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COMMUTING MAPPINGS ON RIGHT IDEALS IN PRIME RINGS

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Abstract

Let R be a prime ring of characteristic different from 2, with extended centroid C, d and g derivations of R, I a non-zero right ideal of R and s_4 the standard identity of degree 4. If $[d([x,y]),[x,y]][x,y] - [x,y][g([x,y]),[x,y]] = 0$, for all $x, y \in I$, then one of the following holds:
(i) $s_4(x_1, x_2, x_3, x_4)x_5$ is an identity for I;
(ii) $d(x) = [a, x]$, with $(a - \alpha)I = 0$ for a suitable $\alpha \in C$ and $g = 0$.

Let R be a prime ring with center $Z(R)$ and extended centroid C, Q its Martindale quotient ring. Here we will consider some related problems concerning derivations in prime rings which satisfy some commuting conditions. Our aim is to study the relationship between the behaviour of such derivations and the structure of R.

Recall that a mapping F from R to R is said to be commuting on R if $[F(x), x] = 0$, for all $x \in R$, and is said to be centralizing on R if $[F(x), x] \in Z(R)$, for all $x \in R$. There has been considerable interest in commuting, centralizing and related mappings in prime and semiprime rings (see for istance [2]).

In [11] Posner proved that the existence of a non centralizing derivation d on a prime ring R, forces R to be commutative. Later in [12] Vukman has proved that in case there exists a non-zero derivation d on R, where R is a prime ring of characteristic different from 2 and 3, such that the mapping $x \longrightarrow [d(x), x]$ is centralizing on R, then R is commutative. In a recent paper [7] Jun and Kim proved that if $d(x)x - xg(x) \in Z(R)$, for d and g derivations of R and

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any $x \in R$ then either R is commutative or d and g must be zero. The main result of this note is then motived by the previous ones. More precisely here we prove the following:

Theorem 1. *Let R be a prime ring of characteristic different from 2, with extended centroid C, d and g derivations of R, I a non-zero right ideal of R and s_4 the standard identity of degree 4. If $[d([x,y]),[x,y]][x,y]-[x,y][g([x,y]),[x,y]] = 0$, for all $x, y \in I$, then one of the following holds:*
(i) $s_4(x_1, x_2, x_3, x_4)x_5$ is an identity for I;
(ii) $d(x) = [a, x]$, with $(a - \alpha)I = 0$ for a suitable $\alpha \in C$ and $g = 0$.

In all that follows R will be a prime ring of characteristic $\neq 2$, d and g derivations of R and I a non-zero right ideal of R.

For any ring S, $Z(S)$ will denote its center, and $[a, b] = ab - ba$. In addition s_4 will denote the standard identity in 4 variables.

The related object we need to mention is the Martindale quotient ring Q of a ring R (sometimes, as in [1], Q is called the maximal two-sided ring of quotients).

The definitions, the axiomatic formulations and the properties of this quotient ring Q can be found in [1].

In any case, when R is a prime ring, all that we need here about Q is that $R \subseteq Q$, Q is a prime ring and the center of Q, denoted by C, is a field which is called the extended centroid of R.

We make also a frequent use of the theory of generalized polynomial identities and differential identities (see [1], [3], [8], [9]). In particular we need to recall that, when R is prime and I a non-zero right ideal of R, then I, IR and IQ satisfy the same generalized polynomial identities [3].

We begin with the following:

Lemma 2. *Let $R = M_k(F)$, the ring of $k \times k$ matrices over the field F, with $k > 1$, a, b non-central elements of R such that $[a,[x,y]]_2[x,y]-[x,y][b,[x,y]]_2 = 0$, for all $x, y \in R$. Then $a, b \in Z(R)$ unless when $k = 2$ and $a + b \in Z(R)$.*

Proof Say $a = \sum_{ij} a_{ij}e_{ij}$, $b = \sum_{ij} b_{ij}e_{ij}$, where $a_{ij}, b_{ij} \in F$, and e_{ij} are the usual unit matrices. Let $[x, y] = [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$, for all $i \neq j$. Thus

$$[a, e_{ii} - e_{jj}]_2(e_{ii} - e_{jj}) - (e_{ii} - e_{jj})[b, e_{ii} - e_{jj}]_2 = 0.$$

Right multiplying by e_{jj} and left multiplying by e_{ii}, we get $-4(a_{ij}+b_{ij})e_{ij} = 0$. Since $char(R) \neq 2$, we have that the matrix $a + b$ is diagonal in $M_k(F)$.

