East-West J. of Mathematics: Vol. 6, No 2 (2004) pp.173-183

IDEAL CO-TRANSFORMS OF LINEARLY COMPACT MODULES

Tran Tuan Nam

Department of Mathematics - Informatics
University of Education of Ho Chi Minh city
280 An Duong Vuong, Ho Chi Minh City, Viet Nam.
namtrantuan@dng.vnn.vn

Abstract

We introduce the concept of ideal co-transforms which is in some sense dual to the concept of ideal transform of M. P. Brodmann. We also study some basic properties of ideal co-transforms of linearly compact modules.

1 Introduction

Throughout, R is a Noetherian commutative ring and has a topological structure. Let I be an ideal of R. It is well-known that in local cohomology theory of A. Grothendieck there is the concept of ideal transform $D_I(M)$ of an $R-$module M with respect to I defined by $D_I(M) = \varinjlim_t \operatorname{Hom}_R(I^t; M)$, which provides a powerful tool in commutative algebra (see [1]). Moreover, we defined in [2] the local homology modules $H_i^I(M)$ of an $R-$module M with respect to I by $H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(R/I^t; M)$. Some basic properties for this local homology modules were shown in [2] when M was Artinian, and in [3] when M was linearly compact. It should be noted that our definition of local homology modules is coincident with that of J. P. C. Greenless and J. P. May [6] in case of linearly compact $R-$modules and in some sense dual to the definition of local cohomology of A. Grothendieck [7]. Therefore, it is natural to define ideal co-transforms $C_i^I(M)$ of an $R-$module M by $C_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(I^t; M)$ $(i \geq 0)$.

Key words: linearly compact module, local homology, local cohomology, ideal transform, ideal co-transform.
2000 AMS Mathematics Subject Classification: 13D45, 16E30, 13J99.

The purpose of this paper is to study the ideal co-transforms $C_i^I(M)$ in case M is a linearly compact $R-$module.

The organization of the paper is as follows. In section 2 we recall briefly some properties of linearly compact modules and local homology modules that we shall use.

In section 3 we introduce the notion of ideal co-transforms. If M is a linearly compact module, then $C_i^I(M)$ $(i \geq 0)$ is also linearly compact and the sequence of functors $\{C_i^I\}$ is a positive strongly connected sequence on the category of linearly compact modules and continuous homomorphisms (Proposition 3.2). For all $i \geq 1$, we have an isomorphism $H_{i+1}^I(M) \cong C_i^I(M)$ and there is a short exact sequence

$$0 \longrightarrow H_1^I(M) \longrightarrow C^I(M) \xrightarrow{\eta_M} M \xrightarrow{\theta_M} \Lambda_I(M) \longrightarrow 0,$$

in which the homomorphisms $\eta_M,\ \theta_M$ are continuous (Theorem 3.3).

Section 4 is devoted to study the relationship between ideal co-transform and co-localization of linearly compact modules. Let S be a multiplicative set of R and ${}_SM = \mathrm{Hom}_R(R_S; M)$ the co-localization of an $R-$module M with respect to S. We have ${}_S(C_i^I(M)) \cong C_i^{IR_S}({}_SM)$ for all $i \geq 0$ (Proposition 4.1). Let $a \in R$, denote by ${}_aM$ the co-localization of M with respect to the multiplicative set $\{1, a, a^2 \ldots\}$. We will show that $C^{aR}(M) \cong {}_aM$ (Theorem 4.4).

2 Preliminaries

To establish the notion and context, we recall briefly definitions and basic properties of linearly compact modules and local homology modules that we shall use.

Let M be a topological $R-$module. M is said to be *linearly topologized* if M has a base of neighborhoods of the zero element $\mathcal{M}$ consisting of submodules. M is called *Hausdorff* if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized $R-$module M is said to be *linearly compact* if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection (see [10, 10]).

It is clear that Artinian $R-$modules are linearly compact and discrete. We have some following properties of linearly compact modules.

Lemma 2.1. (see [10, §3]) *(i) If M is a Hausdorff linearly topologized $R-$module and N a closed submodule of M, then M is linearly compact if and only if N and M/N are linearly compact.*

(ii) Let $f : M \longrightarrow N$ be a continuous homomorphism of Hausdorff linearly topologized $R-$modules. If M is linearly compact, then $f(M)$ is linearly compact and f is a closed map.
(iii) The inverse limit of a system of linearly compact $R-$modules with continuous homomorphisms is linearly compact.

