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A RESULT ON THE INSTABILITY OF SOLUTIONS OF CERTAIN NON-AUTONOMOUS VECTOR DIFFERENTIAL EQUATIONS OF FOURTH ORDER*

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Abstract

The main purpose of this paper is to obtain sufficient conditions under which the trivial solution of the vector differential equations of the form

$$X^{(4)} + A\dddot{X} + B(t)H(X,\dot{X},\ddot{X},\dddot{X})\ddot{X} + C(t)G(X)\dot{X} + D(t)F(X) = 0$$

is unstable.

1 Introduction

We are interested in obtaining a result on the instability behavior of the trivial solution $X = 0$ of the nonlinear vector differential equations of the form:

$$X^{(4)} + A\dddot{X} + B(t)H(X,\dot{X},\ddot{X},\dddot{X})\ddot{X} + C(t)G(X)\dot{X} + D(t)F(X) = 0, \quad (1.1)$$

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in which $t \in R^+$, $R^+ = [0, \infty)$ and $X \in R^n$; A is a constant $n \times n$ -symmetric matrix; B, H, C, G, and D are continuous $n \times n$-symmetric matrices for the arguments displayed explicitly and the dots indicate differentiation with respect to t; $F : R^n \to R^n$ and $F(0) = 0$. It is assumed that the function F is continuous. It should be noted that, through in what follows, we use the following differential system which is equivalent to the differential equation (1.1):

$$\begin{aligned} &\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = W, \\ &\dot{W} = -AW - B(t)H(X,Y,Z,W)Z - C(t)G(X)Y - D(t)F(X) \end{aligned} \tag{1.2}$$

obtained as usual by setting $\dot{X} = Y, \ddot{X} = Z, \dddot{X} = W$ in (1.1).

Let $J_G(X)$ and $J_F(X)$ denote the Jacobian matrices corresponding to the $G(X)$ and $F(X)$, respectively, that is,

$$J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right), \quad J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right),$$

and

$$\frac{d}{dt}C(t) = \dot{C}(t) = \frac{d}{dt}(c_{ik}(t)) \quad (i, j, k = 1, 2, ..., n),$$

where $(x_1, x_2, ..., x_n)$, $(g_1, g_2, ..., g_n)$, $(f_1, f_2, ..., f_n)$ and $(c_{ik}(t))$ are components of X, G, F and C, respectively. Furthermore, it is assumed, as basic throughout in what follows, that the Jacobian matrices $J_F(X)$, $J_G(X)$ and $\dot{C}(t)$ exist and are symmetric and continuous, and that all matrices given in the pairs D, J_F; C, G; C, J_G; $\dot{C}$, G and B, H commute with each others.

As we know, the instability behavior of solutions for various certain second-, third-, fourth-, fifth-, sixth-, seventh- and eighth-order nonlinear differential equations, particulary in the case $n = 1$, have been widely discussed in the literature (see, for example [1-6], [8-12], [14-26]) and refernces quoted therein for some publications on the matter.

Skrapek [17] and Tiryaki [20] investigated the instability of the trivial solution of the fourth order scalar differential equations of the form:

$$x^{(4)} + f(x, \dot{x}, \ddot{x}, \dddot{x}) = 0$$

and

$$x^{(4)} + \psi(\ddot{x})\dddot{x} + \varphi(\dot{x})\ddot{x} + \theta(\dot{x}) + f(x) = 0,$$

respectively. Recently, Sadek [16] and Tunç ([23], [24]) also studied the same subject for the nonlinear vector differential equations:

$$X^{(4)} + A\dddot{X} + H(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X} + G(X)\dot{X} + F(X) = 0,$$

$$X^{(4)} + \Psi(\ddot{X})\dddot{X} + \Phi(\dot{X})\ddot{X} + H(\dot{X}) + F(X) = 0$$

and

$$X^{(4)} + \Psi(\dot{X}, \ddot{X})\, \dddot{X} + \Phi(X, \dot{X})\, \ddot{X} + H(\dot{X}) + F(X) = 0,$$

respectively. According to our observations in the relevant literature, there isn't any research found on the instability of solutions of certain non-autonomous vector differential equations of the fourth order. Especially, the motivation for the present investigation has come from the papers of skrapek [17], Sadek [16], Tiryaki [20] and Tunç ([23], [24]).