For any $\varphi \in Aut_F(R)$, we have

$$[\varphi(a),[\varphi(x),\varphi(y)]]_2[\varphi(x),\varphi(y)] - [\varphi(x),\varphi(y)][\varphi(b),[\varphi(x),\varphi(y)]]_2 = 0$$

for all $x, y \in R$, and so, by the previous case, $\varphi(a) + \varphi(b)$ must be a diagonal matrix in $M_k(F)$ for any $k \geq 2$.

In particular, for any $r \neq s$, if $\varphi(x) = (1 + e_{rs})x(1 - e_{rs})$, then

$$\varphi(a) + \varphi(b) = \varphi(a + b) = (a + b) + e_{rs}(a + b) - (a + b)e_{rs} - e_{rs}(a + b)e_{rs} =$$

$$(a + b) + (a_{ss} + b_{ss} - a_{rr} - b_{rr})e_{rs}.$$

This means $a_{rr} + b_{rr} = a_{ss} + b_{ss}$, for all $r \neq s$, that is $a + b$ must be central.

Let now $k \geq 3$. Since $a + b = c \in Z(R)$, the main assumption says that

$$0 = [a, [x, y]]_2[x, y] - [x, y][b, [x, y]]_2 =$$

$$[-b + c, [x, y]]_2[x, y] - [x, y][b, [x, y]]_2 =$$

$$-[b, [x, y]]_2[x, y] - [x, y][b, [x, y]]_2.$$

As above let $[x, y] = e_{ii} - e_{jj}$, for $i \neq j$. Thus

$$0 = [b, [x, y]]_2[x, y]+[x, y][b, [x, y]]_2 = [b, e_{ii}-e_{jj}]_2(e_{ii}-e_{jj})-(e_{ii}-e_{jj})[b, e_{ii}-e_{jj}]_2.$$

Left multiplying by e_{kk}, for all $k \neq i, j$, and right multiplying by e_{ii}, it follows $e_{kk}be_{ii} = 0$. This means that b is a diagonal matrix. The same above argument, shows that b is central in $M_k(F)$, as well as a. $\square$

The first part of this paper is dedicated to study the case when d and g are both Q-inner derivations, that is there exist $a, b \in Q$ such that $d(x) = [a, x]$ and $g(x) = [b, x]$, for all $x \in R$.

Theorem 3. *Let d and g be Q-inner derivations.*

If $[d([x, y]), [x, y]][x, y] - [x, y][g([x, y]), [x, y]] = 0$, for all $x, y \in R$, then R satisfy the standard identity s_4, unless when $d = g = 0$.

Proof Let d be the inner derivation induced by the element $a \in Q$, and g the one induced by $b \in Q$. Trivially the set $\{a, b\}$ is not contained in the extended centroid C, which is the center of Q, otherwise there is nothing to prove. These assumptions say that R satisfies the generalized polynomial identity $[a, [x, y]]_2[x, y] - [x, y][b, [x, y]]_2 = 0$. By a theorem due to Chuang [3] this generalized polynomial identity is also satisfied by Q. In case C is infinite, we have $[a, [x, y]]_2[x, y] - [x, y][b, [x, y]]_2 = 0$ for all $x, y \in Q \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \bigotimes_C \overline{C}$ are centrally closed, we may replace R by Q or $Q \bigotimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $[a, [x, y]]_2[x, y] - [x, y][b, [x, y]]_2 = 0$, for all $x, y \in R$. By Martindale's theorem [10], R is a primitive ring having a non-zero socle with C as the associated division ring. In light of Jacobson's theorem [6, pag 75] R is isomorphic to a dense ring of linear transformations on some vector space V over C.

Assume first that V is finite-dimensional over C. Then the density of R on V implies that $R \cong M_k(C)$, the ring of all $k \times k$ matrices over C. In this case, by our lemma, $k = 2$ and R satisfies s_4.

