

ON A BOUNDARY VALUE PROBLEM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

New sufficient conditions of the existence and uniqueness of the solution of a boundary problem for a system of ordinary differential equations with certain functional boundary conditions are constructed by the method of a priori estimates.

Introduction

In this paper we give new sufficient conditions for the existence and the uniqueness of the solution of the problem

$$x'_i(t) = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n) \quad (1)$$

$$\Phi_{0i}(x_i) = \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n) \quad (2)$$

where for each $i \in \{1, \dots, n\}$ $f_i : \langle a, b \rangle \times R^n \rightarrow R$ satisfies the Carathéodory conditions, Φ_{0i} - the linear nondecreasing continuous functional on $C(\langle a, b \rangle)$ is concentrated on $\langle a_i, b_i \rangle \subseteq \langle a, b \rangle$ (i.e. the value of Φ_{0i} depends only on functions

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restricted to $\langle a_i, b_i \rangle$ and the segment can be degenerated to a point) and φ_i is a continuous functional on $C_n(\langle a, b \rangle)$. In general $\Phi_{0i}(1) = C_i$ ($i = 1, \dots, n$). Without loss of generality we can suppose $\Phi_{0i} = 1$ ($i = 1, \dots, n$), which simplifies the notation.

Special cases of the conditions (2) are presented by the series of formerly investigated problems e.g.

$$x_i(t_i) = \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n) \quad (3)$$

$$\Phi_{0i}(x_i) = c_i \quad (c_i \in R) \quad (i = 1, \dots, n) \quad (4)$$

and more specialised - Cauchy-Nicoletti problem

$$x_i(t_i) = c_i \quad (c_i \in R) \quad (i = 1, \dots, n) \quad (5)$$

or periodical problem

$$x_i(a) = x_i(b) \quad (i = 1, \dots, n) \quad (6)$$

Problems (1), (5) and (1), (6) were studied in the papers [4], [5]. Problem (1), (3) was studied in [5], [6], [8] and [9], problem (1), (4) in [2], [3], similar results are also published in [1].

Main result

We adopt the following notation:

$\langle a, b \rangle$ - a segment, $-\infty < a \leq a_i \leq b_i \leq b < +\infty$ ($i = 1, \dots, n$), \mathbb{R}^n , the n -dimensional real space with points $x = (x_i)_{i=1}^n$ normed by $\|x\| = \sum_{i=1}^n |x_i|$,

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\},$$

$C_n(\langle a, b \rangle)$ and $AC_n(\langle a, b \rangle)$ are, respectively, the spaces of continuous and absolutely continuous n -dimensional vector-valued functions on $\langle a, b \rangle$ with the norm

$$\|x\|_{C_n(\langle a, b \rangle)} = \max \left\{ \sum_{i=1}^n |x_i(t)| : a \leq t \leq b \right\},$$

$$C^+(\langle a, b \rangle) = \{x \in C(\langle a, b \rangle) : x(t) \geq 0, a \leq t \leq b\}$$

$L^p(\langle a, b \rangle)$ is space of functions integrable on $\langle a, b \rangle$ in p -th power with the norm

$$\|u\|_{L^p} = \begin{cases} \left[\int_a^b |u(t)|^p dt \right]^{1/p} & \text{for } 1 \leq p < \infty \\ \text{vrai max}\{|u(t)| : a \leq t \leq b\} & \text{for } p = +\infty \end{cases}$$

If $x = (x_i(t))_{i=1}^n \in C_n(\langle a, b \rangle)$ and $y = (y_i(t))_{i=1}^n \in C_n(\langle a, b \rangle)$, then $x \leq y$ if and only if $x_i(t) \leq y_i(t)$ for all $t \in \langle a, b \rangle$ and $i = 1, \dots, n$. $K(\langle a, b \rangle)$ is the set of functions $g : \langle a, b \rangle \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying local Carathéodory conditions, i.e. if $g \in K(\langle a, b \rangle)$, $g(\cdot, x)$ is measurable on $\langle a, b \rangle$ for each $x \in \mathbb{R}^n$, $g(t, \cdot)$ is continuous in \mathbb{R}^n for almost all $t \in \langle a, b \rangle$, and

$$\sup\{|g(\cdot, x)| : \|x\| \leq \rho\} \in L(\langle a, b \rangle) \text{ for } \rho \in (0, +\infty)$$

