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A VERSION OF THE DE RHAM LEMMA

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Abstract

In this paper, we consider a version of the de Rham lemma that clarifies the additional information on the trace of the potential. From this we can improve the regularity of solutions of the Stokes system.

1 Introduction

The classical de Rham lemma says that a continuous and linear functional that vanishes on all divergence-free H^1 vector fields that equal zero on the boundary can be represented as a gradient of an L^2 potential function inside the domain. For example, see Boyer and Fabrie [2, Theorem IV. 2.4]. However, the lemma does not provide any information on the trace of the potential function on the boundary.

For example, it is insufficient to show the regularity of solutions for the Stokes problem. Pan [4] considered a version of the de Rham lemma. This asserts that a continuous and linear functional that vanishes on all divergence-free H^1 vector fields that have zero tangential component on the boundary is a gradient of the function $p \in L^2(\Omega)$, and that p has zero trace on the boundary. This additional information on the trace of p makes it possible to improve the regularity of the solution of a Maxwell-Stokes type system.

In this paper, we consider the L^p version of the result obtained by [4]. Though we consult this paper [4], we must treat the arguments more carefully. Our result is useful for the regularity of the Maxwell-Stokes problem in the L^p version, which will appear in a future work .

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The paper is organized as follows. In section 2, we give some preliminaries on the trace and the gradient of functions. In section 3, we give a main theorem and its proof.

2 Preliminaries

In this section, we shall state some preliminaries that are necessary to state a version of the de Rham lemma (Theorem 3.1). Let Ω be a bounded domain in $\mathbb{R}^3$ with a C^2 boundary Γ, let $1 < p < \infty$ and let p' be the conjugate exponent i.e., $(1/p) + (1/p') = 1$. From now on we use $C^1(\overline{\Omega})$, $L^p(\Omega)$, $W^{m,p}(\Omega)$ ($m \geq 0$, integer), $W^{s,p}(\Gamma)$ ($s \in \mathbb{R}$), and so on, for the standard C^1, Sobolev spaces of functions. For any Banach space B, we denote $B \times B \times B$ by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vector and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. Moreover, for the dual space $\boldsymbol{B}'$, we denote the duality between $\boldsymbol{B}'$ and $\boldsymbol{B}$ by $\langle \cdot, \cdot \rangle_{\boldsymbol{B}', \boldsymbol{B}}$. If $Y(\Omega)$ is a space of functions on Ω, we denote

$$\dot{Y}(\Omega) = \{\phi \in Y(\Omega); \int_\Omega \phi dx = 0\}.$$

Define a space

$$\boldsymbol{W}^{1,p}_{t0}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega); \boldsymbol{u}_T = \boldsymbol{0} \text{ on } \Gamma\},$$

where $\boldsymbol{u}_T$ denotes the tangential component of $\boldsymbol{u}$, namely, $\boldsymbol{u}_T = (\boldsymbol{n} \times \boldsymbol{u}) \times \boldsymbol{n}$, $\boldsymbol{n}$ is the outer unit normal vector to the boundary, and we denote its dual space by $\boldsymbol{W}^{1,p}_{t0}(\Omega)'$. Moreover, we define

$$\boldsymbol{W}^{1,p}_{t0}(\Omega, \operatorname{div} 0) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{t0}(\Omega); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}.$$

We define the norm on $\dot{C}^1(\overline{\Omega})$ by

$$\|\phi\|_{p', -1/p'} = \|\phi\|_{L^{p'}(\Omega)} + \|\gamma_0(\phi)\|_{W^{-1/p', p'}(\Gamma)}, \tag{2.1}$$

where γ_0 is the restriction operator to the boundary, and the completion of $\dot{C}^1(\overline{\Omega})$ with respect to this norm by $\dot{L}^{p', -1/p'}(\Omega)$ and define

$$L^{p', -1/p'}(\Omega) = \dot{L}^{p', -1/p'}(\Omega) \oplus \mathbb{R}.$$

Beforehand, we state the celebrated Cattabriga theorem [3] (cf. Amrouche and Girault [1, Theorem 4.1]) associated with the Stokes problem which will be frequently used later. We consider the Stokes problem: for given $\boldsymbol{f}, \varphi, \boldsymbol{g}$, find $(\boldsymbol{u}, \pi)$ such that

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \varphi & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma. \end{cases} \tag{2.2}$$

The compatibility condition is

$$\int_\Omega \varphi dx = \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} dS, \tag{2.3}$$

where dS denotes the surface area of Γ. Then the following holds.