If $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, then $\varprojlim_t{}^1 M_t = 0$ by [8, 7.1]. Therefore we have the following lemma.

Lemma 2.2. *Let*

$$0 \longrightarrow \{M_t\} \longrightarrow \{N_t\} \longrightarrow \{P_t\} \longrightarrow 0$$

be a short exact sequence of inverse systems of $R-$modules. If $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, then the sequence of inverse limits

$$0 \longrightarrow \varprojlim_t M_t \longrightarrow \varprojlim_t N_t \longrightarrow \varprojlim_t P_t \longrightarrow 0$$

is exact

Lemma 2.3. (see [3, §2]) *Let N be a finitely generated $R-$module and M a linearly compact $R-$module. Then for all $i \geq 0$, $\mathrm{Tor}_i^R(N; M)$ is a linearly compact $R-$module. Moreover,*
(i) If $f : N \longrightarrow N'$ is a homomorphism of finitely generated $R-$modules, then the induced homomorphism $f_{i,M} : \mathrm{Tor}_i^R(N; M) \longrightarrow \mathrm{Tor}_i^R(N'; M)$ is continuous.
(ii) If $g : M \longrightarrow M'$ is a continuous homomorphism of linearly compact $R-$modules, then the induced homomorphism $g_{N,i} : \mathrm{Tor}_i^R(N; M) \longrightarrow \mathrm{Tor}_i^R(N; M')$ is also continuous.

Let I be an ideal of R, the *i-th local homology* module $H_i^I(M)$ of an $R-$module M with respect to I is defined in [2] by

$$H_i^I(M) \cong \varprojlim_t \mathrm{Tor}_i^R(R/I^t; M).$$

Denote by $\Lambda_I(M) = \varprojlim_t M/I^tM$ the $I-$adic completion of M, then $H_0^I(M) \cong \Lambda_I(M)$. In case M is a finitely generated $R-$module, $H_i^I(M) = 0$ for all $i > 0$ (see [2, 3.2]). It should be noted that the local homology modules $H_i^I(M)$ of a linearly compact $R-$module M are also linearly compact $R-$modules. Moreover, every short exact sequence of linearly compact $R-$modules induces a long exact sequence of local homology modules (see [3, §3]).

Assume that I is generated by elements $x_1, x_2, \ldots, x_r$. Let $H_i(\underline{x}(t); M)$ be the $i-$th Koszul homology module of M with respect to $\underline{x}(t) = (x_1^t, \ldots, x_r^t)$. Some following properties of local homology modules will be used in next sections.

Theorem 2.4. (see [2, §3]) *Let M be an $R-$module. Then for all $i \geq 0$,*
(i) $H_i^I(M) \cong \varprojlim_t H_i(\underline{x}(t); M)$.
(ii) $\bigcap_{t>0} I^t H_i^I(M) = 0$.

Lemma 2.5. (see [3]) *Let M be a linearly compact $R-$module. Then*
(i) For all $j \geq 0$,

$$H_i^I(H_j^I(M)) \cong \begin{cases} H_j^I(M), & i = 0, \\ 0, & i > 0. \end{cases}$$

(ii) We have

$$H_i^I(\bigcap_{t>0} I^t M) \cong \begin{cases} 0, & i = 0, \\ H_i^I(M), & i > 0. \end{cases}$$

3 Ideal co-transforms

Let I be an ideal of R and M an $R-$module. It is well-known that the ideal transform of M with respect to I is defined by $D_I(M) = \varinjlim_t \mathrm{Hom}_R(I^t; M)$. This suggests the following definition.

Definition 3.1. Let I be an ideal of R and M an $R-$module. The *i-th ideal co-transform* $C_i^I(M)$ of M with respect to I (or *$i-$th $I-$co-transform* of M) is defined by

$$C_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(I^t; M).$$

Especially, $C_0^I(M)$ is called the $I-$co-transform of M and denoted by $C^I(M)$ for simplicity.

If M is a linearly compact $R-$module, then $\{\mathrm{Tor}_i^R(I^t; M)\}$ $(i \geq 0)$ forms an inverse system of linearly compact modules which continuous homomorphisms by 2.3, (i). Therefore by 2.1, (iii), $C_i^I(M)$ is also linearly compact.