Let us consider the problem (1), (2). Under the solution we understand absolutely continuous n -dimensional vector-valued function on $\langle a, b \rangle$, which satisfies the equation (1) for almost all $t \in \langle a, b \rangle$ and fulfils the boundary conditions (2).

Definition Let $G = (g_i)_{i=1}^n : C(\langle a, b \rangle) \rightarrow \mathbb{R}^n$, $H = (h_{ij})_{i,j=1}^n : \langle a, b \rangle \rightarrow \mathbb{R}_+^{n \times n}$ and $\Psi = (\psi_i)_{i=1}^n : C_n(\langle a, b \rangle) \rightarrow \mathbb{R}_+^n$ is a positively homogeneous nondecreasing operator. We say that

$$(G, H, \Psi) \in Nic_0(\langle a, b \rangle; a_1, \dots, a_n, b_1, \dots, b_n) \quad (7)$$

if the system of differential inequalities

$$|x'_i(t) - g_i(t)x_i(t)| \leq \sum_{j=1}^n h_{ij}(t)|x_j(t)| \text{ for } a \leq t \leq b \quad (i = 1, \dots, n) \quad (8)$$

with boundary conditions

$$\min\{|x_i(t)| : a_i \leq t \leq b_i\} \leq \psi_i(|x_1(t)|, \dots, |x_n(t)|) \quad (i = 1, \dots, n) \quad (9)$$

has only trivial solution

Theorem 1. *Let the inequalities*

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - g_i(t)x_i] \text{ sign } x_i &\leq \sum_{j=1}^n h_{ij}(t)|x_j| + \omega_i\left(t, \sum_{j=1}^n |x_j|\right) \\ \text{if } t \in \langle a_i, b \rangle, x \in \mathbb{R}^n &\quad (i = 1, \dots, n) \end{aligned} \quad (10_1)$$

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - g_i(t)x_i] \text{ sign } x_i &\geq -\sum_{j=1}^n h_{ij}(t)|x_j| - \omega_i\left(t, \sum_{j=1}^n |x_j|\right) \\ \text{if } t \in \langle a, b_i \rangle, x \in \mathbb{R}^n &\quad (i = 1, \dots, n) \end{aligned} \quad (10_2)$$

$$|\varphi_i(x_1, \dots, x_n)| \leq \psi_i(|x_1|, \dots, |x_n|) + r_i \left(\sum_{j=1}^n |x_j| \right) \quad (11)$$

for all $x = (x_i)_{i=1}^n \in C_n(\langle a, b \rangle)$ $(i = 1, \dots, n)$.

hold, where $G = (g_i)_{i=1}^n$, $H = (h_{ij})_{i,j=1}^n$ and $\Psi = (\psi_i)_{i=1}^n$ satisfy the condition (7), the functions $\omega_i : \langle a, b \rangle \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ($i = 1, 2, \dots, n$) are measurable with regard to the first and nondecreasing to the second argument, $r_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are nondecreasing and

$$\lim_{\varrho \rightarrow +\infty} \frac{1}{\varrho} \int_a^b \omega_i(t, \varrho) dt = 0 = \lim_{\varrho \rightarrow +\infty} \frac{1}{\varrho} r_i(\varrho) \quad (i = 1, \dots, n) \quad (12)$$

then the problem (1), (2) has at least one solution.

For the proof of the Theorem 1 we need two following assertions and the first is similar to lemma 4.1 from [4] about differential inequality with boundary conditions of Cauchy type.

