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YOUNG DIAGRAMS OF EQUIVALENT FUZZY SUBSETS

V. Murali

Department of Mathematics (Pure & Applied)
Rhodes University, Grahamstown 6140, South Africa

Abstract

We consider in this paper a natural equivalence relation on the set of all fuzzy subsets of a finite set with degree of membership values being taken from the unit interval. This equivalence is a generalization of equality of crisp sets. Maximal chains are called flags and finite chains of real numbers in the unit interval are called keychains. Maximal chains together with keychains twinned in an appropriate manner are called pinned-flags. First we prove that there is a one-to-one correspondence between equivalence classes of fuzzy subsets and the class of pinned-flags. We then represent equivalent classes of fuzzy subsets, using the one-to-one correspondence with pinned-flags, as Young diagrams or as other diagrams arising from Young diagrams.

Introduction.

Equality of two fuzzy subsets of a set X demands the membership values of every element of X to them are equal as real numbers in the unit interval [13]. This is rarely used in any reasonable application. It is only of theoretical value. On the other hand equality of crisp sets as containing the same elements is useful in set theory. In this paper we consider a concept of equivalence of fuzzy subsets that generalizes the equality of crisp sets but weaker than the equality of fuzzy subsets. Through out the paper we assume that X is a finite set of n elements. First we study two related ideas. One is that of representing a

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fuzzy subset by an increasing maximal chain of subsets (flags) [7], with degrees of membership in the decreasing order (keychains). We call such an object a pinned-flag. The other is that of equivalence of fuzzy subsets, [9], [11]. Simply put: two fuzzy sets are equivalent if they maintain the same relative degrees of membership between any pair of elements as in 2.1. The novelty of this paper is in the use of flags and keychains as primary tools to characterize fuzzy subsets. Secondly we prove that the pinned-flags are in one-to-one correspondence with equivalence classes of fuzzy subsets. Thirdly treating keychains as finite chains of real numbers in the unit interval arising from integer partitions of n, we represent them by Young Diagrams which are associated with integer partitions. From all of this we deduce that the equivalence classes of fuzzy subsets are in one-to-one correspondence with certain Young Diagrams.

1 Preliminaries

In this section we gather the basics of fuzzy sets, flags, and keychains. There are excellent books on fuzzy sets such as [5], [14], [8]and for flags and keychains we refer to [10]

1° Fuzzy subsets

We use $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$, the real unit interval as a chain with the usual ordering in which $\wedge$ stands for infimum (or intersection) and $\vee$ stands for supremum (or union) [3]. Throughout this paper we take X to be a non-empty **finite set with n elements** labeled as $\{x_1, x_2, \cdots, x_n\}$.

A *fuzzy subset* μ of a set X is a mapping $\mu : X \to \mathbf{I}$. The number $0 \leq \mu(x) \leq 1$ is known as the degree of membership of $x \in X$ to the fuzzy subset μ. The union, intersection of two fuzzy sets, and complementation of a fuzzy set are defined using sup and inf pointwise, and $1 - \mu$ operator pointwise, respectively. We denote the set of all fuzzy subsets of X by $\mathbf{I}^X$. Further we denote fuzzy sets by the Greek letters μ, ν, η, etc. and the membership values by $\alpha, \beta, \gamma, \delta$ etc. By an α-cut of μ for a real number α in $\mathbf{I}$, we mean a crisp subset $\mu^\alpha = \{x \in X : \mu(x) \geq \alpha\}$ of X. We remark that for $0 \leq \alpha \leq \beta \leq 1$ we have $\mu^\beta \subseteq \mu^\alpha$. For any fuzzy subset μ it can be easily verified that $\mu = \bigvee\{\alpha\chi_{\mu^\alpha} : \alpha \in [0, 1]\}$. By *support* of a fuzzy set μ we mean a crisp subset $supp\,\mu = \{x \in X : \mu(x) > 0\}$ of X. Similarly, by *core* of a fuzzy set μ we mean a crisp subset $core\,\mu = \{x \in X : \mu(x) = 1\}$ of X and by *imageset* of μ denoted by $Im(\mu) = \mu(X)$ we mean the set of membership values of μ including 1.