Proposition 2.1. *Let $m > 0$ integer, $1 < p < \infty$, and let Ω be a bounded domain in $\mathbb{R}^3$ with a boundary Γ that is C^m class if $m \geq 2$ and C^2 class if $m = 1$. Assume that $\boldsymbol{f} \in \boldsymbol{W}^{m-2,p}(\Omega)$, $\varphi \in W^{m-1,p}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{m-1/p,p}(\Gamma)$ satisfy the compatibility condition* (2.3)*. Then the Stokes problem* (2.2) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m,p}(\Omega) \times W^{m-1.p}(\Omega)/\mathbb{R}$, and there exists a constant $C > 0$ depending only on m, r and Ω such that*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{m,p}(\Omega)} + \|\pi\|_{W^{m-1.p}(\Omega)/\mathbb{R}} \\ &\qquad \leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}^{m-2,p}(\Omega)} + \|\varphi\|_{W^{m-1.p}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{m-1/p,p}(\Gamma)}) \end{aligned} \tag{2.4}$$

We give a proposition associated with the trace and the gradient.

Proposition 2.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a C^2 boundary Γ. Then the following holds.*

(i) *There exists a trace map $\gamma : L^{p',-1/p'}(\Omega) \to W^{-1/p',p'}(\Gamma)$ such that if $\phi \in \dot{C}^1(\overline{\Omega})$, then $\gamma(\phi) = \gamma_0(\phi)$.*

(ii) *Given $\phi \in L^{p',-1/p'}(\Omega)$, we define a bounded linear functional on $\boldsymbol{W}^{1,p}_{t0}(\Omega)$, which is denoted by $\nabla\phi$, for all $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_{t0}(\Omega)$,*

$$\begin{aligned} &\langle \nabla\phi, \boldsymbol{w}\rangle_{\boldsymbol{W}^{1,p}_{t0}(\Omega)', \boldsymbol{W}^{1,p}_{t0}(\Omega)} \\ &\qquad = -\int_\Omega \phi \operatorname{div} \boldsymbol{w} dx + \langle \gamma(\phi), \boldsymbol{n} \cdot \boldsymbol{w}\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)}. \end{aligned}$$

If we write

$$\operatorname{grad} L^{p',-1/p'}(\Omega) = \{\nabla\phi \in \boldsymbol{W}^{1,p}_{t0}(\Omega)'; \phi \in L^{p',-1/p'}(\Omega)\},$$

then $\operatorname{grad} L^{p',-1/p'}(\Omega)$ *is a closed subspace of $\boldsymbol{W}^{1,p}_{t0}(\Omega)'$, and it is homeomorphic to $\dot{L}^{p',-1/p'}(\Omega)$.*

Proof. (i) Let $\phi_0 \in \dot{L}^{p',-1/p'}(\Omega)$. Then there exists $\{\phi_j\} \subset \dot{C}^1(\overline{\Omega})$ such that $\|\phi_j - \phi_0\|_{p',-1/p'} \to 0$ as $j \to \infty$. Hence $\phi_j \to \phi_0$ in $L^{p'}(\Omega)$ and $\{\gamma_0(\phi_j)\}$ is a Cauchy sequence in $W^{-1/p',p'}(\Gamma)$. Therefore there exists $\gamma_0 \in W^{-1/p',p'}(\Gamma)$ such that $\gamma_0(\phi_j) \to \gamma_0$ in $W^{-1/p',p'}(\Gamma)$. Here we show that γ_0 is determined independently of the choice of $\{\phi_j\} \subset \dot{C}^1(\overline{\Omega})$ such that $\|\phi_j - \phi_0\|_{p',-1/p'} \to 0$ as $j \to \infty$. In fact, let $\{\phi'_j\} \subset \dot{C}^1(\overline{\Omega})$ be an another sequence such that