Lemma 1. Let $g_i^*(t, y_1, \dots, y_n) \in K(\langle a, b \rangle)$, $g_i^*(t, y_1, \dots, y_n) \text{sign}(t - t_i)$ be nondecreasing to arguments $y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n$ and each solution of the problem

$$y_i' = g_i^*(t, y_1, \dots, y_n) \quad (i = 1, \dots, n) \quad (13)$$

$$y_i(t_i) = c_i \quad (i = 1, \dots, n) \quad (14)$$

where $t_i \in \langle a, b \rangle$, $c_i \in \mathbb{R}$ ($i = 1, \dots, n$) can be extended in the whole segment $\langle a, b \rangle$. Then for each solution $(x_i(t))_{i=1}^n \in AC_n(\langle a, b \rangle)$ of the problem (15), (16)

$$\begin{aligned} x_i'(t) \text{sign}(t - t_i) &\leq g_i^*(t, x_1, \dots, x_n) \times \\ &(\geq) \\ &\times \text{sign}(t - t_i) \text{ for } a \leq t \leq b \quad (i = 1, \dots, n) \end{aligned} \quad (15)$$

$$\begin{aligned} x_i(t_i) &\leq c_i \quad (i = 1, \dots, n) \\ &(\geq) \end{aligned} \quad (16)$$

there exists a solution $(y_i)_{i=1}^n$ defined in the segment $\langle a, b \rangle$ of the problem (13), (14) such that

$$\begin{aligned} x_i(t) &\leq y_i(t) \text{ for } a \leq t \leq b \quad (i = 1, \dots, n) \\ &(\geq) \end{aligned} \quad (17)$$

Lemma 2. Let the condition (7) be satisfied. Then there exists a constant $\varrho > 0$ such that the estimate

$$\|x\|_{C_n(\langle a, b \rangle)} \leq \varrho \left[r_0 + \int_a^b \omega_0(t) dt \right] \quad (18)$$

holds for each constant $r_0 \geq 0$, $\omega_0 \in L(\langle a, b \rangle, \mathbb{R}_+)$ and for each solution $x \in AC_n(\langle a, b \rangle)$ of the differential inequalities

$$\begin{aligned} [x'_i(t) - g_i(t)x_i(t)] \text{ sign } x_i(t) &\leq \\ &\leq \sum_{j=1}^n h_{ij}(t)|x_j(t)| + \omega_0(t) \text{ if } a_i \leq t \leq b \quad (i = 1, \dots, n) \end{aligned} \quad (19_1)$$

$$\begin{aligned} [x'_i(t) - g_i(t)x_i(t)] \text{ sign } x_i(t) &\geq \\ &\geq - \sum_{j=1}^n h_{ij}(t)|x_j(t)| - \omega_0(t) \text{ if } a \leq t \leq b_i \quad (i = 1, \dots, n) \end{aligned} \quad (19_2)$$

with boundary conditions

$$\begin{aligned} \min\{|x_i(t)| : a_i \leq t \leq b_i\} &\leq \\ &\leq \psi_i(|x_1(t)|, \dots, |x_n(t)|) + r_0 \quad (i = 1, \dots, n) \end{aligned} \quad (20)$$

Proof. By contradiction let $r_k \in \mathbb{R}_+$, $\omega_k \in L(\langle a, b \rangle, \mathbb{R}_+)$ and $x_k = (x_{ik})_{i=1}^n \in AC_n(\langle a, b \rangle)$ exist for any natural k , such that

$$\|x_k\|_{C_n(\langle a, b \rangle)} \geq k \left[r_k + \int_a^b \omega_k(t) dt \right]$$

$$\begin{aligned} [x'_{ik}(t) - g_i(t)x_{ik}(t)] \text{ sign } x_{ik}(t) &\leq \sum_{j=1}^n h_{ij}(t)|x_{jk}(t)| + \omega_k(t) \\ &\text{if } a_i \leq t \leq b \quad (i = 1, \dots, n) \end{aligned} \quad (21_1)$$

$$\begin{aligned} [x'_{ik}(t) - g_i(t)x_{ik}(t)] \text{ sign } x_{ik}(t) &\geq - \sum_{j=1}^n h_{ij}(t)|x_{jk}(t)| - \omega_k(t) \\ &\text{if } a \leq t \leq b_i \quad (i = 1, \dots, n) \end{aligned} \quad (21_2)$$

and

$$\min\{|x_{ik}(t)| : a_i \leq t \leq b_i\} \leq \psi_i(|x_{1k}|, \dots, |x_{nk}|) + r_k \quad (i = 1, \dots, n) \quad (22)$$