The symbol $\langle X, Y \rangle$ is used to denote the usual scalar product in R^n for given any X, Y in R^n, that is, $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$, thus $\|X\|^2 = \langle X, X \rangle$. The matrix A is said to be negative-definite, when $\langle AX, X \rangle < 0$ for all non-zero X in R^n, and $\lambda_i(A)$ $(i = 1, 2, ..., n)$ are eigenvalues of the $n \times n-$ matrix A.

2 The main result

We can now state our foremost result:

Theorem *Further to the basic assumptions on A, B, C, D, H, G and F in (1.2), suppose there exist positive constants a_1, a_2 and a_4 such that $a_4 - \frac{1}{4}a_2^2 > 0$ and the following conditions are satisfied:*

(i) $\lambda_i(B(t) \geq 1$, $\lambda_i(\dot{C}\,(t) \leq 0$ *and* $\lambda_i(D(t) \geq 1$ *for all* $t \in R^+$.

(ii) $\lambda_i(A) \geq a_1$, $\lambda_i(H(X, Y, Z, W)) \leq a_2$, $F(X) \neq 0$ *for all* $X \neq 0, X \in R^n$, *and* $\lambda_i(J_F(X)) \geq a_4$ *for all* $X, Y, Z, W \in R^n$.

Then the zero solution $X = 0$ of the system (1.2) is unstable.

Now, we state the following algebraic results required in the proof of the theorem.

Lemma 1 *Let A be a real symmetric $n \times n$ matrix and*

$$a' \geq \lambda_i(A) \geq a > 0 \quad (i = 1, 2, ..., n),$$

where a', a are constants. Then

$$a' \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle.$$

Proof See (Horn and Johnson [7]). □

Lemma 2 *Let Q, D be any two real $n \times n$ commuting symmetric matrices. Then*

(i) *The eigenvalues* $\lambda_i(QD)$ $(i = 1, 2, ..., n)$ *of the product matrices* QD *are real and satisfy*

$$\max_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D) \ge \lambda_i(QD) \ge \min_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D).$$

(ii) *The eigenvalues* $\lambda_i(Q+D)$ $(i = 1, 2, ..., n)$ *of the sum of matrices* Q *and* D *are real and satisfy*

$$\left\{\max_{1\le j\le n} \lambda_j(Q) + \max_{1\le k\le n} \lambda_k(D)\right\} \ge \lambda_i(Q+D) \ge \left\{\min_{1\le j\le n} \lambda_j(Q) + \min_{1\le k\le n} \lambda_k(D)\right\},$$

where $\lambda_j(Q)$ *and* $\lambda_k(D)$ *are, respectively, the eigenvalues of* Q *and* D.

Proof See (Horn and Johnson [7]). □

Proof of the theorem

Our main tool in the proof is the scalar Lyapunov function $V = V(t, X, Y, Z, W)$ defined by:

$$V = \langle Y, Z\rangle + \frac{1}{2}\langle AY, Y\rangle - \langle W, X\rangle - \langle AZ, X\rangle - \int_0^1 \sigma \langle C(t)G(\sigma X)X, X\rangle \, d\sigma. \quad (2.1)$$

It is clear that $V(0, 0, 0, 0, 0) = 0$. Indeed, we also have that

$$\begin{aligned} V(0, 0, \varepsilon, \varepsilon, 0) &= \langle \varepsilon, \varepsilon\rangle + \tfrac{1}{2}\langle A\varepsilon, \varepsilon\rangle \ge \langle \varepsilon, \varepsilon\rangle + \tfrac{1}{2}a_1 \langle \varepsilon, \varepsilon\rangle \\ &= \|\varepsilon\|^2 + \tfrac{1}{2}a_1 \|\varepsilon\|^2 > 0 \end{aligned}$$

for all arbitrary $\varepsilon \ne 0, \varepsilon \in R^n$. Next, let $(X, Y, Z, W) = (X(t), Y(t), Z(t), W(t))$ be an arbitrary solution of the system (1.2). We have from (2.1) and (1.2) that

$$\begin{aligned} \dot{V} = \tfrac{d}{dt}V(t, X, Y, Z, W) &= \langle Z, Z\rangle + \langle B(t)H(X, Y, Z, W)Z, X\rangle + \langle D(t)F(X), X\rangle \\ &+ \langle C(t)G(X)Y, X\rangle - \tfrac{d}{dt}\int_0^1 \sigma \langle C(t)G(\sigma X)X, X\rangle \, d\sigma. \end{aligned}$$