Proposition 3.2. *Let* $0 \longrightarrow M" \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0$ *be a short exact sequence of linearly compact $R-$modules, in which the homomorphisms f and g are continuous. Then we have a long exact sequence of linearly compact $R-$modules*

$$\cdots \longrightarrow C_i^I(M") \xrightarrow{f_i} C_i^I(M) \xrightarrow{g_i} C_i^I(M') \longrightarrow$$

$$\cdots \longrightarrow C_0^I(M") \xrightarrow{f_0} C^I(M) \xrightarrow{g_0} C^I(M') \longrightarrow 0,$$

in which the homomorphisms f_i, g_i are continuous for all $i \geq 0$.

Proof. The short exact sequence $0 \longrightarrow M" \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0$ gives rise to a long exact sequence for all $t > 0$

$$\cdots \longrightarrow \mathrm{Tor}_i^R(I^t; M") \xrightarrow{f_{it}} \mathrm{Tor}_i^R(I^t; M) \xrightarrow{g_{it}} \mathrm{Tor}_i^R(I^t; M') \longrightarrow$$

$$\cdots \longrightarrow I^t \otimes_R M" \xrightarrow{f_{0t}} I^t \otimes_R M \xrightarrow{g_{0t}} I^t \otimes_R M' \longrightarrow 0,$$

in which homomorphisms f_{it}, g_{it} are continuous by 2.3, (ii). Then $\mathrm{Im}\, f_{it}, \ker f_{it}$, $\mathrm{Im}\, g_{it}, \ker g_{it}$ are linearly compact. Thus by 2.2, $\varprojlim_t$ is exact on all of the short exact sequences that arise from the long exact sequence. Therefore we have the long exact sequence in the theorem. We proceed to show that the homomorphisms $f_i,\ g_i$ are continuous. Indeed, since the homomorphisms f_{it}, g_{it} are continuous, the homomorphisms induced on corresponding direct products are also continuous. Therefore the homomorphisms $f_i,\ g_i$ induced on inverse limits are also continuous. The proof is complete. □

For all $t > 0$, the natural homomorphisms $\eta_t : I^t \bigotimes_R M \longrightarrow M$ and the canonical epi-morphisms $\theta_t : M \longrightarrow M/I^t M$ induce corresponding homomorphisms $\eta_M : C^I(M) \longrightarrow M$ and $\theta_M : M \longrightarrow \Lambda_I(M)$. The following theorem gives us the first relation between the local homology modules $H_i^I(M)$ and the $I-$co-transforms $C_i^I(M)$.

Theorem 3.3. *(i) For all $R-$modules M and all $i \geq 1$, $H_{i+1}^I(M) \cong C_i^I(M)$.*
(ii) If M is a linearly compact $R-$module, then there is an exact sequence of linearly compact modules

$$0 \longrightarrow H_1^I(M) \longrightarrow C^I(M) \xrightarrow{\eta_M} M \xrightarrow{\theta_M} \Lambda_I(M) \longrightarrow 0,$$

in which homomorphisms $\eta_M,\ \theta_M$ are continuous.

Proof. For any positive integer t the short exact sequence

$$0 \longrightarrow I^t \longrightarrow R \longrightarrow R/I^t \longrightarrow 0$$

gives isomorphisms $\mathrm{Tor}_{i+1}^R(R/I^t; M) \cong \mathrm{Tor}_i^R(I^t; M)$ for all $i \geq 1$. Thus we have (i). On the other hand, the short exact sequence above induces an exact sequence of linearly compact modules

$$0 \longrightarrow \mathrm{Tor}_1^R(R/I^t; M) \longrightarrow I^t \otimes_R M \xrightarrow{\eta_t} M \xrightarrow{\theta_t} M/I^t M \longrightarrow 0,$$

in which homomorphisms $\eta_t,\ \theta_t$ are continuous by 2.3, (i). It follows from 2.1, (i) and 2.3, (i) that $\{\ker \theta_t\}$ and $\{\mathrm{Tor}_1^R(R/I^t; M)\}$ form inverse systems of linearly compact modules with continuous homomorphisms. Therefore by 2.2, we have an exact sequence

$$0 \longrightarrow H_1^I(M) \longrightarrow C^I(M) \xrightarrow{\eta_M} M \xrightarrow{\theta_M} \Lambda_I(M) \longrightarrow 0.$$

By an argument analogous to the proof of 3.2, we get homomorphisms $\eta_M,\ \theta_M$ are continuous. □