We denote

$$\begin{aligned} \tilde{x}_{ik}(t) &= \frac{x_{ik}(t)}{\|x_k\|_{C_n(\langle a, b \rangle)}} \quad (i = 1, \dots, n), \\ \tilde{\omega}_k(t) &= \frac{\omega_k(t)}{k \left[r_k + \int_a^b \omega_k(t) dt \right]}, \end{aligned} \quad (1)$$

we get

$$\|\tilde{x}_k\|_{C_n((a,b))} = 1 \text{ and } \|\tilde{w}_k\|_{L((a,b))} \leq \frac{1}{k} \quad (23)$$

On the other hand according to (21), (22)

$$\begin{aligned} & [\tilde{x}'_{ik}(t) - g_i(t)\tilde{x}_{ik}(t)] \text{ sign } \tilde{x}_{ik}(t) \leq \\ & \leq \sum_{j=1}^n h_{ij}(t)|\tilde{x}_{jk}(t)| + \tilde{\omega}_k(t) \text{ if } a_i \leq t \leq b \quad (i = 1, \dots, n) \end{aligned} \quad (24_1)$$

$$\begin{aligned} & [\tilde{x}'_{ik}(t) - g_i(t)\tilde{x}_{ik}(t)] \text{ sign } \tilde{x}_{ik}(t) \geq \\ & \geq - \sum_{j=1}^n h_{ij}(t)|\tilde{x}_{jk}(t)| - \tilde{\omega}_k(t) \text{ if } a \leq t \leq b_i \quad (i = 1, \dots, n) \end{aligned} \quad (24_2)$$

and

$$\begin{aligned} & \min\{|\tilde{x}_{ik}(t)| : a_i \leq t \leq b_i\} \leq \\ & \leq \psi_i(|\tilde{x}_{1k}|, \dots, |\tilde{x}_{nk}|) + \frac{1}{k} \quad (i = 1, \dots, n) \end{aligned} \quad (25)$$

Now for any $i \in \{1, \dots, n\}$ and a natural k we choose a point $t_{ik} \in (a_i, b_i)$ such that

$$|\tilde{x}_{ik}(t_{ik})| = \min\{|\tilde{x}_{ik}(t)| : a_i \leq t \leq b_i\} \quad (26)$$

then from (24), (25) and (26), we have

$$\begin{aligned} & [\tilde{x}'_{ik}(t) - g_i(t)\tilde{x}_{ik}(t)] \text{ sign } [(t - t_{ik})\tilde{x}_{ik}(t)] \leq \\ & \leq \sum_{j=1}^n h_{ij}(t)|\tilde{x}_{jk}(t)| + \tilde{\omega}_k(t) \quad (27) \\ & \text{for } a \leq t \leq b \quad (i = 1, \dots, n) \end{aligned}$$

and

$$|\tilde{x}_{ik}(t_{ik})| \leq \psi_i(|\tilde{x}_{1k}|, \dots, |\tilde{x}_{nk}|) + \frac{1}{k} \quad (i = 1, \dots, n) \quad (28)$$

Let $(y_{ik})_{i=1}^n$ be the solution of the Cauchy-Nicoletti problem

$$\begin{aligned} & y'_{ik}(t) = g_i(t)y_{ik}(t) + \left[\sum_{j=1}^n h_{ij}(t)|\tilde{x}_{jk}(t)| + \right. \\ & \left. + \tilde{\omega}_k(t) \right] \text{ sign } (t - t_{ik}) \quad (i = 1, \dots, n) \end{aligned} \quad (29)$$

$$y_{ik}(t_{ik}) = |\tilde{x}_{ik}(t_{ik})| \quad (i = 1, \dots, n) \quad (30)$$

then according to Lemma 1 and to the condition (27)

$$|\tilde{x}_{ik}(t)| \leq y_{ik}(t) \text{ for } a \leq t \leq b \quad (i = 1, \dots, n) \quad (31)$$