$\|\phi'_j - \phi_0\|_{p',-1/p'} \to 0$ as $j \to \infty$. Then $\phi'_j \to \phi_0$ in $L^{p'}(\Omega)$ and there exists a $\gamma'_0 \in W^{-1/p',p'}(\Gamma)$ such that $\gamma_0(\phi'_j) \to \gamma'_0$ in $W^{-1/p',p'}(\Gamma)$. Hence

$$\begin{aligned}\|\gamma_0 - \gamma'_0\|_{W^{-1/p',p'}(\Gamma)} &\le \|\gamma_0 - \gamma_0(\phi_j)\|_{W^{-1/p',p'}(\Gamma)} \\ &\quad + \|\gamma_0(\phi_j) - \gamma_0(\phi'_j)\|_{W^{-1/p',p'}(\Gamma)} + \|\gamma_0(\phi'_j) - \gamma'_0\|_{W^{-1/p',p'}(\Gamma)}\end{aligned}$$

Here

$$\begin{aligned}\|\gamma_0(\phi_j) - \gamma_0(\phi'_j)\|_{W^{-1/p',p'}(\Gamma)} &\le C(p,\Omega)\|\phi_j - \phi'_j\|_{p',-1/p'} \\ &\le C(p,\Omega)(\|\phi_j - \phi_0\|_{p',-1/p'} + \|\phi_0 - \phi'_j\|_{p',-1/p'}) \to 0.\end{aligned}$$

Thus we see that $\gamma_0 = \gamma'_0$ in $W^{-1/p',p'}(\Omega)$.

Define a linear map $\gamma : \dot{L}^{p',-1/p'}(\Omega) \to W^{-1/p',p'}(\Omega)$ by $\gamma(\phi_0) = \gamma_0$ for $\phi_0 \in \dot{L}^{p',-1/p'}(\Omega)$. Then we have

$$\begin{aligned}\|\gamma(\phi_0)\|_{W^{-1/p',p'}(\Gamma)} &= \lim_{j\to\infty} \|\gamma_0(\phi_j)\|_{W^{-1/p',p'}(\Gamma)} \\ &\le \lim_{j\to\infty} \|\phi_j\|_{p',-1/p'} \\ &= \|\phi_0\|_{p',-1/p'}.\end{aligned}$$

If $\phi \in \dot{C}(\overline{\Omega})$, taking $\phi_j = \phi$, it is easily seen that $\gamma(\phi) = \gamma_0(\phi)$.

(ii) For $\phi \in L^{p'-1/p'}(\Omega)$, define a functional T_ϕ on $\boldsymbol{W}^{1,p}_{t0}(\Omega)$ by

$$T_\phi[\boldsymbol{w}] = -\int_\Omega \phi \operatorname{div} \boldsymbol{w} dx + \langle \gamma(\phi), \boldsymbol{n}\cdot\boldsymbol{w}\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)},$$

for $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_{t0}(\Omega)$. We note that if $\phi = c = \text{const.}$, it follows from the divergence theorem that $T_c = 0$. Let $\phi \in \dot{L}^{p',-1/p'}(\Omega)$. Since

$$\begin{aligned}|T_\phi[\boldsymbol{w}]| &\le \|\phi\|_{L^{p'}(\Omega)}\|\operatorname{div}\boldsymbol{w}\|_{L^p(\Omega)} + \|\gamma(\phi)\|_{W^{-1/p',p'}(\Gamma)}\|\boldsymbol{w}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} \\ &\le C\|\phi\|_{p',-1/p'}\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)},\end{aligned}$$

we see that T_ϕ is a bounded linear functional on $\boldsymbol{W}^{1,p}_{t0}(\Omega)$, so $T_\phi \in \boldsymbol{W}^{1,p}_{t0}(\Omega)'$ and

$$\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'} \le C\|\phi\|_{p',-1/p'}. \tag{2.5}$$

For any $\eta \in W^{1-1/p,p}(\Gamma)$, define $h_\eta = \frac{1}{|\Omega|}\int_\Gamma \eta dS = \text{const.}$. If we apply Proposition 2.1 with $m = 1, \boldsymbol{f} = \boldsymbol{0}, \varphi = h_\eta$ and $\boldsymbol{g} = \eta\boldsymbol{n}$, then the compatibility condition trivially holds. Hence the problem

$$\begin{cases} -\Delta\boldsymbol{u}_\eta + \nabla p_\eta = \boldsymbol{0},\ \operatorname{div}\boldsymbol{u}_\eta = h_\eta & \text{in } \Omega, \\ \boldsymbol{u}_\eta = \eta\boldsymbol{n} & \text{on } \Gamma \end{cases}$$

has a unique solution $(\boldsymbol{u}_\eta, p_\eta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$, and there exists a constant $C = C(p, \Omega) > 0$ such that

$$\|\boldsymbol{u}_\eta\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|p_\eta\|_{L^p(\Omega)/\mathbb{R}} \le C(\|h_\eta\|_{L^p(\Omega)} + \|\eta\|_{W^{1-1/p,p}(\Gamma)}).$$