Generally the core of any fuzzy subset μ of any given set may or may not be empty. To facilitate a clear exposition of ideas, we make a **tacit** assumption as stated below about the core. Just as in other branches of Mathematics (such as Topology) where certain mathematical properties are taken to be satisfied vacuously in empty set situation , we propose an Axiom based on vacuous "satisfaction" in the sense that every element of the empty set belongs to the empty set to an absolute degree 1 as a statement of vacuous satisfaction. An accepted definition of empty fuzzy subset of X in the fuzzy literature is a $\chi_\emptyset$ which takes the value 0 for all $x \in X$ provided X is a non-empty set. This is different from considering fuzzy subsets of the empty set $\emptyset$. In this paper we make a clear distinction between the concepts of empty fuzzy subset of a non-empty set and fuzzy subset of the empty set. We state it as

Axiom 1.1. NULSAX *: There is only one fuzzy subset μ of the empty set $\emptyset$ which takes the membership value 1 on the empty set,* $\mu : \emptyset \to I$ *so that* $(\forall x \in \emptyset)(\mu(x) = 1)$*. That is the core of the empty set is the empty set and that the core of every non-empty set always includes the empty set as a subset.*

2° Flags and keychains

We define and briefly discuss flags and keychains in this subsection. We refer to [10] and [11] for further details.

Definition 1.2. By a *keychain* $\boldsymbol{\ell}$ of length n, we mean a set of (n+1) real numbers $\lambda_i \in \mathbf{I}, i = 0, 1, \cdots, n$ of the form

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \tag{1.1}$$

where the λ_i's are not all necessarily distinct. The λ_i's are called *pins* . The set of all keychains is denoted by $\mathcal{K}$.

A matter of notation: When pins repeat we denote them by the same symbol. For instance the keychain $1 > \lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 = \lambda_5 = \lambda_6 > \lambda_7 = \lambda_8 = 0$ of length 8 is denoted by $1\alpha\alpha\beta\gamma\gamma\gamma00$. Similarly by $111\alpha\beta\gamma\gamma\delta\delta$ we mean the keychain (length 8, of course) $1 = \lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 = \lambda_6 > \lambda_7 = \lambda_8 > 0$ with obvious meaning to α, β etc., namely, they are real numbers in the unit interval satisfying $1 > \alpha > \beta > \gamma > \delta > 0$.

Definition 1.3. (i) By a *flag* $\mathcal{C}$ on X, we mean a maximal chain $\mathcal{C}$ of subsets of X of the form

$$X_0 \subset X_1 \subset X_2 \cdots \subset X_n = X \tag{1.2}$$

In terms of labeled elements, X_i can be taken to be a subset $\{x_{j_1}, x_{j_2}, \cdots, x_{j_i}\}$ of i elements from X without repetition. We call various X_i's the *components* of the flag $\mathcal{C}$. The above inclusions in the flag are always taken to be strict.

The set of all flags on X is denoted by $\mathcal{M}$ and the set of all permutations on n symbols $\underline{\mathbf{n}} = \{1, 2. \cdots, n\}$ is familiarly denoted by S_n. Clearly there is a one-to-one correspondence between the elements of the sets $\mathcal{M}$ and S_n. Therefore we can index $\mathcal{M}$ by elements of S_n. Thus we talk of a flag $\mathcal{C}_\sigma$ for any permutation $\sigma \in S_n$. Hence $\mathcal{M} = \{\mathcal{C}_\sigma : \sigma \in S_n\}$.

2 Equivalence and pinned-flags

In this section we introduce the equivalence of fuzzy subsets and the associated concept of pinned-flags.

1°. Equivalence relation on fuzzy sets

An equivalence relation $\sim$ on $\mathbf{I}^X$ is defined as follows, see [1],[4]:

$$\mu \sim \nu \text{ if and only if}$$

$$\begin{array}{llll} (i) & \text{for all } x, y \in X \ \mu(x) > \mu(y) & \text{if and only if} & \nu(x) > \nu(y) \\ (ii) & \mu(x) = 1 & \text{if and only if} & \nu(x) = 1 \\ (iii) & \mu(x) = 0 & \text{if and only if} & \nu(x) = 0. \end{array} \tag{2.1}$$

It is easily checked that this relation is indeed an equivalence relation on I^X and, when restricted to $\mathbf{2}^X$, where $\mathbf{2} = \{0, 1\}$, coincides with equality of sets [9]. Under this equivalence relation, the equivalence class containing μ is denoted by $[\mu]$. We denote the set of all equivalence classes of fuzzy subsets by $\mathcal{E}$. The following facts have been established about this equivalence, (see [9], [10]). As before μ and ν are two fuzzy subsets of X.