Formulae (29), (30) and (31) yield

$$\begin{aligned} y_{ik}(t) &\leq \exp\left(\int_{t_{ik}}^t g_i(s) ds\right) |\tilde{x}_{ik}(t_{ik})| + \\ &+ \left| \int_{t_{ik}}^t e^{\int_{\tau}^t g_i(s) ds} \left[\sum_{j=1}^n h_{ij}(\tau) |\tilde{x}_{ik}(\tau)| + \tilde{\omega}_k(\tau) \right] d\tau \right| \end{aligned} \quad (32)$$

and

$$\begin{aligned} y_{ik}(t) &\leq \exp\left(\int_{t_{ik}}^t g_i(s) ds\right) y_{ik}(t_{ik}) + \\ &+ \left| \int_{t_{ik}}^t e^{\int_{\tau}^t g_i(s) ds} \left[\sum_{j=1}^n h_{ij}(\tau) y_{ik}(\tau) + \tilde{\omega}_k(\tau) \right] d\tau \right| \end{aligned} \quad (33)$$

According to (23), (29) and (32), we obtain

$$|y_{ik}(t)| \leq r \text{ for } a \leq t \leq b, \quad (i = 1, \dots, n) \quad (k = 1, 2, \dots) \quad (34)$$

and

$$|y'_{ik}(t)| \leq \tilde{g}(t) + \tilde{\omega}_k(t) \text{ for } a \leq t \leq b \quad (i = 1, \dots, n), \quad (k = 1, \dots, n) \quad (35)$$

where

$$r = \left(2 + \sum_{i,j=1}^n \int_a^b h_{ij}(t) dt \right) \max \left\{ \exp \int_a^b |g_i(t)| dt : i = 1, \dots, n \right\}$$

and

$$\tilde{g}(t) = \sum_{i,j=1}^n h_{ij}(t) + r \max \{ |g_i(t)| : i = 1, \dots, n \}$$

Formulae (23), (28), (30) and (31) imply, that

$$\|(y_{ik})_{i=1}^n\|_{C_n((a,b))} \geq 1, \quad (k = 1, 2, \dots) \quad (36)$$

$$|y_{ik}(t_{ik})| \leq \psi_i(y_{ik}, \dots, y_{nk}) + \frac{1}{k} \quad (i = 1, \dots, n), \quad (k = 1, 2, \dots) \quad (37)$$

From (34) and (35), it follows that the sequences $\{y_{ik}\}_{k=1}^{\infty}$ ($i = 1, \dots, n$) are uniformly bounded and uniformly continuous. According to the Lemma of Arzela-Ascoli, we can suppose without the loss of generality that these sequences uniformly converge. The sequences of points $\{t_{ik}\}_{k=1}^{\infty}$ ($i = 1, \dots, n$) can be taken convergent as well. Denoting

$$\lim_{k \rightarrow +\infty} t_{ik} = t_{i0} \quad (i = 1, \dots, n) \quad (38)$$

and

$$\lim_{k \rightarrow +\infty} y_{ik}(t) = y_{i0}(t) \quad (i = 1, \dots, n)$$

Clearly

$$t_{i0} \in \langle a_i, b_i \rangle \quad (i = 1, \dots, n) \quad (39)$$

Passing to the limit in the inequalities (33) and (37), using (23) we obtain

$$y_{i0}(t) \leq \exp\left(\int_{t_{i0}}^t g_i(s) ds\right) y_{i0}(t_{i0}) + \left| \int_{t_{i0}}^t e^{\int_{t_{i0}}^{\tau} g_i(s) ds} \left[\sum_{j=1}^n h_{ij}(\tau) y_{i0}(\tau) \right] d\tau \right| \quad (i = 1, \dots, n) \quad (40)$$

$$y_{i0}(t_{i0}) \leq \psi_i(y_{i0}, \dots, y_{n0}) \quad (i = 1, \dots, n) \quad (41)$$