Corollary 3.4. *Let M be a linearly compact $R-$module. Then the homomorphism $\eta_M : C^I(M) \longrightarrow M$ is an isomorphism if and only if $H_1^I(M) = \Lambda_I(M) = 0$.*

Proof. It is immediately induced from 3.3, (ii). □

Corollary 3.5. *Let M be a linearly compact $R-$module. There are two short exact sequences*

$$0 \longrightarrow H_1^I(M) \longrightarrow C^I(M) \longrightarrow \bigcap_{t>0} I^t M \longrightarrow 0,$$

$$0 \longrightarrow \bigcap_{t>0} I^t M \longrightarrow M \xrightarrow{\theta_M} \Lambda_I(M) \longrightarrow 0.$$

Proof. It follows from 3.3, (ii), since $\ker \theta_M = \bigcap_{t>0} I^t M$. □

Proposition 3.6. *Let M be a linearly compact $R-$module. Then*
(i) $C^I(H_i^I(M)) = 0,\ i \geq 0$.
(ii) $C^I(\bigcap_{t>0} I^t M) \cong C^I(M)$.
(iii) $C^I(M) \cong C^I(C^I(M))$.
(iv) $H_i^I(C^I(M)) \cong H_i^I(M),\ i \geq 2$.
(v) $\Lambda_I(C^I(M)) = H_1^I(C^I(M)) = 0$.

Proof. (i). We first note that $H_i^I(M)$ is a linearly compact $R-$module. Therefore from the exact sequence of 3.3, (ii), replace M by $H_i^I(M)$, we have an exact sequence

$$0 \longrightarrow H_1^I(H_i^I(M)) \longrightarrow C^I(H_i^I(M)) \longrightarrow H_i^I(M) \longrightarrow \Lambda_I(H_i^I(M)) \longrightarrow 0.$$

Since $H_1^I(H_i^I(M)) = 0$ and $\Lambda_I(H_i^I(M)) \cong H_i^I(M)$ by 2.5, (i), we get $C^I(H_i^I(M)) = 0$ for all $i \geq 0$.
(ii). The second exact sequence of 3.5, in which the homomorphisms are continuous, induces by 3.2 an exact sequence

$$C_1^I(\Lambda_I(M)) \longrightarrow C^I(\bigcap_{t>0} I^t M) \longrightarrow C^I(M) \longrightarrow C^I(\Lambda_I(M)) \longrightarrow 0.$$

In virtue of 3.3, (i) and 2.5, (i), we have $C^I_1(\Lambda_I(M)) \cong H^I_2(\Lambda_I(M)) = 0$. Moreover, $C^I(\Lambda_I(M)) = 0$ by (i), we get $C^I(\bigcap_{t>0} I^tM) \cong C^I(M)$.

(iii). We consider the first short exact sequence of 3.5. The epi-morphism is continuous, because η_M is continuous. Moreover, replacing $H^I_1(M)$ by $\ker \eta_M$, we may assume that the mono-morphism of the short exact sequence is also continuous. Therefore we have an exact sequence by 3.2

$$C^I(H^I_1(M)) \longrightarrow C^I(C^I(M)) \longrightarrow C^I(\bigcap_{t>0} I^tM) \longrightarrow 0.$$

In view of 2.5, (i), and (i), we have $C^I(H^I_1(M)) \cong C^I(\Lambda_I(H^I_1(M)) = 0$. Hence $C^I(C^I(M)) \cong C^I(\bigcap_{t>0} I^tM)$, which together with (ii) implies (iii).

(iv). The first exact sequence of 3.5 induces a long exact sequence of local homology modules

$$\cdots \longrightarrow H^I_i(H^I_1(M)) \longrightarrow H^I_i(C^I(M)) \longrightarrow H^I_i(\bigcap_{t>0} I^tM) \longrightarrow \cdots .$$

Since $H^I_i(H^I_1(M)) = 0,\ i \geq 2$, by 2.5, (i), we have an isomorphism

$$H^I_i(C^I(M)) \cong H^I_i(\bigcap_{t>0} I^tM).$$

Thus (iv) follows by 2.5, (ii).
(v) follows from (iii) and 3.4. □

Proposition 3.7. *Let $f : M' \longrightarrow M$ be a continuous homomorphism of linearly compact $R-$modules such that $\Lambda_I(\ker f) \cong \ker f,\ \Lambda_I(\operatorname{coker} f) \cong \operatorname{coker} f$. Let $\varphi : K \longrightarrow M$ be a further homomorphism of linearly compact $R-$modules. Then*
(i) The homomorphism $C^I(f) : C^I(M') \longrightarrow C^I(M)$ is an isomorphism.
(ii) There is a unique homomorphism $\psi : C^I(K) \longrightarrow M'$ such that the diagram