Since it follows from the Hölder inequality that

$$\|h_\eta\|_{L^p(\Omega)} \le C(\Omega)\|\eta\|_{W^{1-1/p,p}(\Gamma)},$$

we can see that

$$\|\boldsymbol{u}_\eta\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C(p, \Omega)\|\eta\|_{W^{1-1/p,p}(\Gamma)}. \tag{2.6}$$

Since clearly $\boldsymbol{u}_{\eta,T} = \boldsymbol{0}$ on Γ, we see that $\boldsymbol{u}_\eta \in \boldsymbol{W}^{1,p}_{t0}(\Omega)$. Moreover, since

$$\int_\Omega \phi \operatorname{div} \boldsymbol{u}_\eta dx = h_\eta \int_\Omega \phi dx = 0,$$

it follows from (2.6) that

$$\begin{aligned} \langle \gamma(\phi), \eta\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} &= \langle \gamma(\phi), \boldsymbol{n}\cdot\boldsymbol{u}_\eta\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \\ &= T_\phi[\boldsymbol{u}_\eta] \\ &\le \|T_\phi\|_{\boldsymbol{W}^{1,p}(\Omega)'}\|\boldsymbol{u}_\eta\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)} \\ &\le C(p,\Omega)\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}\|\eta\|_{W^{1-1/p,p}(\Gamma)}. \end{aligned}$$

Since $\eta \in W^{1-1/p,p}(\Gamma)$ is arbitrary, we have

$$\|\gamma(\phi)\|_{W^{-1/p',p'}(\Gamma)} \le C(p, \Omega)\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}. \tag{2.7}$$

On the other hand, let $\phi \in \dot{L}^{p',-1/p'}(\Omega)$. For any $\varphi \in \dot{L}^p(\Omega)$, we consider the following Stokes problem: to find $(\boldsymbol{v}_\varphi, q_\varphi)$ such that

$$\begin{cases} -\Delta \boldsymbol{v}_\varphi + \nabla q_\varphi = \boldsymbol{0}, \ \operatorname{div} \boldsymbol{v}_\varphi = \varphi & \text{in } \Omega, \\ \boldsymbol{v}_\varphi = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{2.8}$$

The compatibility condition (2.3) clearly holds. If we apply Proposition 2.1 with $m = 1, \boldsymbol{f} = \boldsymbol{0}, \varphi \in L^p(\Omega)$ and $\boldsymbol{g} = \boldsymbol{0}$, the equation (2.8) has a unique solution $(\boldsymbol{v}_\varphi, q_\varphi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$, and there exists a constant $C = C(p, \Omega) > 0$ such that

$$\|\boldsymbol{v}_\varphi\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|q_\varphi\|_{L^p(\Omega)/\mathbb{R}} \le C(p, \Omega)\|\varphi\|_{L^p(\Omega)}. \tag{2.9}$$

From (2.8), since $\boldsymbol{v}_\varphi = \boldsymbol{0}$ on Γ, we have

$$\begin{aligned} \left|\int_\Omega \phi\,\varphi dx\right| &= \left|\int_\Omega \phi \operatorname{div} \boldsymbol{v}_\varphi dx\right| = |T_\phi[\boldsymbol{v}_\varphi]| \\ &\le \|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}\|\boldsymbol{v}_\varphi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)} \le C(p,\Omega)\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}\|\varphi\|_{L^p(\Omega)}. \end{aligned} \tag{2.10}$$

For any $\varphi \in L^p(\Omega)$, $\varphi - c_\varphi \in \dot{L}^p(\Omega)$, where $c_\varphi = \frac{1}{|\Omega|}\int_\Omega \varphi dx$. Since $\phi \in \dot{L}^{p',-1/p'}(\Omega)$, we have

$$\int_\Omega \phi\varphi dx = \int_\Omega \phi(\varphi - c_\varphi)dx + c_\varphi \int_\Omega \phi dx = \int_\Omega \phi(\varphi - c_\varphi)dx.$$