Let us introduce the functions

$$y_i(t) = \exp\left(\int_{t_{i0}}^t g_i(s) ds\right) y_{i0}(t_{i0}) + \left| \int_{t_{i0}}^t e^{\int_{t_{i0}}^{\tau} g_i(s) ds} \left[\sum_{j=1}^n h_{ij}(\tau) y_{i0}(\tau) \right] d\tau \right| \quad (i = 1, \dots, n),$$

then

$$y_i(t_{i0}) = y_{i0}(t_{i0}) \quad (i = 1, \dots, n) \quad (42)$$

and

$$y_i'(t) = g_i(t) y_i(t) + \left[\sum_{j=1}^n h_{ij}(t) y_{i0}(t) \right] \times \times \text{sign}(t - t_{i0}) \quad (i = 1, \dots, n) \quad (43)$$

From (39) - (43) it follows that $(y_i(t))_{i=1}^n$ is a solution of the problem (8), (9). Therefore according to the condition (7)

$$y_i(t) \equiv 0 \quad (i = 1, \dots, n)$$

On the other hand, (36) and (40) imply

$$\|(y_i(t))_{i=1}^n\|_{C_n((a,b))} \geq 1,$$

which is a contradiction and the lemma is proved. \square

Proof of Theorem 1. Let ϱ be a constant from Lemma 1. Firstly, we want to show that there exists a constant $\varrho_0 > 0$ such that

$$\varrho \left[r(2\varrho_0) + \int_a^b \omega(t, 2\varrho_0) dt \right] \leq \varrho_0 \quad (44)$$

where for any $\eta \in (0, +\infty)$

$$r(\eta) = \max\{r_i(\eta) : i = 1, \dots, n\}$$

$$\omega(t, \eta) = \max\{\omega_i(t, \eta) : i = 1, \dots, n\}$$

Suppose (44) is not valid, then for any $\eta \in (0, +\infty)$

$$\eta \leq \varrho \left[r(2\eta) + \int_a^b \omega(t, 2\eta) dt \right]$$

On the other hand, (12) implies that for any $k \geq \varrho$ there exists a constant $\eta_0 > k$ such that

$$\varrho \left[r(2\eta) + \int_a^b \omega(t, 2\eta) dt \right] < \frac{\varrho}{k} \cdot \eta < \eta$$

for all $\eta \geq \eta_0$, which is a contradiction and (44) is valid.

Now we put

$$\chi(s) = \begin{cases} 1 & \text{if } |s| \leq \varrho_0 \\ 2 - \frac{|s|}{\varrho_0} & \text{if } \varrho_0 < |s| < 2\varrho_0 \\ 0 & \text{if } |s| \geq 2\varrho_0 \end{cases}$$

$$\tilde{f}_i(t, x_1, \dots, x_n) = \chi(\|x\|)[f(t, x_1, \dots, x_n) - g_i(t)x_i] \text{ for } x = (x_i)_{i=1}^n \in \mathbb{R}^n, \quad (i = 1, \dots, n) \quad (45)$$

$$\begin{aligned}\tilde{\varphi}_i(x_1, \dots, x_n) &= \chi(\|x\|_{C_n(\langle a, b \rangle)})\varphi_i(x_1, \dots, x_n) \\ \text{for } x &= (x_i)_{i=1}^n \in C_n(\langle a, b \rangle), \quad (i = 1, \dots, n)\end{aligned}\quad (46)$$

We consider the problem

$$y'_i = g_i(t)y_i + \tilde{f}_i(t, y_1, \dots, y_n), \quad (i = 1, \dots, n) \quad (47)$$

$$\Phi_{0i}(y_i) = \tilde{\varphi}_i(y_1, \dots, y_n) \quad (i = 1, \dots, n) \quad (48)$$