Remark 2.1.

1. In condition (i) of the above equivalence relation, we can replace the strict inequality by $\geq$ inequality without affecting the equivalence.
2. Condition (iii) of equation 2.1 above says that the supports of equivalent fuzzy subsets are the same.
3. From condition (ii) of equation 2.1 we can easily deduce that $\mu(x) = 1$ if and only if $\nu(x) = 1$, that is, the top cuts, that is the core of two equivalent fuzzy subsets are the same.
4. If $\mu \sim \nu$, then $|Im(\mu)| = |Im(\nu)|$.
5. The converse is not true, viz. if $|Im(\mu)| = |Im(\nu)|$ or even if $Im(\mu) = Im(\nu)$,$core\,\mu = core\,\nu$ and $supp\,\mu = supp\,\nu$, it is not necessary to have $\mu \sim \nu$.
6. $\mu \sim \nu$ if and only if for each $\alpha > 0$ there exists an $\beta > 0$ such that $\mu^\alpha = \nu^\beta$.

2°. Pinned-flags

We describe below how pinned-flags and fuzzy subsets determine each other.

Definition 2.2. By a *pinned-flag* on X, we mean a pair ($\mathcal{C}$, $\boldsymbol{\ell}$), of a flag $\mathcal{C}$ on X and a keychain $\boldsymbol{\ell}$ from $\mathbf{I}$, written suggestively as follows:

$$X_0^1 \subset {X_1}^{\lambda_1} \subset {X_2}^{\lambda_2} \cdots \subset {X_n}^{\lambda_n} \tag{2.2}$$

By eliminating the repetitive pins we get a sequence

$$X_0^1 \subset {X_1}^{\lambda_1} \subset {X_2}^{\lambda_2} \cdots \subset {X_k}^{\lambda_k}, \quad 1 \le k \le n \tag{2.3}$$

which we call a *reduced pinned-flag*. Note that the pins in a reduced pinned-flag are all distinct from each other. Now, we can associate a fuzzy subset μ of X with such a pinned-flag ($\mathcal{C}$, $\boldsymbol{\ell}$) as follows:

$$\mu(x) = \begin{cases} 1, & x \in X_0 \\ \lambda_1, & x \in X_1 \setminus X_0 \\ \lambda_2, & x \in X_2 \setminus X_1 \\ \vdots & \\ \lambda_n & x \in X_n \setminus X_{n-1} \end{cases} \tag{2.4}$$

where the component X_n is the full set X. We denote this simply by ${X_n}^{\lambda_n} = X^{\lambda_n}$. Note that λ_n may or may not be 0. If $\lambda_n = 0$, then $supp\,\mu$ is strictly contained in X, and if $\lambda_n \neq 0$ then $supp\,\mu = X$. Similarly if $\lambda_1 = \lambda_0 = 1$ and $\lambda_2 \neq 0$, then the top cut $\mu^1 = X_1$. If not, i. e. if $\lambda_1 \neq 1$, the top cut μ^1 is the empty set X_0 and so on. Further we note that the α-cuts of μ corresponding to $\lambda_{i-1} > \alpha \ge \lambda_i$ are X_i for $i = 1, 2, \cdots, n$. The μ as defined above is a fuzzy subset of X.
Conversely,

Proposition 2.3. *Suppose μ is a fuzzy subset of X, then we can decompose μ into a pinned-flag*

$$(\mathcal{C}, \boldsymbol{\ell}) : \; X_0^1 \subset {X_1}^{\lambda_1} \subset {X_2}^{\lambda_2} \cdots \subset {X_n}^{\lambda_n}$$

that represents μ as in equation 2.4.

Sketch of proof

Since X is finite, $\mu(X) \subset \mathbf{I}$ is finite. Let $\mu(X) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ where the sequence is decreasing (strictly). Let $Y_i = \mu^{\lambda_i}$ be the α-cut corresponding to $\alpha = \lambda_i$ for $i = 1, 2, \cdots, k$. Then three facts are well-known:
(i) every Y_i is a subset of X,
(ii) $\lambda_i > \lambda_j$ implies that $Y_i \subset Y_j$ for $1 \le i, j \le k$.
(iii) The chain $\mathcal{C}_1 : Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_k$ can be refined to yield a flag

$$\mathcal{C} : X_0 \subset X_1 \subset X_2 \cdots \subset X_n = X$$

As we refine $\mathcal{C}_1$ to $\mathcal{C}$ we may have to repeat some of the pins λ_i's correspondingly. Once this process carried out, we arrive at a pinned-flag

$$X_0^1 \subset {X_1}^{\lambda_1} \subset {X_2}^{\lambda_2} \cdots \subset X^{\lambda_n} \tag{2.5}$$

which obviously represents μ as in equation 2.4.