Hence from (2.10), we have

$$\left|\int_\Omega \phi\varphi dx\right| \le C(p,\Omega)\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}\|\varphi - c_\varphi\|_{L^p(\Omega)}.$$

Since

$$\|c_\varphi\|_{L^p(\Omega)} = |c_\varphi||\Omega|^{1/p} \le \|\varphi\|_{L^p(\Omega)},$$

we have

$$\left|\int_\Omega \phi\varphi dx\right| \le 2C(p,\Omega)\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}\|\varphi\|_{L^p(\Omega)}$$

for all $\varphi \in L^p(\Omega)$. Thus we have

$$\|\phi\|_{L^{p'}(\Omega)} \le 2C(p,\Omega)\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}. \tag{2.11}$$

Summing up (2.7) and (2.11), we have

$$\|\phi\|_{L^{p',-1/p'}(\Omega)} \le C(p,\Omega)\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}. \tag{2.12}$$

From (2.5) and (2.12), there exist positive constants C_1 and C_2 such that

$$C_1\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'} \le \|\phi\|_{L^{p',-1/p'}(\Omega)} \le C_2\|T_\phi\|_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'}.$$

If we put $Y = \{T_\phi; \phi \in L^{p',-1/p'}(\Omega)\}$, then Y is a linear subspace of $\boldsymbol{W}^{1,p}_{t0}(\Omega)'$ and it is an isomorphism onto $L^{p',-1/p'}(\Omega)$. Since $L^{p',-1/p'}(\Omega)$ is complete, we see that Y is a closed subspace of $\boldsymbol{W}^{1,p}_{t0}(\Omega)'$. If we write T_ϕ by $\nabla\phi$, the conclusion of (ii) holds. □

3 The main theorem and the proof

In this section we give a L^p version of the de Rham lemma and its proof.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a C^2 boundary Γ. Assume that $T \in \boldsymbol{W}^{1,p}_{t0}(\Omega)'$ satisfies*

$$\langle T, \boldsymbol{w}\rangle_{\boldsymbol{W}^{1,p}_{t0}(\Omega)',\boldsymbol{W}^{1,p}_{t0}(\Omega)} = 0 \textit{ for all } \boldsymbol{w} \in \boldsymbol{W}^{1,p}_{t0}(\Omega, \operatorname{div} 0). \tag{3.1}$$

Then there exists $p \in L^{p',-1/p'}(\Omega)$ with $\gamma(p) = 0$ on Γ such that $T = \nabla p$ on $\boldsymbol{W}^{1,p}_{t0}(\Omega)'$.

Proof. Step 1. We first show that there exists $p \in \dot{L}^{p',-1/p'}(\Omega)$ such that $T = \nabla p$.

In general, let X be a Banach space with its dual X'. For $A \subset X$, we write

$$A^{\perp} = \{f \in X'; \langle f, x\rangle_{X',X} = 0 \text{ for all } x \in A\}.$$

Let $X = \boldsymbol{W}^{1,p}_{t0}(\Omega)$ and $Z = \boldsymbol{W}^{1,p}_{t0}(\Omega, \operatorname{div} 0)$. Then Z is a closed subspace of X and

$$Z^{\perp} = \{T \in \boldsymbol{W}^{1,p}_{t0}(\Omega)'; \langle T, \boldsymbol{w}\rangle_{\boldsymbol{W}^{1,p}_{t0}(\Omega)', \boldsymbol{W}^{1,p}_{t0}(\Omega)} = 0 \text{ for all } \boldsymbol{w} \in Z\}.$$

Hence it suffices to prove $Z^{\perp} \subset Y := \operatorname{grad} L^{p',-1/p'}(\Omega)$. Since X is reflexive, we can write

$$Y^{\perp} = \{\boldsymbol{w} \in \boldsymbol{W}^{1,p}_{t0}(\Omega); \langle T, \boldsymbol{w}\rangle_{\boldsymbol{W}^{1,p}_{t0}(\Omega)'. \boldsymbol{W}^{1,p}_{t0}(\Omega)} = 0 \text{ for all } T \in Y\}.$$