From (45) and (46), it follows immediately that $\tilde{f}_i : \langle a, b \rangle \times \mathbb{R}^n \rightarrow R$ ($i = 1, \dots, n$) satisfy the local Carathéodory conditions, $\tilde{\varphi}_i : C_n(\langle a, b \rangle) \rightarrow R$ ($i = 1, \dots, n$) are continuous functionals,

$$\begin{aligned}f_{0i}(t) &= \text{suf}\{|\tilde{f}_i(t, x_1, \dots, x_n)| : (x_i)_{i=1}^n \in \mathbb{R}^n\} \in \\ &\in L(\langle a, b \rangle) \quad (i = 1, \dots, n)\end{aligned}\quad (49)$$

and

$$r_i = \text{sup}\{|\tilde{\varphi}_i(x_1, \dots, x_n)| : x \in C_n(\langle a, b \rangle)\} < +\infty \quad (i = 1, \dots, n) \quad (50)$$

We want to show that the homogeneous problem

$$y'_i = g_i(t)y_i \quad (i = 1, \dots, n) \quad (47_0)$$

$$\Phi_{0i}(y_i) = 0 \quad (i = 1, \dots, n) \quad (48_0)$$

has only trivial solution. Let $\tilde{y} = (\tilde{y}_i)_{i=1}^n$ be an arbitrary solution of this problem. Then $\tilde{y}_i(t) = c_i \exp\left(\int_a^t g_i(\tau)d\tau\right)$, where $c_i = \text{const}$ ($i = 1, \dots, n$). According to (48₀)

$$c_i \cdot \Phi_{0i}\left(\exp \int_a^t g_i(\tau)d\tau\right) = 0 \quad (i = 1, \dots, n)$$

However, if Φ_{0i} ($i = 1, \dots, n$) are nondecreasing functionals and $\Phi_{0i}(1) = 1$ ($i = 1, \dots, n$), we have

$$\Phi_{0i}\left(\exp \int_a^t g_i(\tau)d\tau\right) \geq \exp\left(-\int_a^b |g_i(t)|dt\right)\Phi_{0i}(1) > 0 \quad (i = 1, \dots, n)$$

Consequently

$$\tilde{y}_i(t) \equiv 0 \quad (i = 1, \dots, n)$$

Using Lemma 2.1 from [3], we obtain that the conditions (49) and (50) and the unicity of trivial solution of the problem (47₀), (48₀) guarantee the existence of solutions of the problem (47), (48). Let $(y_i(t))_{i=1}^n$ be the solution of the problem (47), (48), then

$$\begin{aligned} [y'_i(t) - g_i(t)y_i(t)] \operatorname{sign} y_i(t) &= \tilde{f}_i(t, y_1(t), \dots, y_n(t)) \operatorname{sign} y_i(t) \\ &= \chi \left(\sum_{i=1}^n |y_i(t)| \right) [f_i(t, y_1(t), \dots, y_n(t)) - g_i(t)y_i(t)] \operatorname{sign} y_i(t) \\ &\quad \text{for } a \leq t \leq b, \quad (i = 1, \dots, n) \end{aligned}$$

and

$$\begin{aligned} \min\{|y_i(t)| : a_i \leq t \leq b_i\} &\leq |\Phi_{0i}(y_i(t))| = |\tilde{\varphi}_i(y_1, \dots, y_n)| \\ &= \chi(\|y\|_{C_n((a,b))}) |\varphi_i(y_1, \dots, y_n)| \quad (i = 1, \dots, n) \end{aligned}$$

From here taking into consideration (10_{1,2}) and (11), we obtain inequalities (19_{1,2}) and (20), where

$$\omega_0(t) = \chi \left(\sum_{i=1}^n |y_i(t)| \right) \omega \left(t, \sum_{i=1}^n |y_i(t)| \right) \leq \omega(t, 2\varrho_0)$$

and

$$r_0 = \chi(\|y\|_{C_n((a,b))}) r \left(\sum_{i=1}^n |y_i(t)| \right) \leq r(2\varrho_0)$$

