons between these ideas with the corresponding classical ideas on density point, density topology etc. have been elicited.

1. Introduction

The primary object of this survey article is to highlight the main works on ψ-density topology, where ψ is a suitable continuous real valued function of a real variable, found recently by workers in this field. In fact, the paper [18] is the beginning of this theory the authors of which started the investigations and published several papers after the paper of Taylor [17] on Lebesgue density theorem. Therefore a collective information of works done so far is warranted for a total view of the subject and to encourage researchers in the related fields to resume their investigations in this branch of modern analysis. It is now well-known that the theory of classical density topology has a prominent position in the literature of real analysis, topology and measure theory in which various workers are currently interested to pursue their investigations from various angles of this fine branch of modern analysis. Attention of interested readers may be drawn to two survey articles { [10], [11]} which focussed on the basic ideas related to classical density topology.

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It appears that the theory of ψ-density topology, so far established, is analogous in nature to the corresponding theory of classical density topology, but the proofs in the new theory are entirely different and highly sophisticated. For convenience of readers, in Section 2 we summarise in brief, the basic definitions and ideas on classical density theory to have an idea how the comparable results on ψ-density theory has been generated. In Section 3, we explain the queries of Taylor [17] on Lebesgue density theorem which ultimately becomes the basis of ψ-density subject. Section 4 outlines the details of ψ-density points and ψ-density topology. In Section 5 we present comparisons of ψ-density topologies for various functions ψ. Section 6 treats with the interior operation in a ψ-density topology. Section 7 discusses the nature of ψ-density topology for function ψ with continuity dropped. In Section 8, we observe how the idea of ψ-approximately continuous functions are introduced and properties of such functions are found out. Various properties of the real line equipped with the ψ-density topology are presented in section 9, and we discuss the union of ψ-density topologies in Section 10.

Notations : (a) $\mathbb{R}$ stands for the set of all real numbers with the usual topology, τ says, unless otherwise stated, (b) $\mathbb{R}_+$ is the set of all positive real numbers and $\mathbb{Q}_+$ is the set of all positive rational numbers, (c) sets are always subsets of $\mathbb{R}$ and points are in $\mathbb{R}$, (d) $\mathbb{N}$ stands for the set of all positive integers, (e) if A is a set, then A^c stands for the complement of A in $\mathbb{R}$, (f) $A - B$ is the set of points in A which are not in B, (g) $A\Delta B$ is the symmetric difference of the sets A and B, i.e. $A\Delta B = (A - B) \cup (B - A)$, (h) if A is a set and x is a point then $A - x = \{a - x; a \in A\}$, (i) measurable means Lebesgue measurable, (j) $m^*(A)$ denotes the Lebesgue outer measure of A and $m(A)$ is the Lebesgue measure of A.

2. Classical density topology and approximate continuity

If $E \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$, then the *outer right upper (lower) density* of E at x_0 is defined as

$$\limsup_{x \to x_0^+} \frac{m^*(E \cap [x_0, x])}{|x - x_0|} \left(\liminf_{x \to x_0^+} \frac{m^*(E \cap [x_0, x])}{|x - x_0|} \right). \tag{1}$$

Similarly the *outer left upper (lower) density* of E at x_0 is defined as

$$\limsup_{x \to x_0^-} \frac{m^*(E \cap [x, x_0])}{|x - x_0|} \left(\liminf_{x \to x_0^-} \frac{m^*(E \cap [x, x_0])}{|x - x_0|} \right). \tag{2}$$

If all the four densities of E at x_0 are equal then the common value is written as

$$\lim_{h\to 0} \frac{m^*(E\cap[x_0-h,x_0+h])}{2h} \tag{3}$$

and is denoted by $D^*(E,x_0)$ and the *outer density* of E is said to exist at x_0. Similarly, we denote by $\overline{D}^*(E,x_0)$ and $\underline{D}^*(E,x_0)$ respectively the *outer upper* and *outer lower densities* of E at x_0 when they exist. If $D^*(E,x_0)=1$ then x_0 is called an *outer density point* of E and if $D^*(E,x_0)=0$ then x_0 is called an *outer dispersion point* of E. If E is measurable then in (1) - (3) m^* is replaced by m and the term outer is dropped everywhere. $D^*(E,x_0)$, $\overline{D}^*(E,x_0)$ and $\underline{D}^*(E,x_0)$ are then denoted by $D(E,x_0)$, $\overline{D}(E,x_0)$ and $\underline{D}(E,x_0)$ respectively.

In the theory of density of sets, the following theorem plays a prominent role.