$$\begin{array}{ccc} M' & \xrightarrow{f} & M \\ \uparrow \psi & & \uparrow \varphi \\ C^I(K) & \xrightarrow{\eta_K} & K, \end{array}$$

is commutative, i. e., $f \circ \psi = \varphi \circ \eta_K$.
(iii) If $\varphi : K \longrightarrow M$ and $\eta_{M'} : C^I(M') \longrightarrow M'$ are both isomorphisms, then the homomorphism ψ of part (ii) is also an isomorphism.

Proof. (i) We have short exact sequences of linearly compact modules

$$0 \longrightarrow \ker f \longrightarrow M' \xrightarrow{\alpha} \operatorname{Im} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} f \xrightarrow{\beta} M \longrightarrow \operatorname{coker} f \longrightarrow 0$$

in which $f = \beta\alpha$ and homomorphisms are continuous. It is therefore enough to show that $C^I(\alpha)$ and $C^I(\beta)$ are both isomorphisms.

The first short exact sequence above induces by 3.2 an exact sequence

$$C^I(\ker f) \longrightarrow C^I(M') \xrightarrow{C^I(\alpha)} C^I(\operatorname{Im} f) \longrightarrow 0.$$

From 3.6, (i) and the hypothesis $\ker f \cong \Lambda_I(\ker f)$ we have $C^I(\ker f) = 0$. Hence $C^I(\alpha)$ is an isomorphism. Next, from the second short exact sequence we get an induced exact sequence

$$C_1^I(\operatorname{coker} f) \longrightarrow C^I(\operatorname{Im} f) \xrightarrow{C^I(\beta)} C^I(M) \longrightarrow C^I(\operatorname{coker} f) \longrightarrow 0.$$

We have $C^I(\operatorname{coker} f) \cong C^I(\Lambda_I(\operatorname{coker} f)) = 0$. Moreover, combining 3.3, (i), 2.5, (i) and the hypothesis $\Lambda_I(\operatorname{coker} f) \cong \operatorname{coker} f$, we abtain

$$C_1^I(\operatorname{coker} f) \cong H_2^I(\Lambda_I(\operatorname{coker} f)) = 0.$$

Therefore $C^I(\beta)$ is an isomorphism.

(ii) We have a commutative diagram

$$\begin{array}{ccccc} M' & \xrightarrow{f} & M & \xleftarrow{\varphi} & K \\ \uparrow \eta_{M'} & & \uparrow \eta_M & & \uparrow \eta_K \\ C^I(M') & \xrightarrow{C^I(f)} & C^I(M) & \xleftarrow{C^I(\varphi)} & C^I(K). \end{array}$$

By (i), $C^I(f)$ is an isomorphism. Set $\psi = \eta_{M'} \circ C^I(f)^{-1} \circ C^I(\varphi)$, we have

$$f\psi = f\eta_{M'} C^I(f)^{-1} C^I(\varphi) = \eta_M C^I(\varphi) = \varphi\eta_K.$$

Assume that there is a homomorphism $\psi' : C^I(K) \longrightarrow M'$ such that $f \circ \psi' = \varphi \circ \eta_K$. We have another commutative diagram

$$\begin{array}{ccccc} M' & \xleftarrow{\eta_{M'}} & C^I(M') & \xrightarrow{C^I(f)} & C^I(M) \\ \uparrow \psi' & & \uparrow C^I(\psi') & & \uparrow C^I(\varphi) \\ C^I(K) & \xleftarrow{\eta_{C^I(K)}} & C^I(C^I(K)) & \xrightarrow{C^I(\eta_K)} & C^I(K). \end{array}$$

is an isomorphism by 3.6, (iii), we get

$$\begin{aligned} \psi' &= \psi' \circ \eta_{C^I(K)} \circ \eta_{C^I(K)}^{-1} \\ &= \eta_{M'} \circ C^I(\psi') \circ C^I(\eta_K)^{-1} \\ &= \eta_{M'} \circ C^I(f)^{-1} \circ C^I(\varphi) = \psi. \end{aligned}$$

as required.