Therefore it suffices to prove $Y^{\perp} \subset Z$. Let $\boldsymbol{u} \in Y^{\perp}$. For any $\phi \in L^{p',-1/p'}(\Omega)$, we have $T_{\phi} = \nabla\phi \in Y$. Therefore, we have

$$\begin{aligned}
&\langle T_{\phi}, \boldsymbol{u}\rangle_{\boldsymbol{W}^{1,p}_{t0}(\Omega)', \boldsymbol{W}^{1,p}_{t0}(\Omega)} \\
&\qquad = -\int_{\Omega} \phi \operatorname{div} \boldsymbol{u} dx + \langle \gamma(\phi), \boldsymbol{n}\cdot\boldsymbol{u}\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} = 0
\end{aligned}$$

Taking $\phi \in W^{1,p}_0(\Omega)$ in the above equality, we have

$$-\int_{\Omega} \phi \operatorname{div} \boldsymbol{u} dx = 0 \text{ for all } \phi \in W^{1,p}_0(\Omega).$$

Hence $\operatorname{div} \boldsymbol{u} = 0$ in Ω, so $\boldsymbol{u} \in W^{1,p}_{t0}(\Omega, \operatorname{div} 0) = Z$.

Step 2. We show that $\gamma(p) = p_0$ is a constant. In fact, since $T = \nabla p$, for any $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_{t0}(\Omega, \operatorname{div} 0)$,

$$\begin{aligned}
0 &= \langle \nabla p, \boldsymbol{w}\rangle_{\boldsymbol{W}^{1,p}_{t0}(\Omega)', \boldsymbol{W}^{1,p}_{t0}(\Omega)} \\
&= -\int_{\Omega} p \operatorname{div} \boldsymbol{w} dx + \langle \gamma(p), \boldsymbol{n}\cdot\boldsymbol{w}\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \\
&= \langle \gamma(p), \boldsymbol{n}\cdot\boldsymbol{w}\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)}.
\end{aligned} \tag{3.2}$$

Define

$$p_0 = \frac{1}{|\Gamma|}\langle \gamma(p), 1\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)}.$$

For any $\eta \in W^{1-1/p,p}(\Gamma)$, define

$$\eta_0 = \frac{1}{|\Gamma|}\langle 1, \eta\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)}.$$

We consider the following Stokes problem: to find $(\boldsymbol{w}_\eta, q_\eta)$ such that

$$\begin{cases} -\Delta \boldsymbol{w}_\eta + \nabla q_\eta = \mathbf{0}, \ \operatorname{div} \boldsymbol{w}_\eta = 0 & \text{in } \Omega, \\ \boldsymbol{w}_\eta = (\eta - \eta_0)\boldsymbol{n} & \text{on } \Gamma. \end{cases}$$

Since $\int_\Gamma (\eta - \eta_0) dS = 0$, the compatibility condition (2.3) holds. Therefore if we apply Proposition 2.1 with $m = 1, \boldsymbol{f} = \mathbf{0}, \varphi = 0$ and $\boldsymbol{g} = (\eta - \eta_0)\boldsymbol{n}$, then there exists a unique solution $(\boldsymbol{w}_\eta, q_\eta) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Clearly $\boldsymbol{w}_\eta \in \boldsymbol{W}^{1,p}_{t0}(\Omega, \operatorname{div} 0)$. From (3.1), we see

$$\begin{aligned} 0 = \langle \gamma(p), \boldsymbol{n} \cdot \boldsymbol{w}_\eta \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \\ = \langle \gamma(p), \eta - \eta_0 \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)}. \end{aligned}$$

Since

$$\begin{aligned} &\langle \gamma(p), \eta \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \\ &\quad = \langle \gamma(p), \eta_0 \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \\ &\quad = \eta_0 \langle \gamma(p), 1 \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \\ &\quad = \frac{1}{|\Gamma|} \langle 1, \eta \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \langle \gamma(p), 1 \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \\ &\quad = \langle p_0, \eta \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)}, \end{aligned}$$

where

$$p_0 = \frac{1}{|\Gamma|} \langle \gamma(p), 1 \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} = \text{ const.},$$

we have

$$\langle \gamma(p) - p_0, \eta \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} = 0$$

for all $\eta \in W^{1-1/p,p}(\Gamma)$. Hence $\gamma(p) = p_0$ is a constant. Since $p - p_0 \in L^{p',-1/p'}(\Omega)$ and $\gamma(p - p_0) = 0$ on Γ, and $T = \nabla(p - p_0)$, the conclusion of this theorem follows.

$\square$

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