Theorem 1 (Lebesgue Density Theorem, [15], p. 17) *If $E\subset\mathbb{R}$, then almost all points of E are points of outer density of E. Further if E is measurable then almost all points of E^c are points of dispersion of E.*

If $E\subset\mathbb{R}$ is measurable, let

$$\phi(E)=\{x\in\mathbb{R},\ D(E,x)=1\}.$$

By Theorem 1, $\phi(E)$ is measurable and the set function $\phi(E)$ has the following properties :

Theorem 2 (cf. [15], p. 88) *If A,B are measurable sets and if $A\sim B$ means $m(A\Delta B)=0$, then*

(i) $\phi(A)\sim A$.

(ii) $A\sim B$ *implies* $\phi(A)=\phi(B)$.

(iii) $\phi(\Phi)=\Phi$ *and* $\phi(\mathbb{R})=\mathbb{R}$ *where* Φ *is the empty set.*

(iv) $\phi(A\cap B)=\phi(A)\cap\phi(B)$.

(v) $A\subset B$ *implies* $\phi(A)\subset\phi(B)$.

Such a function ϕ is called a *lower density.*

Definition 1 { [7]; [15], p. 90}. If $\mathcal{D}=\{A\subset\mathbb{R}; A$ is measurable and $A\subset\phi(A)\}$, then $\mathcal{D}$ is a topology called the *density topology* on $\mathbb{R}$ (T_d - topology in short).

It is known that T_d-topology is finer than the Euclidean topology τ and thus T_d-topology is Hausdorff [7].

Theorem 3 ([7], [8]) *The density topology is regular, not normal, neither first countable, nor Lindelöff.*

Definition 2 (cf. [7]) If E is measurable, then the *interior* of E in the T_d-topology is the set of all density points of E which are contained in E.

In other words $T_d - Int(E) = E \cap \phi(E)$.

The following theorem which is proved for $\mathbb{R}$ in [13] and for higher dimensional Euclidean spaces in [8] is the well-known Lusin-Menchoff theorem (see [19]).

Theorem 4 *Let $E \subset \mathbb{R}$ be a Borel set and let $X \subset E$ be a closed set such that $D(E, x) = 1$ for every $x \in X$. Then there is a perfect set P such that*

(1) $X \subset P \subset E$,

(2) $D(P, x) = 1$ for every $x \in X$.

Next theorem heavily depends on Theorem 4.

Theorem 5 ([8]) *The T_d-topology on $\mathbb{R}$ is completely regular.*

Definition 3 ([7]) A mapping f from $\mathbb{R}$ to a topological space is called *approximately continuous* at a point $p \in \mathbb{R}$ if $D^*(f^{-1}(G), p) = 1$ for every open set G containing $f(p)$.

It is clear that an approximately continuous function is continuous in the T_d-topology and that any continuous function is approximately continuous but the converse is not true.

Definition 4 A subset E of a topological space is said to have the *property of Baire* [9, p. 87] if $E = G \Delta P$ where G is open and P is of first category. A mapping $f : \mathbb{R} \to X$ where X is a metric space is said to have the *property of Baire* if for every open set U in X, $f^{-1}(U)$ has the property of Baire.

We now observe that an approximately continuous function enjoys the following fundamental properties.

Theorem 6 ([7]) *An approximately continuous function is of Baire class 1 and also has the property of Baire.*

Theorem 7 ([7]) *The image of $\mathbb{R}$ under an approximately continuous mapping to a metric space is separable. Also such a mapping takes d-regular sets (regular sets in T_d-topology) into connected sets.*

The following theorem shows that the measurability of a mapping has a close relationship with its approximate continuity.

Theorem 8 ([16], p. 132) *A function $f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if it is approximately continuous almost everywhere.*

Most of the above descriptions may be found in the survey articles [10] and [11].

3. Basis of ψ-density ideas

We note that in the definition of outer density of sets (3), one can replace the symmetric closed intervals by closed intervals I containing x_0 without any material change so that we may write the definition as

$$D^*(E,x_0) = \lim_{x_0\in I, m(I)\to 0} \frac{m^*(E\cap I)}{m(I)} \tag{4}$$

provided the limit exists. In view of (4) the second part of Lebesgue density theorem (Theorem 1) infers that if E is measurable, then

$$\lim_{x\in I, m(I)\to 0} \frac{m(E^c\cap I)}{m(I)} = 0 \tag{5}$$

for all points x of E except for a possible subset E' of E of measure zero.

Taylor [17] asked whether the above result (i.e. (5)) can be improved either for a particular measurable set E or uniformly for all measurable sets. He proposed four problems in this direction, of which we are interested for the present on the following two problems.

Problem 1 : Given a measurable set $E \subset \mathbb{R}$, whether there exists a real function $\psi(x)$, depending on E, monotone increasing, and defined for positive x with $\lim_{x\to 0^+} \psi(x) = 0$ such that

$$\lim_{x\in I, m(I)\to 0} \frac{m(E^c\cap I)}{m(I).\psi(m(I))} = 0 \tag{6}$$

for almost all points x of E.

The answer to this question is affirmative as the following theorem shows.