But

$$\begin{aligned}\frac{d}{dt}\int_0^1 \sigma \quad & \langle C(t)G(\sigma X)X, X\rangle \, d\sigma = \\ & = \int_0^1 \sigma \left\langle \dot{C}\ (t)G(\sigma X)X, X \right\rangle d\sigma + \int_0^1 \langle \sigma C(t)G(\sigma X)Y, X\rangle \, d\sigma \\ & + \int_0^1 \sigma^2 \langle C(t)J_G(\sigma X)XY, X\rangle \, d\sigma + \int_0^1 \sigma \langle C(t)G(\sigma X)X, Y\rangle \, d\sigma \\ & = \int_0^1 \sigma \left\langle \dot{C}\ (t)G(\sigma X)X, X \right\rangle d\sigma + \int_0^1 \langle \sigma C(t)G(\sigma X)Y, X\rangle \, d\sigma \\ & + \int_0^1 \sigma \frac{\partial}{\partial\sigma} \langle \sigma C(t)G(\sigma X)Y, X\rangle \, d\sigma \\ & = \int_0^1 \sigma \left\langle \dot{C}\ (t)G(\sigma X)X, X \right\rangle d\sigma + \sigma^2 \langle C(t)G(\sigma X)Y, X\rangle \Big|_0^1 \\ & = \int_0^1 \sigma \left\langle \dot{C}\ (t)G(\sigma X)X, X \right\rangle d\sigma + \langle C(t)G(X)Y, X\rangle \, .\end{aligned}$$

By noting the assumptions of the theorem, it follows that

$$\begin{aligned}\dot{V} = \quad & \langle Z, Z\rangle + \langle X, B(t)H(X,Y,Z,W)Z\rangle + \langle D(t)F(X), X\rangle \\ & \qquad + \int_0^1 \left\langle \sigma\ \dot{C}\ (t)G(\sigma X)X, X \right\rangle d\sigma \\ & \geq \langle Z, Z\rangle + \langle X, B(t)H(X,Y,Z,W)Z\rangle + \langle D(t)F(X), X\rangle \, .\end{aligned}$$

Since $\frac{\partial}{\partial\sigma}F(\sigma X) = J_F(\sigma X)X$ and $F(0) = 0$, then

$$F(X) = \int_0^1 J_F(\sigma X)X d\sigma.$$

Therefore, the assumptions of the theorem show that

$$\langle D(t)F(X), X\rangle = \int_0^1 \langle D(t)J_F(\sigma X)X, X\rangle \, d\sigma \geq a_4 \int_0^1 \langle X, X\rangle \, d\sigma = a_4 \langle X, X\rangle \, .$$

Hence

$$\begin{aligned}\dot{V} &\geq \left\|Z+\tfrac{1}{2}B(t)H(X,Y,Z,W)X\right\|^2\\ &\quad -\tfrac{1}{4}\langle B(t)H(X,Y,Z,W)X, B(t)H(X,Y,Z,W)X\rangle + a_4\langle X,X\rangle\\ &\geq a_4\langle X,X\rangle - \tfrac{1}{4}\langle B(t)H(X,Y,Z,W)X, B(t)H(X,Y,Z,W)X\rangle\\ &\geq a_4\|X\|^2 - \tfrac{1}{4}a_2^2\|X\|^2 = \left(a_4-\tfrac{1}{4}a_2^2\right)\|X\|^2 > 0.\end{aligned}$$

Thus, the assumptions of the theorem show that $\dot{V}\geq 0$ for all $t\geq 0$, that is, $\dot{V}$ is positive semi-definite. Furthermore, $\dot{V}=0$ $(t\geq 0)$ necessarily implies that $Y=0$ for all $t\geq 0$, and therefore also that $X=\xi$ (a constant vector),

$$Z=\dot{Y}=0,\ W=\ddot{Y}=0,\ \ \dot{W}=\dddot{Y}=0\ \text{ for all } t\geq 0.$$

The substitution of the estimates

$$X=\xi, Y=Z=W=0$$

in the system (1.2) leads to the result $F(\xi)=0$ which by assumption (ii) of the theorem implies (only) that $\xi=0$. For this reason $\dot{V}=0$ $(t\geq 0)$ implies that

$$X=Y=Z=W=0 \ \text{ for all } t\geq 0.$$

Therefore, the function V has the entire requisite Krasovskii criterion [9] if the conditions of the theorem hold. Thus, the basic properties of the function V, which are proved just above verify that the zero solution of the system (1.2) is unstable. (See Theorem 1.15 in Reissig and et al [14] and Krasovskii [9]). The system of equations (1.2) is equivalent to the differential equation (1.1). This complates the proof of the theorem. □

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