Therefore by Lemma 2 and the inequality (44) we get

$$\|y\|_{C_n((a,b))} \leq \varrho [r(2\varrho_0) + \int_a^b \omega(t, 2\varrho_0) dt] \leq \varrho_0$$

Consequently $\chi(\sum_{i=1}^n |y_i(t)|) = 1$ when $a \leq t \leq b$ and

$$\chi(\|y\|_{C_n((a,b))}) = 1$$

Putting these equalities into (45) - (48), we obtain that $(y_i)_{i=1}^n$ is a solution of the problem (1), (2). The Theorem 1 is proved. \square

Theorem 2. *Let the inequalities*

$$\begin{aligned} &\{[f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] - \\ &-g_i(t)[x_{1i} - x_{2i}]\} \operatorname{sign} [x_{1i} - x_{2i}] \leq \sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}| \end{aligned} \quad (51_1)$$

$$\begin{aligned}
& \text{if } a_i \leq t \leq b, \quad x_1 = (x_{1i})_{i=1}^n, \quad x_2 = (x_{2i})_{i=1}^n \in \mathbb{R}^n \quad (i = 1, \dots, n), \\
& \quad \{[f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] - \\
& \quad - g_i(t)[x_{1i} - x_{2i}]\} \operatorname{sign} [x_{1i} - x_{2i}] \geq - \sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}| \quad (51_2)
\end{aligned}$$

$$\begin{aligned}
& \text{if } a \leq t \leq b_i, \quad x_1 = (x_{1i})_{i=1}^n, \quad x_2 = (x_{2i})_{i=1}^n \in \mathbb{R}^n \quad (i = 1, \dots, n) \\
& \quad |\varphi_i(x_{11}, \dots, x_{1n}) - \varphi_i(x_{21}, \dots, x_{2n})| \leq \psi_i(|x_{11} - x_{12}|, \dots, |x_{1n} - x_{2n}|) \quad (52) \\
& \quad \text{for all } x_1 = (x_{1i})_{i=1}^n, x_2 = (x_{2i})_{i=1}^n \in C_n((a, b)) \quad (i = 1, \dots, n)
\end{aligned}$$

hold, where $G = (g_i)_{i=1}^n$, $H = (h_{ij})_{i,j=1}^n$ and $\Psi = (\psi_i)_{i=1}^n$ satisfy the condition (7). Then the problem (1), (2) has unique solution.

Proof. From (51_{1,2}) and (52) the conditions (10_{1,2}) and (11) follow, where $\omega_i(t, \varrho) = |f_i(t, 0, \dots, 0)|$ and $r_i(\varrho) = |\varphi_i(0, \dots, 0)|$ ($i = 1, \dots, n$). Therefore, by Theorem 1 the problem (1), (2) has a solution. We shall prove its uniqueness.

Let $(x_{1i})_{i=1}^n$ and $(x_{2i})_{i=1}^n$ be arbitrary solutions of the problem (1), (2). Let us put

$$y_i(t) = x_{1i}(t) - x_{2i}(t) \quad (i = 1, \dots, n)$$

The assumptions (51_{1,2}) guarantee that vector function $(y_i)_{i=1}^n$ is a solution of the system of the differential inequalities (8) satisfying the conditions

$$|\Phi_{0i}(y_i)| \leq \psi_i(|y_1|, \dots, |y_n|) \quad (i = 1, \dots, n)$$

However

$$\begin{aligned}
|\Phi_{0i}(y_i)| & \geq \Phi_{0i}(1) \min\{|y_i(t)| : a_i \leq t \leq b_i\} = \\
& = \min\{|y_i(t)| : a_i \leq t \leq b_i\} \quad (i = 1, \dots, n)
\end{aligned}$$

Consequently the inequalities (9) are satisfied and according to the condition (7) the equalities

$$y_i(t) \equiv 0 \quad (i = 1, \dots, n)$$

hold, i.e. $(x_{1i})_{i=1}^n = (x_{2i})_{i=1}^n$. Theorem is proved. \square

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