Theorem 9 ([17]) *Given any Lebesgue measurable set E, there exists a function $\psi(x)$ which is defined for positive x, is continuous and decreases to zero, such that*

$$\lim_{x\in I, m(I)\to 0} \frac{m(E^c\cap I)}{m(I).\psi(m(I))} = 0$$

for all x in E except for a subset of Lebesgue measure zero.

The above result implies that for a given measurable set E, the Lebesgue density theorem (see (5)) can be strengthened in the sense that the conclusion remains the same even with an additional factor in the denominator that tends to zero.

Problem 2 : Taylor [17] asked if a fixed function $\psi(x)$ of the above type exists such that (6) holds for almost all points of every measurable set E.

He answered this query by proving that no such function $\psi(x)$ exists.

Theorem 10 ([17]) *Given any function $\psi(x)$, defined for $0 < x < 1$, which decreases to zero as x decreases to zero, and a real number α, $0 < \alpha < 1$, there exists a perfect set $E \subset [0,1]$ with $m(E) = \alpha$ such that*

$$\lim_{x\in I, m(I)\to 0} \frac{m(E^c \cap I)}{m(I).\psi(m(I))} = \propto$$

for all $x \in E$.

Theorems 9 and 10 appear to be the basis of the ideas of ψ-density points, ψ-density topology etc. initiated by Terepeta and Bojakowska [18].

4. ψ-density points and ψ-density topology

Let C denote the family of all continuous non-decreasing functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim\limits_{x\to 0^+} \psi(x) = 0$. The concept of ψ-density points and eventually ψ-density topology was introduced by Terepeta and Bojakowska [18] which arises from relation (6) formulated by Taylor [17]. To keep the analogy more closely with the classical case, instead of the closed interval I as in (6), they considered only symmetric closed intervals i.e. if $x \in \mathbb{R}$ and $E \subset \mathbb{R}$ is measurable then the following relation

$$\lim_{h\to 0^+} \frac{m(E^c \cap [x-h, x+h])}{2h.\psi(2h)} = 0 \tag{7}$$

where $\psi \in C$ is considered instead of (6).

Clearly (6) implies (7) because $\lim\limits_{x\to 0^+} \psi(x) = 0$. But (7) need not imply (6) as shown by the following theorem.

Theorem 11 ([18]) *There exists a measurable set A and a function $\psi \in C$ such that for $x = 0$ the condition (7) is fulfilled but (6) does not hold.*

Taking account of the smallest symmetric interval centered at x, Theorem 10 has the following analogous version.

Theorem 12 ([18]) *For each function $\psi \in C$ and real number $\alpha, 0 < \alpha < 1$, there exists a perfect set $E \subset [0,1]$ such that $m(E) = \alpha$ and*

$$\limsup_{h\to 0^+} \frac{m(E^c \cap [x-h, x+h])}{2h.\psi(2h)} = \propto$$

for all $x \in E$.

Throughout this section as well as for the rest ψ will be a function from C, unless otherwise stated.

We are now in a position to present the definition of ψ-density point and related concepts.

Definition 5 ([18]) We say that 0 is a ψ-*density point* of a measurable set A if and only if

$$\lim_{h\to 0^+} \frac{m(A^c \cap [-h,h])}{2h.\psi(2h)} = 0.$$

$x \in \mathbb{R}$ is called a ψ-*density point* of A if and only if 0 is a ψ-density point of $A - x$. x is a ψ-*dispersion point* of A if and only if x is a ψ-density point of A^c i.e.

$$\lim_{h\to 0^+} \frac{m(A \cap [x-h,x+h])}{2h.\psi(2h)} = 0.$$

For a measurable set A, let

$$\phi_\psi(A) = \{x \in \mathbb{R};\ x \text{ is a } \psi - \text{ density point of } A\}.$$

Then $\phi_\psi(A) \subset \phi(A)$ (for $\phi(A)$, see Section 2) and so

$$\phi_\psi(A) - A \subset \phi(A) - A.$$

Hence it follows from Theorem 1 that

$$m(\phi_\psi(A) - A) = 0.$$

In view of the definition of $\phi_\psi(A)$, Theorem 12 may be restated as follows: Given $\psi \in C$ and $\alpha \in (0,1)$ there exists a perfect set $E \subset [0,1]$ such that

$$m(E) = \alpha \text{ and } \phi_\psi(E) = \Phi$$

and thus

$$m(E\Delta\phi_\psi(E)) = \alpha. \tag{8}$$

This again shows that for an arbitrary $\psi \in C$, Lebesgue density theorem (Theorem 1) does not hold uniformly for all measurable sets.

We observe that ϕ_ψ transforms measurable sets into measurable sets.