Assume next that V is infinite-dimensional over C. We will prove that in this case we get a contradiction. Since V is infinite dimensional over C then, as in lemma 2 in [13], the set $[R, R]$ is dense on R and so from $[a, [x, y]]_2[x, y] - [x, y][b, [x, y]]_2 = 0$, for all $x, y \in R$, we have $[a, r]_2 r - r[b, r]_2 = 0$, for all $r \in R$. Suppose there exists $v \in V$ such that $\{v, va\}$ are linearly C-independent. By the density of R, there exist $w \in V$ and $x_0 \in R$ such that $\{v, va, w\}$ are linearly C-independent and $vx_0 = 0$, $vax_0 = w$, $wx_0 = va$. From this, we have the contradiction

$$0 = v([a, x_0]_2 x_0 - x_0[b, x_0]_2) = w \neq 0.$$

Therefore $\{v, va\}$ are linearly C-dependent, for all $v \in V$, which implies that $a \in C$, a contradiction.

The previous step implies that $r[b, r]_2 = 0$, for all $r \in R$. Suppose that there exists $v \in V$ such that $\{v, vb\}$ are linearly C-independent. Again by the density of R, there exists $x_0 \in R$ such that $vx_0 = v$, $vbx_0 = 0$ so that $0 = vx_0[b, x_0]_2 = vb \neq 0$. Therefore $\{v, vb\}$ are linearly C-dependent, for all $v \in V$, and also $b \in C$, a contradiction. $\square$

As a consequence we get:

Corollary 4. *Let g be a Q-inner derivation.*

If $[x, y][g([x, y]), [x, y]] = 0$, for all $x, y \in R$, then R satisfy the standard identity s_4, unless when $g = 0$.

We will extend the previous theorem to one-sided case, as follows:

Theorem 5. *Let d and g be inner derivations induced respectively by the elements a and b in Q. If $[d([x, y]), [x, y]][x, y] - [x, y][g([x, y]), [x, y]] = 0$, for all $x, y \in I$, a non-zero right ideal of R, then either I satisfy the identity $s_4(x_1, x_2, x_3, x_4)x_5$, or there exist $\alpha, \beta \in C$ such that $(a - \alpha)I = 0$ and $b = \beta$.*

Proof We suppose that the conclusion $s_4(x_1, x_2, x_3, x_4)x_5 = 0$ in I does not occur and prove that, in this case, there exist $\alpha, \beta \in C$ such that $(a - \alpha)I = 0$ and $b = \beta$.

Our first aim is to show that R is a GPI-ring, that is it satisfies a non-trivial generalized polynomial identity.

Let $u \in I$. If a and au are linearly C-dependent, then $(a - \gamma)u = 0$ for a suitable $\gamma \in C$. Since $a - \gamma$ and a induce the same inner derivation d, we have that

$$0 = [d([ux, uy]), [ux, uy]][ux, uy] - [ux, uy][g([ux, uy]), [ux, uy]] =$$

$$-[ux, uy][b, [ux, uy]]_2 \qquad \textbf{(1)}$$

for all $x, y \in R$. If b and bu are linearly C-independent, the (1) is a non trivial GPI for R. In the either case, there exists $\gamma_1 \in C$ such that $bu = b\gamma_1$ and the equation (1) becomes

$$-\gamma_1[ux, uy]u[ux, uy]^2 - [ux, uy]^3 b + 2\gamma_1[ux, uy]^2 u[ux, uy]$$

which is again a non trivial GPI for R. A parallel proof shows that R is a GPI-ring also when a and au are linearly C-independent.

Since R is GPI, by [10] RC is primitive with non-zero socle H. It follows from [3] that $[a,[x,y]]_2[x,y]-[x,y][b,[x,y]]_2$ is a generalized polynomial identity for IH. Let $r_1, r_2, r_3, r_4, r_5, w \in I$ such that $s_4(r_1, r_2, r_3, r_4)r_5 \neq 0$. The first aim is here to prove that $(a-\alpha)I = 0$ for a suitable $\alpha \in C$.

If, for any $x \in I$ there exists $\alpha_x \in C$ such that $ax = \alpha_x x$, then standard well known arguments show that $ax = \alpha x$ for all x, where α is not depending on the choice of x, so we are done.

Therefore suppose that there exists $w \in I$ such that $aw \neq \gamma)w$ for all $\gamma \in C$.