(iii) Since φ is an isomorphism, $C^I(\varphi)$ is also an isomorphism. Therefore (iii) follows immediately from the fact that $\psi = \eta_{M'} \circ C^I(f)^{-1} \circ C^I(\varphi)$. □

4 Co-localization

Let S be multiplicative set of R. For an $R-$mdule M the module $\mathrm{Hom}_R(R_S; M)$ is called co-localization of M with respect to S (see [11]). We denote it briefly by ${}_SM$. If M is a linearly compact $R-$module, then ${}_SM$ is also a linearly compact $R-$module by [5, 2.4]. The following proposition says that the co-localization can "commute" to the ideal co-transform of a linearly compact $R-$module

Proposition 4.1. *Let M be a linearly compact $R-$module. Then*

$$ {}_SC_i^I(M) \cong C_i^{IR_S}({}_SM) $$

for all $i \geq 0$.

Proof. We have

$$ \begin{aligned} {}_SC_i^I(M) &= \mathrm{Hom}_R(R_S; \varprojlim_t \mathrm{Tor}_i^R(I^t; M)) \\ &\cong \varprojlim_t \mathrm{Hom}_R(R_S; \mathrm{Tor}_i^R(I^t; M)) = {}_S(\mathrm{Tor}_i^R(I^t; M)) \end{aligned} $$

by [8, 2.5]. On the other hand by [4, 3.9],

$$ {}_S(\mathrm{Tor}_i^R(I^t; M)) \cong \mathrm{Tor}_i^{R_S}(I^tR_S; {}_SM). $$

Therefore

$$ {}_SC_i^I(M) \cong C_i^{IR_S}({}_SM) $$

as required. ☐

Let a be an element in R, the notation ${}_aM$ means that the co-localization of M with respect to the multiplicative set $\{1, a, a^2, \ldots\}$. We consider the canonical homomorphism $\delta : {}_aM = \mathrm{Hom}_R(R_a; M) \longrightarrow M, f \longmapsto f(1)$.

Lemma 4.2. *Let M be a linearly compact $R-$module and a an element of R. Then the canonical homomorphism δ is continuous.*

Proof. We consider the direct product M^{R_a}, where R_a the index set. Let $j : {}_aM \hookrightarrow M^{R_a}, f \longmapsto (f_x)_{x \in R_a}$, where $f_x = f(x) \in M$ be the inclusion and $\pi : M^{R_a} \longrightarrow M, (m_x)_{x \in R_a} \longmapsto m_1$ the natural projection. Then $\delta = \pi \circ j$. Therefore δ is continuous, since j and π are continuous. ☐

Lemma 4.3. *Let M be a linearly compact $R-$module. Then*

$$ \Lambda_{aR}(\ker \delta) \cong \ker \delta \text{ and } \Lambda_{aR}(\operatorname{coker} \delta) \cong \operatorname{coker} \delta. $$

Proof. Note by 3.3, (ii) that for a linearly compact $R-$module L, $\Lambda_{aR}(L) \cong L$ if and only if $\bigcap_{t>0} a^t L = 0$. Set $K = \ker \delta$ and $N = \operatorname{coker} \delta$, then K and N are linearly compact by 4.2 and 2.1, (i). Therefore it is sufficient to show that $\bigcap_{t>0} a^t K = \bigcap_{t>0} a^t N = 0$. For an element $f \in \bigcap_{t>0} a^t K$, we have for all $t > 0$, $f = a^t f_t$ for some $f_t \in K$. Hence $f(1/a^t) = a^t f_t(1/a^t) = f_t(a^t/a^t) = f_t(1) = 0$ for all $t > 0$. Therefore $f = 0$, thus $\bigcap_{t>0} a^t K = 0$.

On the other hand, since $\operatorname{Im} \delta = \bigcap_{t>0} a^t M$ by [5, 4.1], we get

$$\begin{aligned} \bigcap_{t>0} a^t N &= \bigcap_{t>0} a^t (M/\operatorname{Im}\delta) \\ &= \bigcap_{t>0} (a^t M/\operatorname{Im}\delta) \\ &= (\bigcap_{t>0} a^t M)/\operatorname{Im}\delta = 0. \end{aligned}$$

This finishes the proof. □

Theorem 4.4. *Let M be a linearly compact $R-$module and a an element in R. There is an isomorphism*

$$C^{aR}(M) \cong {}_aM.$$

Proof. From 3.7, (iii) and 4.3, replace the homomorphism f in 3.7 by the canonical homomorphism δ, we only need to show that $C^{aR}({}_aM) \cong {}_aM$, this is the case if and only if $\Lambda_{aR}({}_aM) = H_1^{aR}({}_aM) = 0$ by 3.4.