Theorem 13 ([18]) *If A is measurable then $\phi_\psi(A)$ is also measurable.*

The proof follows from the fact that if we put

$$F(x,h) = \frac{m(A^c \cap [x-h,x+h])}{2h.\psi(2h)}$$

for $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$, then $\phi_\psi(A)$ can be shown to be of the form

$$\phi_\psi(A) = \bigcap_{n\in N} \bigcup_{\delta\in Q_+} \bigcap_{0<h<\delta} \{x \in \mathbb{R};\ F(x,h) \leq \frac{1}{n}\}$$

so that $\phi_\psi(A)$ is a set of $F_{\sigma\delta}$ type.

Theorem 14 ([18]) *If A, B are measurable, then* *(i)* $A \subset B$ *implies* $\phi_\psi(A) \subset \phi_\psi(B)$*;*

(ii) If $A \sim B$ *then* $\phi_\psi(A) = \phi_\psi(B)$*;*

(iii) $\phi_\psi(A \cap B) = \phi_\psi(A) \cap \phi_\psi(B)$*;*

(iv) $\phi_\psi(\Phi) = \Phi$ *and* $\phi_\psi(\mathbb{R}) = \mathbb{R}$ *where* Φ *is the empty set and* $A \sim B$ *means* $m(A\Delta B) = 0$*.*

The operation ϕ_ψ is not Oxtoby's "lower density" (see Theorem 2) because as was observed in (8), $A \sim \phi_\psi(A)$ does not hold for all measurable sets A.

Definition 6 ([18]) Let $T_\psi = \{A \subset \mathbb{R}; A$ is measurable and $A \subset \phi_\psi(A)\}$.

Theorem 15 ([18]) *T_ψ is a topology on the real line called ψ-density topology which is stronger than the Euclidean topology but weaker than the density topology.*

Since open sets in the T_d-topology are measurable [7], from Theorem 15 it therefore follows that open sets in T_ψ-topology are also measurable.

5. Comparison of ψ-density topologies and translation of sets

The primary object in this section is to present investigations on comparisons of ψ-density topologies for various ψ's, all members of C. To study this phenomenon, comparisons of ψ's from different aspects are necessary.

If $\psi_1, \psi_2 \in C$ and for measurable A, the fact that x is a ψ_1 dispersion point of A implies that x is a ψ_2-dispersion point of A, then clearly $T_{\psi_1} \subset T_{\psi_2}$. For example if $\psi_1(x) \le \psi_2(x)$ or more generally if $\psi_1(x) \le k.\psi_2(x)$ for some $k \in \mathbb{R}_+$ and for all $x \in \mathbb{R}_+$ then $T_{\psi_1} \subset T_{\psi_2}$.

Definition 7 ([18]) If $\psi_1, \psi_2 \in C$, then ψ_1 is said to precede ψ_2 if and only if

$$\limsup_{h \to 0^+} \frac{\psi_1(h)}{\psi_2(h)} < \propto .$$

This fact is denoted by the notation $\psi_1 \prec \psi_2$.

Theorem 16 ([18]) *If $\psi_1 \prec \psi_2$, then $T_{\psi_1} \subset T_{\psi_2}$.*

Definition 8 ([18]) Two functions $\psi_1, \psi_2 \in C$ are said to be equivalent if and only if there exist positive numbers α, β, δ such that for each $h \in (0, \delta)$,

$$\alpha < \frac{\psi_1(h)}{\psi_2(h)} < \beta.$$

Clearly, ψ_1, ψ_2 are equivalent if and only if

$$\limsup_{h\to 0^+} \frac{\psi_1(h)}{\psi_2(h)} < \propto$$

and

$$\liminf_{h\to 0^+} \frac{\psi_1(h)}{\psi_2(h)} > 0.$$

We see that equivalent functions generate the same ψ-density topologies.

Theorem 17 ([18]) *If the functions $\psi_1, \psi_2 \in C$ are equivalent then $T_{\psi_1} = T_{\psi_2}$.*

However the equivalence condition is only sufficient but not necessary as shown by

Theorem 18 ([18]) *There exist two functions $\psi_1, \psi_2 \in C$ such that*

$$\liminf_{h\to 0^+} \frac{\psi_1(h)}{\psi_2(h)} = 0$$

and

$$0 < \limsup_{h\to 0^+} \frac{\psi_1(h)}{\psi_2(h)} < \propto$$

for which $T_{\psi_1} = T_{\psi_2}$.

If $\lim_{h\to 0^+} \frac{\psi_1(h)}{\psi_2(h)} = 0$, then the situation becomes more clear as it follows from the following theorem.

Theorem 19 ([18]) *Let $\psi_1, \psi_2 \in C$. If*

$$\lim_{h\to 0^+} \frac{\psi_1(h)}{\psi_2(h)} = 0$$

then there exists a measurable set $A \subset \mathbb{R}_+$ such that 0 is a ψ_2-dispersion point of A but it is not a ψ_1-dispersion point of A.

We observe an interesting phenomenon below how the ψ-density topology generated by an arbitrary $\psi \in C$ identifies with the ψ-density topology generated by some piece-wise linear function.