Since RC is a regular ring, there exists $e^2 = e \in IH$ such that $eRC = \sum_{i=1}^{n+2} r_i RC + wRC$ and $r_i = er_i$ for $i = 1, .., 5$, $w = ew$.

Since $[a,[ex,ey]]_2[ex,ey] - [ex,ey][b,[ex,ey]]_2$ is satisfied by RC, left multiplying by $(1-e)$, we get that RC satisfies $(1-e)a[ex,ey]^3$. By [4] it follows that either $(1-e)ae = 0$ or $[ex,ey]e$ is a generalized identity for RC. On the other hand this last case cannot occur, since $0 \neq s_4(er_1, er_2, er_3, er_4)er_5 = s_4(r_1, r_2, r_3, r_4)r_5$, thus $(1-e)ae = 0$, that is $ae = eae$. Therefore $[a,[x,y]]_2[x,y]-[x,y][b,[x,y]]_2$ is satisfied by $eRCe$.

By theorem 3, since $0 \neq s_4(er_1, er_2, er_3, er_4)er_5 = s_4(r_1, r_2, r_3, r_4)r_5$, we get $ae, ebe \in Ce$. In particular $ae = \alpha e$ for some $\alpha \in C$. But this drives to the contradiction $aw = aew = \alpha w \neq aw$.

Hence we have that $(a-\alpha)I = 0$, for a suitable $\alpha \in C$. Since a and $a-\alpha$ induce the same inner derivation d, we have that $eRCe$ satisfies

$$[a,[x,y]]_2[x,y] - [x,y][b,[x,y]]_2 = [a-\alpha,[x,y]]_2[x,y] - [x,y][b,[x,y]]_2 =$$

$$-[x,y][b,[x,y]]_2.$$

Since $eRCe$ does not satisfy s_4, the previous corollary says that $ebe \in Ce$, that is $e(b-\beta)e = 0$ for a suitable $\beta \in C$.

If $[b, I]I = 0$, then $0 = [b,e]e = [b-\beta, e]e = (b-\beta)e$. Since b and $b-\beta$ induce the same inner derivation g, it follows that RC satisfies

$$[ex,ey][b-\beta,[ex,ey]]_2 = [ex,ey]^3(b-\beta).$$

Again by [4] either $[ex,ey]e = 0$ or $b = \beta$. Since the first conclusion contradicts with $s_4(er_1, er_2, er_3, er_4)er_5 \neq 0$, then $b \in C$ and we are done.

Finally consider the case when there exist $u, v \in I$ such that $[b,u]v \neq 0$, in particular we may assume $u, v \in eRC$. By Litoff's theorem [5] there exists an idempotent $f \in H \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C, such that

$$e, be, eb, u, v, bu, ub, bv, vb \in f(H \bigotimes_C \overline{C})f \cong M_m(\overline{C}) \qquad m > 2.$$

For $x, y \in ef(H \bigotimes_C \overline{C})f$, by assumption we have

$$0 = f[x, y][b, [x, y]]_2 f = [x, y][fbf, [x, y]]_2.$$

By our lemma, for $m > 2$, $[fbf, ef(H \bigotimes_C \overline{C})f]ef(H \bigotimes_C \overline{C})f = 0$, but $0 \neq [b, u]v = [fbf, efuf]efvf \in [fbf, ef(H \bigotimes_C \overline{C})f]ef(H \bigotimes_C \overline{C})f = 0$. This gives a contradiction and the theorem is finished. $\square$

Now we premit a simple result which will be useful in the proof of main theorem:

Theorem 6. *Let R be a prime ring of characteristic different from 2. Define the following polynomials on R:*

$$f_1 = [[x_1, x_2], [x_3, x_2]][x_3, x_2] - \alpha[x_3, x_2][[x_1, x_2], [x_3, x_2]] \quad -1 \neq \alpha \in Z(R);$$

$$f_2 = [[x_1, x_2], [x_3, x_2]][x_3, x_2];$$

$$f_3 = [x_1, x_2][x_3, x_2]^2 - [x_3, x_2]^2[x_1, x_2].$$

If R satisfies f_1 or f_2 then it is commutative. If R satisfies f_3 then it satisfies the standard identity s_4.