We first show that $a({}_aM) = {}_aM$. Indeed, for each $f \in {}_aM$ we can write $f = a.g$, in which g is defined by $g(r/a^t) = f(r/a^{t+1})$ for all $r \in R$ and all $t \geq 0$ ($a^0 = 1$). Then $a({}_aM) = {}_aM$ and we get $\Lambda_{aR}({}_aM) = 0$. We now have by 2.4, (i), $H_1^{aR}({}_aM) \cong \varprojlim_t H_1(a^t; {}_aM) \cong \varprojlim_t 0 :_{{}_aM} a^t$. Since $a({}_aM) = {}_aM$, we have

$$0 :_{{}_aM} a^t \subseteq a(0 :_{{}_aM} a^{t+1}) \subseteq 0 :_{{}_aM} a^{t+1}.$$

From the left exactness of inverse limit we get

$$\varprojlim_t 0 :_{{}_aM} a^t \subseteq a \varprojlim_t 0 :_{{}_aM} a^{t+1} \subseteq \varprojlim_t 0 :_{{}_aM} a^{t+1}.$$

Thus

$$H_1^{aR}({}_aM) \subseteq aH_1^{aR}({}_aM) \subseteq H_1^{aR}({}_aM).$$

It follows that $H_1^{aR}({}_aM) = aH_1^{aR}({}_aM)$. Therefore $H_1^{aR}({}_aM) = \bigcap_{t>0} a^t H_1^{aR}({}_aM) = 0$ by 2.4, (ii). This finishes the proof. □

From 3.3, (ii) and 4.4, there is an exact sequence

$$0 \longrightarrow H_1^{aR}(M) \longrightarrow {}_aM \xrightarrow{\delta} M \xrightarrow{\theta_M} \Lambda_{aR}(M) \longrightarrow 0.$$

Therefore, we have the following immediate consequence.

Corollary 4.5. *Let M be a linearly compact $R-$module and $\delta : {}_aM \longrightarrow M$ the canonical homomorphism. Then*

$$\ker \delta \cong H_1^{aR}(M), \;\; \operatorname{coker} \delta \cong \Lambda_{aR}(M).$$

Acknowledgments. The author is greatly indebted to Prof. Dr. Nguyen Tu Cuong, his supervisor, for suggesting the problems and ideas and for all his support. He also acknowledges the suppport of the National Basis Research Programme in Natural Science of Vietnam and The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy under the Post-doctoral fellowship.

References

[1] Brodmann M. P., Sharp R. Y., "Local cohomology: an algebraic introduction with geometric applications", Cambridge University Press 1998.

[2] N. T. Cuong and T. T. Nam, *The $I-$adic completion and local homology for Artinian modules* Math. Proc. Camb. Phil. Soc. **131**(2001), 61-72.

[3] N. T. Cuong and T. T. Nam, *A local homology theory for linearly compact modules* (preprint).

[4] N. T. Cuong and T. T. Nam, *On the Co-localization, Cosupport and Co-associated primes of local homology modules* Vietnam J. Math **29:4** (2001) 359-368.

[5] Nguyen Tu Cuong, Le Thanh Nhan, "On linearly compact modules which are representable", Proc. AMS **130** (2001), 1927-1936.

[6] J. P. C. Greenlees and J. P. May, *Derived functors of $I-$adic completion and local Homology,* J. Algebra **149**(1992), 438-453.

[7] A. Grothendieck, "Local cohomology", Lect. Notes in Math., **20**, Springer-Verlag, Berlin-Heidelberg-New York, 1966.

[8] C. U. Jensen, "Les Foncteurs Dérivés de $\varprojlim$ et leurs Applications en Théorie des Modules," Springer-Verlag, Berlin-Heidelberg-New York 1972.

[9] Lefschetz, "Algebraic Topology", Colloq. Lect. Amer. Soc. **27**, 1942.

[10] I. G. Macdonald, *Duality over complete local rings,* Topology **1**(1962), 213-235.

[11] L. Melkersson and P. Schenzel, *The co-localization of an Artinian module,* proceed. Ed. Math. Soc. **38** (1995), 121-131.

[12] D. Zelinsky, *Linearly compact modules and rings,* Amer. Math. **75**(1953), 79-90.