Definition 9 ([18]) A function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is *piecewise linear* if and only if there exists a decreasing sequence $\{t_n\}$ convergent to 0 such that L is linear on each interval $[t_{n+1}, t_n]$ for $n \in \mathbb{N}$ and L is arbitrary for $x > t_1$.

Theorem 20 ([18]) *For arbitrary $\psi \in C$ there exists a piecewise linear function $L \in C$ such that $T_\psi = T_L$.*

If $U \subset \mathbb{R}$ is open in the Euclidean topology and Z is a subset of $\mathbb{R}$ of measure zero, let O^* be the collection of all sets of the form $U - Z$ which forms a topology. This topology is known as *Hashimoto topology.* Interestingly, the intersection of all ψ-density topologies is a Hashimoto topology. More precisely.

Theorem 21 ([18]) $\bigcap_{\psi \in C} T_\psi = O^*$.

Definition 10 If $\alpha \in \mathbb{R}$ and $A \subset \mathbb{R}$, then $\alpha A = \{\alpha a; a \in A\}$.

So αA is a translation of the set A in the multiplicative sense. We notice now that ψ-dispersion point is invariant with respect to translation of sets.

Theorem 22 ([18]) *Let A be measurable, $\psi \in C$ and $\alpha \geq 1$. If 0 is a ψ-dispersion point of A, then 0 is a ψ-dispersion point of αA.*

However, for $\alpha < 1$, we have

Theorem 23 ([18]) Let $\psi \in C$ and $\alpha < 1$. If,

$$\liminf_{x \to 0^+} \frac{\psi(\alpha x)}{\psi(x)} = 0,$$

then there exists a measurable set A such that 0 is a ψ-dispersion point of A, but is not a ψ-dispersion point of αA.

Further study is made in [2] on the comparison of ψ-dispersion points and ψ-density topologies for various ψ's by constructing suitably different sets of points in $\mathbb{R}_+$.

For $\psi_1, \psi_2 \in C$ let

$$A_k^+ = \{x \in \mathbb{R}_+;\ \psi_1(2x) < \frac{1}{k}\psi_2(2x)\},$$

$$B_k^+ = \{x \in \mathbb{R}_+;\ \psi_2(2x) < \frac{1}{k}\psi_1(2x)\}.$$

$A_k = A_k^+ \cup (-A_k^+), B_k = B_k^+ \cup (-B_k^+)$ for $k \in \mathbb{N}$ where for a set $A, -A = \{-a; a \in A\}$.

Using A_k, the following is the first comparison theorem.

Theorem 24[2]. Let $\psi_1, \psi_2 \in C$ and

$$\in_k = \limsup_{x \to 0^+} \frac{m(A_k \cap [-x, x])}{2x.\psi_1(2x)}$$

for $k \in N$. If $\lim\limits_{k \to \propto} \in_k = 0$ and 0 is a ψ_2-dispersion point of a measurable set E, then 0 is a ψ_1 dispersion point of E.

Corollary 1 ([2]) *Under the assumptions of Theorem 24, $T_{\psi_2} \subset T_{\psi_1}$.*

Corollary 2 ([2]) *Let $\psi_1, \psi_2 \in C$,*

$$\in_k = \limsup_{x \to 0^+} \frac{m(A_k \cap [-x, x])}{2x.\psi_1(2x)}$$

and

$$\eta_k = \limsup_{x \to 0^+} \frac{m(B_k \cap [-x, x])}{2x.\psi_2(2x)}.$$

If $\lim_{k \to \propto} \in_k = \lim_{k \to \infty} \eta_k = 0$, then $T_{\psi_1} = T_{\psi_2}$.

If however, $\lim_{k \to \infty} \in_k > 0$ then we have

Theorem 25([2])*Let $\psi_1, \psi_2 \in C$ and*

$$\in_k = \limsup_{x \to 0^+} \frac{m(A_k \cap [-x, x])}{2x.\psi_1(2x)}$$

for $k \in N$. If $\lim_{k \to \infty} \in_k > 0$, then there exists a measurable set $E \subset \mathbb{R}$ such that 0 is a ψ_2-dispersion point of E, but it is not a ψ_1-dispersion point of E.

Corollary 3 ([2]) *Under the assumptions of Theorem 25, $T_{\psi_1} - T_{\psi_2} \neq \Phi$.*

Ultimately we observe the necessary and sufficient condition for two topologies T_{ψ_1} and T_{ψ_2} to be identical in terms of $\in_k$ and η_k.