Note 2.4. If $k = n$, then all the λ_i's are distinct. In this case the number of pins is n. If $k < n$, some pins are repeating. First we collect the distinct pins. Secondly we count the number of times each distinct pin is repeated. If some pins do not repeat then they are single pins. With single pins we associate 1. Thus we arrive at $\mathbf{i} = (l_1, l_2, \cdots, l_k)$ where some of the l_i's may be 1, that is, those corresponding to non-repeating single pins. In any case $\sum_{i=1}^{k} l_i = n$. The k-tuple of positive integers $\mathbf{i}$ is known as the *index* of μ or of the keychain of μ. See corollary 2.7

3° Equivalent fuzzy sets and pinned-flags

The following proposition expresses the equivalence of fuzzy subsets in terms of pinned-flags.

Proposition 2.5. *Suppose the reduced pinned-flags corresponding to two fuzzy subsets μ and ν of the same set X are given by*

$$\begin{array}{c} (\mathcal{C}_\mu, \boldsymbol{\ell}_\mu) : X_0^1 \subset X_1^{\lambda_1} \subset \cdots \subset X_r^{\lambda_r} \\ \textit{and} \\ (\mathcal{C}_\nu, \boldsymbol{\ell}_\nu) : Y_0^1 \subset Y_1^{\beta_1} \subset \cdots \subset Y_s^{\beta_s} \end{array} \tag{2.6}$$

Then $\mu \sim \nu$ on X if and only if :
(i) $r = s$;
(ii) $X_i = Y_i$ for $i = 0, 1, \cdots, r$;
(iii) $\lambda_i > \lambda_j$ if and only if $\beta_i > \beta_j$ for $1 \le i, j \le r$ and $\lambda_t = 0$ if and only if $\beta_t = 0$ for all t between r and n.

Proof ($\Rightarrow$) (i) For $x \in X$, define a function $f : I^X \to I^X$ by $f(\mu(x)) = \nu(x)$. Then it is easy to check that f is firstly well defined and secondly bijective. Thus $|Im(\mu)| = |Im(\nu)|$. Therefore $r = s$.

(ii) We prove by induction on r. For $r = 0$, $X_0 = Y_0$ since each set is the empty set. Suppose $X_k = Y_k$ for $k \ge 0$, and $g \in X_{k+1}$. If $g \in X_k$, then $g \in Y_k$ which is contained in Y_{k+1}. There is nothing to prove in this case. If not, $\mu(g) = \lambda_{k+1}$. Suppose $g \notin Y_{k+1}$, then $\nu(g) < \beta_{k+1}$. Choose $x \in Y_{k+1}$ but $x \notin Y_k$. Then $\nu(g) < \nu(x)$ which implies $\mu(g) < \mu(x)$ by equivalence. Hence $\lambda_{k+1} = \mu(g) < \mu(x) = \alpha$ where $\lambda_k \le \alpha < \lambda_{k+1}$. This implies $x \in \mu^\alpha \subseteq X_k = Y_k$, a contradiction. Therefore $X_{k+1} \subset Y_{k+1}$. Similarly we can show that $Y_{k+1} \subset X_{k+1}$. This completes the induction.
(iii) Follows from (i) and (ii) and from the definition of equivalence pertaining to support.

($\Leftarrow$) Suppose μ and ν are two fuzzy subsets such that (i), (ii) and (iii) are valid. Then from (iii) $\lambda_t = 0$ if and only if $\beta_t = 0$ for all t between r and n is true. By (i) $r = s$. Therefore $supp\,\mu = supp\,\nu$. For $x, y \in supp\,\mu$, suppose

$\mu(x) > \mu(y)$. Then $\mu(x) = \lambda_i$ and $\mu(y) = \lambda_j$ for some i, j. But from (iii) $\lambda_i > \lambda_j$ if and only if $\beta_i > \beta_j$ for $1 \leq i, j \leq r$. Now, (ii) and (iii) together imply $\nu(x) > \nu(y)$. Thus $\mu \sim \nu$ as required.