Proof Since R is a ring satisfying a polynomial identity, it is well known that there exists a field F such that $R \subseteq M_k(F)$, moreover $M_k(F)$ satisfies the same identities of R. If $k = 1$ there is nothing to prove. Let $k \geq 2$.

Fix $x_1 = e_{12}, x_2 = e_{11} - e_{22}, x_3 = e_{21}$, then we get the contradictions

$$f_1 = 8e_{21} + 8\alpha e_{21} \neq 0 \qquad f_2 = 8e_{21} \neq 0.$$

On the other hand, for $k \geq 3$, let $x_1 = e_{23} - e_{22}, x_2 = e_{12}, x_3 = e_{21}$, and again we have the contradiction

$$f_3 = e_{13} \neq 0.$$

$\square$

Before beginnig the proof of the main theorem, for the sake of completeness, we prefer to recall some basic notations, definitions and some easy consequences of the result of Kharchenko [8] about the differential identities on a prime ring R. We refer to [1, Chapter 7] for a complete and detaleid description of the theory of generalized polynomial identities involving derivations.

It is well known that any derivation of a prime ring R can be uniquely extended to a derivation of its Martindale quotients ring Q, and so any derivation of R can be defined on the whole Q [1, pg. 87].

Now, we denote by $\mathrm{Der}(Q)$ the set of all derivations on Q. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 d_2 \dots d_m$, with each $d_i \in \mathrm{Der}(Q)$. Then a differential polynomial is a generalized polynomial, with coefficents in Q, of the form $\Phi({}^{\Delta_j}x_i)$ involving noncommutative indeterminates x_i on which the derivations words Δ_j act as unary operations. The differential polynomial $\Phi({}^{\Delta_j}x_i)$ is said a differential identity on a subset T of Q if it vanishes for any assignment of values from T to its indeterminates x_i.

Let D_{int} be the C-subspace of $\mathrm{Der}(Q)$ consisting of all inner derivations on Q and let d and g be two non-zero derivations on R. By [8, Theorem 2] we have the following result (see also [9, Theorem 1]):

Remark 7. Let R be a prime ring of characteristic different from 2, if d and g are $C-$linearly indipendent modulo D_{int} and $\Phi({}^{\Delta_j}x_i)$ is a differential identity on R, where Δ_j are derivations words of the following form g, d, then $\Phi(y_{ji})$ is a generalized polynomial identity on R, where y_{ji} are distinct indeterminates.

As a particular case, we have:

Remark 8. If d is a non-zero derivation on R and $\Phi(x_1, .., x_n, {}^dx_1, .., {}^dx_n)$ is a differential identity on R, then one of the following holds:
(i) either $d \in D_{\mathrm{int}}$
(ii) or R satisfies the generalized polynomial identity

$$\Phi(x_1, .., x_n, y_1, .., y_n).$$

The following two results will help us in the proof of main theorem:

Theorem 9. *Let R be a prime ring of characteristic different from 2, d and g derivations of R, I a non-zero right ideal of R such that $[d([x,y]),[x,y]][x,y] - [x,y][g([x,y]),[x,y]] = 0$, for all $x, y \in I$. If d ang g are linearly C-independent modulo D_{int} then $[x_1,x_2]x_3$ is an identity for I.*

Proof Let $u \in I$, then R satisfies the following

$$[d([ux,uy]),[ux,uy]][ux,uy] - [ux,uy][g([ux,uy]),[ux,uy]] =$$

$$[[d(u)x + ud(x), uy] + [ux, d(u)y + ud(y)], [ux,uy]][ux,uy]-$$

$$[ux,uy][[g(u)x + ug(x), uy] + [ux, g(u)y + ug(y)], [ux,uy]].$$

Since d and g are linearly C-independent modulo D_{int}, it follows that R satisfies

$$[[d(u)x_1 + ux_2, ux_3] + [ux_1, d(u)x_3 + ux_4], [ux_1,ux_3]][ux_1,ux_3]-$$

$$[ux_1,ux_3][[g(u)x_1 + ux_5, ux_3] + [ux_1, g(u)x_3 + ux_6], [ux_1,ux_3]]$$

and in particular it satisfies the blended component

$$[[ux_2,ux_3],[ux_1,ux_3]][ux_1,ux_3].$$

Hence R is a GPI-ring and, by [10], RC is primitive with non-zero socle H. It follows from [3] that

$$[[x_2, x_3], [x_1, x_3]][x_1, x_3]$$

is a generalized polynomial identity for IH. Suppose by contradiction that there exist $r_1, r_2, r_3 \in I$ such that $[r_1, r_2]r_3 \neq 0$.