Theorem 26 ([2]) *Let $\psi_1, \psi_2 \in C$,*

$$\in_k = \limsup_{x \to 0^+} \frac{m(A_k \cap [-x, x])}{2x.\psi_1(2x)} \text{ and } \eta_k = \limsup_{x \to 0^+} \frac{m(B_k \cap [-x, x])}{2x.\psi_2(2x)}.$$

The topologies T_{ψ_1} and T_{ψ_2} are equal if and only if

$$\lim_{k \to \infty} \in_k = \lim_{k \to \infty} \eta_k = 0.$$

It is not difficult to see that the above conclusions hold if in the definitions of A_k^+, B_k^+, A_k, B_k we consider an arbitrary increasing sequence $\{a_k\}_{k \in \mathbb{N}}$ of positive numbers tending to infinity instead of positive integers k.

We consider now the family $\mathbf{A} = \{T_\psi; \psi \in C\}$ of all ψ-density topologies. One can introduce a partial order relation in $\mathbf{A}$ using inclusion relation. Further it can be shown that for arbitrary sequence $\{\psi_n\}_{n \in \mathbb{N}}$ of functions from C there exists a function $\psi \in C$ such that $\lim_{x \to 0^+} \frac{\psi_n(x)}{\psi(x)} = 0$ for $n \in \mathbb{N}$ and consequently $T_{\psi_n} \subset T_\psi$ $\forall n \in \mathbb{N}$ (see [18, Theorem 20]). So for each countable subset of $\mathbf{A}$ there exists the upper bound of this set in $\mathbf{A}$. Further this partial order in $\mathbf{A}$ is dense as shown by

Theorem 27 ([2]) *For arbitrary $\psi_1, \psi_2 \in C$ such that $\psi_1(x) \leq \psi_2(x)$ for $x \in \mathbb{R}_+$ and $T_{\psi_1} \subset T_{\psi_2}$, there exists a function $\psi_3 \in C$ such that $T_{\psi_1} \subset T_{\psi_3} \subset T_{\psi_2}$.*

Corollary 4 ([2])*If $T_{\psi_1} \subset T_{\psi_2}$, then there exists a function $\psi_3 \in C$ such that $T_{\psi_1} \subset T_{\psi_3} \subset T_{\psi_2}$.*

Finally one sees that there exists an uncountable number of functions from C such that the corresponding density topologies are not comparable.

Theorem 28 ([2]) *There exists a subfamily $C_0 \subset C$ such that card $(C_0) = c$ where c is the power of the continuum and for each $\psi_1, \psi_2 \in C_0, \psi_1 \neq \psi_2$, the topologies T_{ψ_1} and T_{ψ_2} are not comparable by inclusion.*

6. The interior operation in a ψ-density topology

It is known [14] that the interior of an arbitrary set $A \subset \mathbb{R}$ in the density topology can be shown to be equal to $A \cap \phi(B)$ where $B \subset A$ is a measurable kernel of A and $\phi(B)$ is the set of all density points of B. However the problem of obtaining the interior of a set in a ψ-density topology appears to be more complicated because the proof of the result of [14] referred to above uses Lebesgue density theorem while ϕ_ψ does not have the Lebesgue property (see Section 4). Let $\psi \in C$ be fixed. For convenience of notations, we shall denote in this section ϕ_ψ by ϕ and T_ψ by T. If A is measurable, let $\phi^1(A) = \phi(A)$. If α is an ordinal number, $1 < \alpha < \Omega$, where Ω is the ordinal number of the set of all order types of countable well ordered sets, an operator ϕ^α is defined in the following way :

(i) If α has a predecessor i.e. $\alpha = \beta + 1$, where $1 \leq \beta < \Omega$, then

$$\phi^\alpha(A) = \phi(\phi^\beta(A)).$$

(ii) If α is a limit number, then

$$\phi^\alpha(A) = \bigcap_{1 \leq \beta < \alpha} \phi^\beta(A).$$

With this provision the following theorem is true.

Theorem 29 ([3]) *For each measurable A and each countable ordinal α greater than zero*

$$T - Int(A) \subset A \cap \phi^\alpha(A).$$

There exists an ordinal $\beta, 1 \leq \beta < \Omega$ such that

$$T - Int(A) = A \cap \phi^\beta(A).$$

If the measurability of A is dropped, then an analogous theorem for the expression of interior of a set in a ψ-density topology is true.

Theorem 30 ([3]) *For arbitrary $A \subset \mathbb{R}$, we have*

$$T - Int(A) = A \cap \phi^{\beta}(B),$$

where $B \subset A$ is a measurable kernel of A and β is some countable ordinal greater than or equal to 1.

7. ψ-density topology for discontinuous regulator functions

We observed in the preceding sections that the ψ-density topology has been generated for ψ's such that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing *continuous* and $\lim_{x\to 0^+} \psi(x) = 0$. Aversa and Wilczynski [1] investigated if the continuity condition on ψ may be dropped, but still generating the same family of ψ-density topologies. As such let C^* denote the class of all non-decreasing functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to 0^+} \psi(x) = 0$. For a $\psi \in C^*$ one can define analogously the ψ-density points, ψ-density topology etc. The following theorem is the basis to draw the conclusion in this respect.