We may take the above proposition as the definition of equivalence of pinned-flags. That is $(\mathcal{C}_\mu, \boldsymbol{\ell}_\mu) \sim (\mathcal{C}_\nu, \boldsymbol{\ell}_\nu)$ if and only if conditions (i), (ii) and (iii) are satisfied. Also two keychains are equivalent if conditions (i) and (iii) are satisfied. Therefore we call two keychains distinct if they are not equivalent. With these definitions of equivalence of pinned-flags and fuzzy sets, we have the following

Note 2.6. With each fuzzy subset μ of X we can associate one and only one pinned-flag $(\mathcal{C}, \boldsymbol{\ell})$ which we denote by $(\mathcal{C}_\mu, \boldsymbol{\ell}_\mu)$, or simply by $\mathcal{C}_\mu$ if the keychain $\boldsymbol{\ell}_\mu$ can be determined from the context, or simply by $\boldsymbol{\ell}_\mu$ if $\mathcal{C}_\mu$ is known, and conversely, with each pinned-flag $(\mathcal{C}_\mu, \boldsymbol{\ell}_\mu)$ we can associate a fuzzy subset μ as above. *Thus we can speak of fuzzy subsets and their associated pinned-flags (or keychains only if the underlying flags are understood from the context) interchangeably as referring to the same construct.*

Thus not only we can associate the pinned-flag $(\mathcal{C}_\mu, \boldsymbol{\ell}_\mu)$ with a given μ but we can also associate the same pinned-flag with an equivalence class $[\mu]$ of μ provided we take into account the properties (i), (ii) and (iii) of the above proposition pertaining to $\mathcal{C}_\mu$ and $\boldsymbol{\ell}_\mu$, i.e. equivalence of pinned-flags. With this in mind , we state the following corollary :

Corollary 2.7. *There is a one-to-one correspondence between pinned-flags on X and equivalence classes of fuzzy subsets subject to (i),(ii), and (iii) of proposition 2.5.*

By abuse of notation, we refer to the pinned-flag associated with an equivalence class $[\mu]$ containing fuzzy subset μ simply by $\boldsymbol{\ell}_\mu$ when the underlying flag is understood from the context. Thus, in terms of the notion of index as introduced in note 2.4 we can refer to the index of a fuzzy subset to be the same as the index of the underlying keychain of the fuzzy subset. The index is an invariant for all fuzzy subsets in the same equivalence class.

3 Young Diagrams of Fuzzy subsets

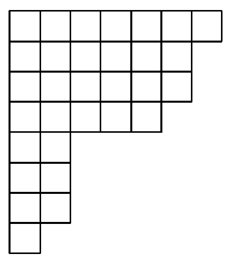
In this section we briefly discuss partitions and their Young diagrams [2], [6], [12] permuted diagrams and the way in which diagrams are augmented with further four squares. We describe the steps to arrive at diagrams from keychains, and finally picture the diagrams of pinned-flags.

1° Partitions and Young Diagrams

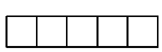
A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $l_1, l_2, \cdots, l_r$ such that $\sum_{i=1}^{r} l_i = n$. The l_i's are called *parts* of the partition. As is customary we shall write $\ell \vdash n$ for the partition $\ell = (l_1, l_2, \cdots, l_r)$ whose parts are the l_i's. Sometimes it is usual to use a notation that makes explicit the number of times that a particular integer occurs as a part. Thus if ℓ is a partition of n we write

$$\ell = (1^{f_1} 2^{f_2} \cdots)$$

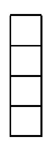
where exactly f_k number of times k appears in the partition ℓ of n, so that $\sum_{i=1}^{m} f_i i = n$. We remark that there are several equivalent useful ways of forming the graphical representation of a partition. We follow the most popular representation known as the Standard Young diagram, using appropriate number of square boxes for the various parts of the partition. We illustrate the partition of $31 = 7 + 6 + 6 + 5 + 2 + 2 + 2 + 1$ as a Young diagram ℓ_{31}.