Since RC is a regular ring, there exists $e^2 = e \in IH$ such that $eRC = \sum_{i=1}^{3} r_i RC$ and $r_i = er_i$ for $i = 1, 2, 3$.

Since $eRCe$ satisfies $[[x_2, x_3], [x_1, x_3]][x_1, x_3]$, by theorem 6 $eRCe$ is commutative. This contradicts with $[er_1, er_2]er_3 = [r_1, r_2]r_3 \neq 0$. □

Theorem 10. *Let R be a prime ring of characteristic different from 2, d and g non-zero derivations of R, I a non-zero right ideal of R such that $[d([x, y]), [x, y]][x, y] - [x, y][g([x, y]), [x, y]] = 0$, for all $x, y \in I$. If $g = \alpha d$, for some $\alpha \in C$, then $s_4(x_1, x_2, x_3, x_4)x_5$ is an identity for I.*

Proof In this case I satisfies

$$[d([x, y]), [x, y]][x, y] - \alpha[x, y][d([x, y]), [x, y]].$$

If d and g are both inner derivations, then we end up by theorem 5.

Let both d and $g = \alpha d$ be outer derivations. For $u \in I$, R satisfies the following

$$[d([ux, uy]), [ux, uy]][ux, uy] - \alpha[ux, uy][d([ux, uy]), [ux, uy]] =$$

$$[[d(u)x + ud(x), uy] + [ux, d(u)y + ud(y)], [ux, uy]][ux, uy]-$$

$$\alpha[ux, uy][[d(u)x + ud(x), uy] + [ux, d(u)y + ud(y)], [ux, uy]].$$

By Kharchenko's result in [8], as in theorem 9, R satisfies

$$[[d(u)x_1 + ux_2, ux_3] + [ux_1, d(u)x_3 + ux_4], [ux_1, ux_3]][ux_1, ux_3]-$$

$$\alpha[ux_1, ux_3][[d(u)x_1 + ux_2, ux_3] + [ux_1, d(u)x_3 + ux_4], [ux_1, ux_3]]$$

and in particular it satisifes the blended component

$$[[ux_2, ux_3], [ux_1, ux_3]][ux_1, ux_3] - \alpha[ux_1, ux_3][[ux_2, ux_3], [ux_1, ux_3]] \qquad (\mathbf{2}).$$

Hence R is a GPI-ring and RC is primitive with non-zero socle H. Suppose that there exist $r_1, r_2, r_3, r_4, r_5 \in I$ such that $s_4(r_1, r_2, r_3, r_4)r_5 \neq 0$. So there exists $e^2 = e \in IH$ such that $eRC = \sum_{i=1}^{5} r_i RC$ and $r_i = er_i$ for $i = 1, .., 5$. Moreover

$$[[x_2, x_3], [x_1, x_3]][x_1, x_3] - \alpha[x_1, x_3][[x_2, x_3], [x_1, x_3]]$$

is polynomial identity for $eRCe$.

By theorem 6, since $s_4(eRCe, eRCe, eRCe, eRCe) \neq 0$, we have $\alpha = -1$, that is $g = -d$ and the equation (2) must be rewrite as follows:

$$[ux_2, ux_3][ux_1, ux_3]^2 - [ux_1, ux_3]^2[ux_2, ux_3] = 0$$

for all $u \in I$ and $x_1, x_2, x_3 \in RC$. In particular

$$[ex_2, ex_3][ex_1, ex_3]^2 - [ex_1, ex_3]^2[ex_2, ex_3] = 0.$$

Again by theorem 6, $eRCe$ satisifes $s_4(x_1, x_2, x_3, x_4)x_5$, a contradiction. $\square$

In light of previous results we may finally prove the following:

Theorem 11. *Let R be a prime ring of characteristic different from 2, with extended centroid C, d and g derivations of R, I a non-zero right ideal of R and s_4 the standard identity of degree 4. If $[d([x,y]), [x,y]][x,y] - [x,y][g([x,y]), [x,y]]$, for all $x, y \in I$, then one of the following holds:*
(i) $s_4(x_1, x_2, x_3, x_4)x_5$ is an identity for I;
(ii) $d(x) = [a, x]$, with $(a - \alpha)I = 0$ for a suitable $\alpha \in C$ and $g = 0$.