Theorem 31 ([1]) *If $\psi \in C^*$, then there exist two functions $F, G \in C$ such that*

1) *For each $x \in (0, 2], F(x) \le \psi(x) \le G(x)$ (or, for each $x \in (0, 1], F(2x)$ $\le \psi(2x) \le G(2x)$);*
2) *if $A_2^+ = \{x \in \mathbb{R}_+ : F(2x) < \frac{1}{2}G(x)\}$ and $A_2 = A_2^+ \cup (-A_2^+)$ then*

$$\limsup_{x\to 0^+} \frac{m(A_2 \cap [-x, x])}{2x.F(2x)} = 0.$$

Using Theorem 31, the following theorem follows.

Theorem 32 ([1]) *If $\psi \in C^*$, then there exists $F \in C$ such that $T_\psi = T_F$.*

As a final conclusion, one sees that the family of ψ-density topologies generated by functions from C is identical with that generated by functions from C^*.

Corollary 5 ([1]) $\{T_\psi; \psi \in C\} = \{T_\psi; \psi \in C^*\}$.

8. ψ-approximately continuous functions

To define ψ-approximately continuous functions, we need the idea of inner ψ-density point. As usual ψ always is a member of C.

Definition 11 ([18]) We say that 0 is the *inner ψ-density point* of $A \subset \mathbb{R}$ if and only if there exists a measurable set B such that $B \subset A$ and 0 is a ψ-density point of B.

Definition 12 ([18]) We say that x is the *inner ψ-density point* of $A \subset \mathbb{R}$ if and only if 0 is the inner ψ-density point of $A - x$.

It is therefore clear that for a measurable set A, the notion of ψ-density point and inner ψ-density point are coincided. The following theorem now follows easily.

Theorem 33 ([18]) *A set A is open in the topology T_ψ if and only if each point of A is the inner ψ-density point of A.*

We now give the definition of ψ-approximately continuous functions.

Definition 13 ([18]) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *ψ-approximately continuous at* x_0 if and only if x_0 is the inner ψ-density point of $f^{-1}[(f(x_0) - \in, \ f(x_0) + \in)]$ for each $\in > 0$.

Definition 14 ([18]) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *ψ-approximately continuous* if and only if f is ψ-approximately continuous at each point.

Theorem 34 ([18]) *A function $f : \mathbb{R} \to \mathbb{R}$ is ψ-approximately continuous if and only if for each interval (a, b) the set $f^{-1}(a, b) \in T_\psi$.*

Analogous to the classical situation (see Theorem 8), measurability and ψ-approximate continuity are closely connected.

Theorem 35 ([18])*A function $f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if there exists a function $\psi \in C$ such that f is ψ-approximately continuous almost everywhere.*

It is known { [6], p. 21; [7]} that an approximately continuous function is Darboux Baire 1 function. Since ψ-approximate continuity implies approximate continuity, it follows therefore that every ψ-approximately continuous function is Darboux Baire 1 function. However the family of all ψ-approximately continuous functions is not contained in the class of Baire* 1 functions.

Definition 15 ([12]) A function $f : [0, 1] \to \mathbb{R}$ is called Baire ^{*1} function (in short B^*1) if for every closed set F there is an open interval (a, b) with $(a, b) \cap F \neq \Phi$ such that f/F is continuous on (a, b).

We now see that the family of ψ-approximately continuous functions is not a subset of the family of Baire* 1 functions.

Theorem 36 ([4]) *There exists a ψ-approximately continuous function f such that $f \notin B^*1$.*

In analogy with the classical case (see Section 2), a ψ-approximately continuous function is characterised by the following theorem.

Theorem 37 ([4]) *A function $f : \mathbb{R} \to R$ is ψ-approximately continuous at x_0 if and only if there exists a measurable set E such that $x_0 \in E \cap \phi_\psi(E)$ and f/E is continuous at x_0.*

9. Properties of the topological space $(\mathbb{R}, T_\psi)$

In this section we are concerned with the real number space $\mathbb{R}$ equipped with the ψ-density topology. To show that $(\mathbb{R}, T_\psi)$ is completely regular, analogue of Lusin-Menchoff type theorem { [6], p. 26; [8]} is needed which Bojakowska [4] proved in the following theorem.

Theorem 38 ([4]) *Let $E \subset \mathbb{R}$ be a Borel set and let $F \subset E$ be a closed set such that $F \subset \phi_\psi(E)$. Then there exists a perfect set P such that*

1) $F \subset P \subset E$ and

2) $F \subset \phi_\psi(P)$.

The following Lemmas, which are analogous to Lemmas 11 and 12 of [19], are helpful to derive several topological properties of $(\mathbb{R}, T_\psi)$.

Lemma 1 ([4]) *Let $E \subset \mathbb{R}$ be a T_ψ-open set and of type F_σ. Then there exists a ψ-approximately continuous function f such that*

$$0 < f(x) \leq 1$$

for $x \in E$ and

$$f(x) = 0$$

for $x \notin E$.