The Young diagram corresponding to the partition of $\ell = (n)$ of n is represented by n boxes in a row. The other extreme case of the partition $n = \underbrace{1 + 1 + \cdots + 1}_{n-times}$ is represented by a column of n boxes. These are illustrated by the following diagrams:

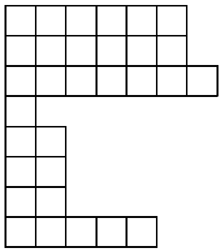


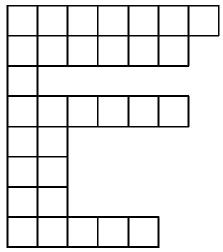
($\ell = (5)$)

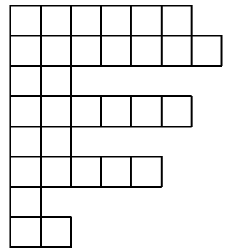


($\ell =$ (1,1,1,1)= (1^4))

Together with Standard Young diagram, one can construct new diagrams, called *permuted diagrams* , with rows permuted such as the following diagrams, (only a sample shown below, there are a number of other possiblities) arising from ℓ_{31}







Also one can attach two new boxes to a given diagram, one at the beginning and one at the end, in four different positions at the top left and bottom right as shown below to the standard ℓ_{31}, with only one exception. We call these the *augmented diagrams*. These augmentations correspond to having a fuzzy subset μ with *core* μ empty or not and *supp* μ is X or strictly contained in X. Section 3 below will clarify further.

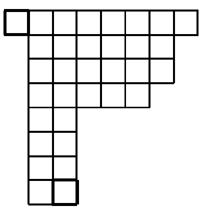


Diagram 1

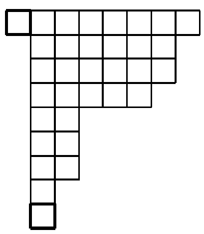


Diagram 2

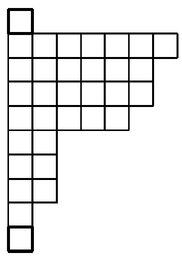


Diagram 3

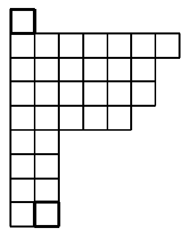


Diagram 4

Now we come to deal with the only exceptional case that was mentioned above. Suppose $\ell = (n)$. Then there are only three augmented diagrams. We illustrate for the case $\ell_5 = (5)$.

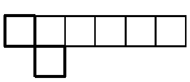


Diagram 5

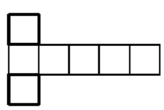


Diagram 6

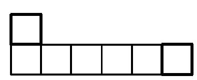


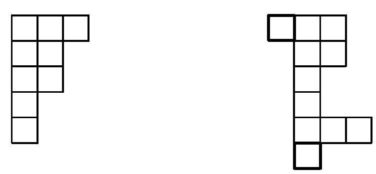
Diagram 7

2° Diagrams of keychains

It is clear that there is a one-to-one correspondence between keychains of n-chains and their indices. But indices are nothing but ordered partitions of n. Consequently augmented permuted diagrams of Young diagrams are associated with keychains. We now describe the method how to realize this correspondence between keychains. First recall that a keychain of length $n+1$ is represented by equation 1.1 where we could take one ("$=$") or the other " $>$ " choice in $n+1$ places. Secondly consider $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ omitting the first and last entries. We draw a blank square for each λ_i for $i = 1, 2, \cdots, n$. We place them in a row adjecent to each other with each choice of "$=$" . As soon as a choice of " $>$ " occurs a new row is created with squares aligned with the previous row and so on arriving at a permuted diagram. We now augment this diagram with the appropriate square attachment for the first and last choice of equality or inequality as the case may be, as shown in section 4.1 above. We have drawn below the augmented permuted diagrams as well as standard Young Diagram of two keychains as illustration.

Suppose $\boldsymbol{\ell_1} : 1\alpha\alpha\beta\gamma\gamma\gamma00$ has length 8 with index $(2,1,3,2)$ and $\boldsymbol{\ell_2} : 111\alpha\alpha\beta\gamma\delta\delta\delta$ has length 9 with index $(2,2,1,1,3)$ with the tacit assumption $1 > \alpha > \beta > \gamma > \delta > 0$. The associated partitions of $\boldsymbol{\ell_1}$ and $\boldsymbol{\ell_2}$ are $8 = 3+2+2+1$ and $9 = 2+2+1+1+3$, respectively. The following are the augmented permuted diagrams (right) and the standard Young diagrams (left) of keychains $\boldsymbol{\ell_1}$ and $\boldsymbol{\ell_1}$ respectively :