Proof Thanks to theorems 9 and 10, we may consider the only case when d and g are linearly C-dependent modulo D_{int}, that is there exist $\alpha \in C$ and $b \in Q$ such that $g = \alpha d + ad(b)$, where $ad(b)(x) = [b, x]$, for all $x \in R$. Moreover $b \notin C$, since $g \notin Cd$.

If d is an inner derivation, then so is also g and we end up by theorem 5. Therefore let d be an outer derivation. We prove that if $s_4(I, I, I, I)I \neq 0$ then a contradiction occurs.

For all $u \in I$, R satisifes

$$[d([ux, uy]), [ux, uy]][ux, uy] - [ux, uy][\alpha d([ux, uy]) + [b, [ux, uy]], [ux, uy]] =$$

$$[[d(u)x + ud(x), uy] + [ux, d(u)y + ud(y)], [ux, uy]]\, [ux, uy] -$$

$$[ux, uy]\, [[\alpha d(u)x + [b, u]x + u\alpha d(x) + u[b, x], uy] +$$

$$[ux, \alpha d(u)y + [b, u]y + u\alpha d(y) + u[b, y]], [ux, uy]].$$

Since d is outer, by Kharchenko's result in [8], R satisfies

$$[[d(u)x_1 + ux_2, ux_3] + [ux_1, d(u)x_3 + ux_4], [ux_1, ux_3]]\, [ux_1, ux_3] -$$

$$[ux_1, ux_3]\, [[\alpha d(u)x_1 + [b, u]x_1 + u\alpha x_2 + u[b, x_1], ux_3] +$$

$$[ux_1, \alpha d(u)x_3 + [b, u]x_3 + u\alpha x_4 + u[b, x_3]], [ux_1, ux_3]].$$

In particular R satisifes the blended component

$$[[ux_2, ux_3], [ux_1, ux_3]][ux_1, ux_3] - \alpha[ux_1, ux_3][[ux_2, ux_3], [ux_1, ux_3]].$$

Hence R is a GPI-ring and RC is primitive with non-zero socle H. Suppose that there exist $r_1, r_2, r_3, r_4, r_5 \in I$ such that $s_4(r_1, r_2, r_3, r_4)r_5 \neq 0$. So there exists $e^2 = e \in IH$ such that $eRC = \sum_{i=1}^{5} r_i RC$ and $r_i = er_i$ for $i = 1, .., 5$. Moreover

$$[[x_2, x_3], [x_1, x_3]][x_1, x_3] - \alpha[x_1, x_3][[x_2, x_3], [x_1, x_3]]$$

is polynomial identity for $eRCe$.

By theorem 6, since $s_4(eRCe, eRCe, eRCe, eRCe) \neq 0$, we have $\alpha = -1$, that is $g = -d + ad(b)$. Then R satisifes

$$[d([ex, ey]), [ex, ey]][ex, ey] + [ex, ey][d([ex, ey]) + [b, [ex, ey]], [ex, ey]] =$$

$$([d(e)x + ed(x), ey] + [ex, d(e)y + ed(y)])\,[ex, ey]^2-$$

$$[ex, ey]^2\,([d(e)x + ed(x), ey] + [ex, d(e)y + ed(y)]) +$$

$$[ex, ey][b, [ex, ey]]_2.$$

Again it follows that R satisfies

$$([d(e)x_1 + ex_2, ex_3] + [ex_1, d(e)x_3 + ex_4])\,[ex_1, ex_3]^2-$$

$$[ex_1, ex_3]^2\,([d(e)x_1 + ex_2, ex_3] + [ex_1, d(e)x_3 + ex_4]) +$$

$$[ex_1, ex_3][b, [ex_1, ex_3]]_2$$

and also the blended component

$$[ex_1, ex_3]^2[ex_2, ex_3] - [ex_2, x_3][ex_1, ex_3]^2.$$

By theorem 6, since $s_4(eRCe, eRCe, eRCe, eRCe) \neq 0$, we get a contradiction. $\square$

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