Lemma 2 ([4]) *Let E_1, E_2, H be pairwise disjoint subsets of $\mathbb{R}$ such that*

(i) $E_1 \cup E_2 \cup H = R$,

(ii) $E_1 \cup H, E_2 \cup H$ are T_ψ-open and of type F_σ. Then there exists a ψ-approximately continuous function f such that

(iii) $f(x) = 0$ for $x \in E_1$,

(iv) $0 < f(x) < 1$ for $x \in H$,

(v) $f(x) = 1$ for $x \in E_2$.

We are now in a position to state the following theorem.

Theorem 39 ([4]) *The space $(\mathbb{R}, T_\psi)$ is completely regular.*

We see that if $m(N) = 0$ then N is T_ψ-closed, $\phi_\psi(N) = \Phi$ and (for ψ-interior see Section 6)

$$T_\psi - Int(N) \subset N \cap \phi_\psi(N) = \Phi.$$

Thus N is nowhere dense in T_ψ.

But not only sets of measure zero are nowhere dense in T_ψ. Because referring to [17, Theorem 4] and [18, Theorem 0.2], we see that there exists a perfect nowhere dense set E of positive measure such that $\phi_\psi(E) = \Phi$. Then E is T_ψ-closed and $T_\psi - Int(E) = \Phi$ and thus E is nowhere dense in the topology T_ψ.

This section is concluded with the following remarkable theorem.

Theorem 40 ([4]) *The space $(\mathbb{R}, T_\psi)$ is of the first category.*

10. Union of ψ-density topologies

We observed in Theorem 15 that for arbitrary $\psi \in C$ the ψ-density topology T_ψ is stronger than the Euclidean topology but is weaker than the density topology T_d. We have also seen that $\bigcap_{\psi \in C} T_\psi$ is the Hashimoto topology on $\mathbb{R}$ (Theorem 21). Theorem 15 tells that for a measurable set A, every ψ-density point of A is a density point of A. It therefore follows that

$$\bigcup_{\psi \in C} T_\psi \subset T_d.$$

Bojakowska and Wilczynski [5] asked if the above inclusion is proper and answered the query in the affirmative by showing that for a measurable set A and for any $\psi \in C$ there exists a point $x_\psi \in A$ such that x_ψ is a density point of A but x_ψ is not a ψ-density point of A.

For this, they first constructed a Cantor type set of positive measure in $\mathbb{R}$ in the following way.

Put

$$I_1^0 = [0, 1]$$

and

$$\in_0 = \frac{m(I_1^0)}{2^3 + 2}.$$

Let I_1^1, I_2^1 be two closed intervals obtained by removing from the centre of $[0, 1]$ an open interval (a_1^0, b_1^0) of length $\in_0$.

Now assume that the intervals (a_i^j, b_i^j) for $j = 0, 1, \ldots, n-1$ and $i = 1, \ldots, 2^j$ have been defined. Let $\{I_i^n\}, i = 1, \ldots, 2^n$ be the sequence of equal component closed intervals (numbered from left to right) of the set

$$[0,1]/\bigcup_{j=0}^{n-1}\bigcup_{i=1}^{2^j}(a_i^j, b_i^j).$$

Let

$$\in_n = \frac{m(I_i^n)}{2^{n+3}+2}.$$

Let (a_i^n, b_i^n) be the open interval of length $\in_n$ centered in the middle of $I_i^n, i = 1, 2, \ldots, 2^n$. We denote by I_{2i-1}^{n+1} and I_{2i}^{n+1} the left and the right part of I_i^n respectively, obtained after removing the interval (a_i^n, b_i^n) from I_i^n.

Continuing in this way, finally we let

$$A = [0,1]/\bigcup_{n=0}^{\infty}\bigcup_{i=1}^{2^n}(a_i^n, b_i^n) = \bigcap_{n=0}^{\infty}\bigcup_{i=1}^{2^n} I_i^n.$$

Then A is a perfect nowhere dense set with $m(A) > 0$.

After constructing the set A the following theorem is obtained.

Theorem 41 ([5]) *For each $\psi \in C$ there exists a point $x_\psi \in A$ such that x_ψ is a density point of A but x_ψ is not a ψ-density point of A.*

Corollary 6 ([5]) *There exists a set A_0 such that*

$$A_0 \in T_d / \bigcup_{\psi \in C} T_\psi.$$

The proof of the corollary follows on using Theorem 41 and taking A_0 to be the T_d-interior of the set A.

However if $T(\mathcal{A})$ denotes the coarsest topology including an arbitrary family $\mathcal{A}$ of subsets of the real line, then [5] ultimately contains the following interesting theorem.

Theorem 42 ([5]) $T_d = T(\bigcup_{\psi \in C} T_\psi)$.

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