$\boldsymbol{\ell_1}$ Standard and Augmented diagram



$\boldsymbol{\ell_2}$ Standard and Augmented diagram

3° Diagrams of pinned-flags

To get a pinned-flag from a keychain, we only have to consider any given flag on X which twins with the given keychain as in definition 2.6. But flags on X are obtained by permuting the given set of elements with identity permutation taken as the given natural order. Leaving out the very first and the last square from the augmented permuted diagram and filling the rest of the squares with one number from 1 to n inclusive in any order, we get a corresponding pinned-flag. With that pinned-flag is associated an equivalent class of fuzzy subsets of X. Consider the example of section 2 with keychain $\boldsymbol{\ell_2}$. Suppose $X = \{1, 2, \cdots, 9\}$ is a set with 9 elements in its natural order. Apply a permutation σ to X to get $X_\sigma = \{4,3,5,9,6,2,8,1,7\}$. Consider $\boldsymbol{\ell_\mu} = \boldsymbol{\ell_2}$ and $\mathcal{C}_\mu : X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_9$ where $X_0 = \emptyset$, $X_1 = \{4\}, X_2 = \{4,3\}, X_3 = \{4,3,5\}, \cdots, X_9 = \{4,3,5,9,6,2,8,1,7\} = X$. Then the pinned -flag $(\mathcal{C}_\mu, \boldsymbol{\ell_\mu})$

corresponds to the equivalent fuzzy subset μ of X given by the following:

$$\mu(x) = \begin{cases} 1, & x = \emptyset = Y_0 \\ 1, & x = \{4,3\} = X_2 = Y_1 \\ \alpha, & x = \{5,9\} = X_4 \setminus X_2 = Y_2 \\ \beta, & x = \{6\} = X_5 \setminus X_4 = Y_3 \\ \gamma, & x = \{2\} = X_6 \setminus X_5 = Y_4 \\ \delta, & x = \{8,1,7\} = X_9 \setminus X_6 = Y_5 \end{cases} \tag{3.1}$$

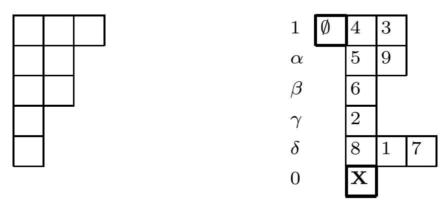
The actual pinned-flag $(\mathcal{C}_\mu, \boldsymbol{\ell}_\mu)$ is

$$X_0^1 \subset {X_1}^1 \subset {X_2}^1 \subset {X_3}^\alpha \subset {X_4}^\alpha \subset {X_5}^\beta \subset {X_6}^\gamma \subset {X_7}^\delta \subset {X_8}^\delta \subset {X_9}^\delta \tag{3.2}$$

which can be written as

$$Y_0^1 \subset {Y_1}^1 \subset {Y_2}^\alpha \subset {Y_3}^\beta \subset {Y_4}^\gamma \subset {Y_5}^\delta \tag{3.3}$$

In the following we have drawn the standard Young diagram on the left and on the right augmented permuted diagram with the assigned elements in the squares representing the pinned-flag $(\mathcal{C}_\mu, \boldsymbol{\ell}_\mu)$.



With the above procedures for associating fuzzy subsets to augmented permuted diagrams we can interpret diagrams 1 to 7 of section 4.1. The augmentation of diagrams says something about the core and the support of associated fuzzy subsets. For instance in diagram 1 of section 1, the associated fuzzy subset has non-empty core and the support strictly contained in X. Similarly in the second diagram, the core is non-empty , but the support is whole of X. In diagrams 3 and 4, the core is empty but in the third diagram support is whole of X whereas in the fourth the support is strictly contained in X. The fuzzy subset associated with diagram 5 of section 1 is the whole of crisp set X, and the diagram 7 corresponds to the empty subset of X. Diagram 6 represents a fuzzy subset μ for which every element of X has membership value α to μ where α is a fixed non-unit, non-zero number.

We hope to study the operations such as union, intersection, product, group operations etc of fuzzy subsets in a subsequent papers. We could compute their supplemented permuted diagrams of standard Young diagrams and interpret the fuzzy set theoretic properties by means of their diagrams.

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