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SEMIDIRECT PRODUCT OF A MONOID AND A Γ-SEMIGROUP

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Abstract

In this paper the notion of the semidirect product of a monoid and a Γ-semigroup has been introduced and studied. Necessary and sufficient conditions for this semidirect product to be right (left) orthodox Γ-semigroup and right (left) inverse Γ-semigroup has been obtained.

1. Introduction

The notion of Γ-semigroup has been introduced by Sen and Saha [5] in the yaer 1986. Many classical notions of semigroup has been extended to Γ-semigroup. In [3]and [4] we have introduced the notions of right inverse Γ-semigroup and right orthodox Γ-semigroup. In the present paper we have introduced the notions of semidirect product of a monoid and a Γ-semigroup which may be considered as a generalization of the semidirect product of a monoid and a semigroup introduced by T. Saito [2]. We obtain necessary and sufficient conditions for the semidirect product of the monoid and a Γ-semigroup to be right (left) orthodox Γ-semigroup and right (left) inverse Γ-semigroup.

Key words: Γ-semigroup, regular Γ-semigroup, orthodox Γ-semigroup, right orthodox Γ-semigroup, orthodox semigroup, right inverse semigroup.
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2. Preliminaries

Definition 2.1 [5] Let $M = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. M is called a Γ-*semigroup* if $a\alpha b \in M$ for $\alpha \in \Gamma$ and $a, b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$.

Definition 2.2 [5] Let M be a Γ-semigroup. An element a of M is said to be *regular* if $a \in a\Gamma M\Gamma a$ where $a\Gamma M\Gamma a = \{a\alpha b\beta a : b \in M, \alpha, \beta \in \Gamma\}$. If all elements of M are regular then M is called a *regular* Γ-*semigroup.*

Definition 2.3 [5] let M be a Γ-semigroup. An element $e \in M$ is said to be an α-*idempotent* if $e\alpha e = e$ for some $\alpha \in \Gamma$.

Definition 2.4 [7] Let $a \in M$ and $\alpha, \beta \in \Gamma$. An element $b \in M$ is called (α, β)-*inverse* of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. In this case we write $b \in V_\alpha^\beta(a)$.

Definition 2.5 [4] A regular Γ-semigroup M is called a *right* (*left*) *orthodox* Γ-*semigroup* if for any α-idempotent e and β-idempotent f, $e\alpha f$ (resp. $f\alpha e$) is a β-idempotent.

Theorem 2.6 [4] *A regular* Γ-*semigroup* M *is a right orthodox* Γ-*semigroup if and only if for* $a, b \in M, V_{\alpha_1}^\beta(a) \cap V_\alpha^\beta(b) \neq \phi$ *for some* $\alpha, \alpha_1, \beta \in \Gamma$ *implies that* $V_{\alpha_1}^\delta(a) = V_\alpha^\delta(b)$ *for all* $\delta \in \Gamma$.

Definition 2.7 [3] A regular Γ-semigroup is called a *right* (*left*) *inverse* Γ-*semigroup* if for any α-idempotent e and for any β-idempotent f, $e\alpha f\beta e = f\beta e$ ($e\beta f\alpha e = e\beta f$).

Definition 2.8 [9] A regular Γ-semigroup S is said to be an *orthodox semigroup* if $E(S)$, the set of all idempotents of S forms a subsemigroup of S.

Definition 2.9 [8] A regular semigroup S is said to be a right (left) inverse semigroup if for any $e, f \in E(S)$, $efe = fe(efe = ef)$.

3. Semidirect product of a monoid and a Γ-semigroup

Definition 3.1 Let S be a monoid and T be a Γ-semigroup. Let $End(T)$ denote the set of all endomorphisms on T i.e., the set of all mappings $f : T \to T$ satisfying $(a\alpha b)f = af\alpha bf$ for all $a, b \in T$, $\alpha \in \Gamma$. Again let $\phi : S \to End(T)$ be a given 1-preserving homomorphism. If $s \in S$ and $t \in T$, we write t^s for $(t)\phi(s)$. Let $S \times_\phi T = \{(s, t) : s \in S, t \in T\}$. We define a multiplication on

$S \times_\phi T$ by $(s_1, t_1)\,\alpha(s_2, t_2) = (s_1 s_2, t_1^{s_2}\,\alpha t_2)$. Then $S \times_\phi T$ is a Γ-semigroup. This Γ-semigroup $S \times_\phi T$ is called the semidirect product of the monoid S and the Γ-semigroup T.

Lemma 3.2 *Let S be a monoid and T be a Γ-semigroup, $\phi : S \to End(T)$ be a given 1-preserving homomorphism. Then if the semideirect product is*

(i) *right (left) orthodox Γ-semigroup then S is an orthodox semigroup and T is a right (left) orthodox Γ-semigroup.*

(ii) *right (left) inverse Γ-semigroup then S is a right (left) inverse semigroup and T is a right (left) inverse Γ-semigroup.*

Proof (i) Suppose that $S \times_\phi T$ is a right orthodox Γ-semigroup. Since it is regular, for $(s, t) \in S \times_\phi T$, there exists $(s', t') \in S \times_\phi T$ and $\alpha, \beta \in \Gamma$ such that

$$(s, t) = (s, t)\alpha(s', t')\beta(s, t) = (ss's, t^{s's}\alpha(t')^s\beta t)$$

and

$$(s', t') = (s', t')\beta(s, t)\alpha(s', t') = (s'ss', (t')^{ss'}\beta t^{s'}\alpha t').$$

This implies $s' \in V(s)$. Again if we take $s = 1$ then $s' = 1$ and we get $t' \in V_\alpha^\beta(t)$. Thus S is a regular semigroup and T is a regular Γ-semigroup.

Let t_1 be an α-idempotent and t_2 be a β-idempotent in T and $e, g \in E(S)$. Then $(1, t_1)\alpha(1, t_1) = (1, t_1\alpha t_1) = (1, t_1)$. Hence $(1, t_1)$ is an α-idempotent in $S\times_\phi T$. Similarly $(1, t_2)$ is a β-idempotent in $S\times_\phi T$. Now $\Big(1, (t_1\alpha t_2)\beta(t_1\alpha t_2)\Big) = (1, t_1\alpha t_2)\beta(1, t_1\alpha t_2) = \Big((1, t_1)\alpha(1, t_2)\Big)\beta\Big((1, t_1)\alpha(1, t_2)\Big) = (1, t_1)\alpha(1, t_2) = (1, t_1\alpha t_2)$. Thus $t_1\alpha t_2$ is a β-idempotent. So, T is a right orthodox Γ-semigroup.

Again (e, t_1^e) is an α-idempotent since $(t_1^e)^e = t_1^e$ and (g, t_2^g) is a β-idempotent. Now $(eg, t_1^{eg}\alpha t_2^g) = (e, t_1^e)\alpha(g, t_2^g) = \Big((e, t_1^e)\alpha(g, t_2^g)\Big)\,\beta\Big((e, t_1^e)\alpha(g, t_2^g)\Big) = (eg, t_1^{eg}\alpha t_2^g)$ $\beta(eg, t_1^{eg}\alpha t_2^g) = \Big((eg)^2, (t_1^{eg}\ \alpha t_2^g)\beta(t_1^{eg}\alpha t_2^g)\Big)$. Hence $(eg)^2 = eg$. So, S is an orthodox semigroup.

(ii) Let $S \times_\phi T$ is a right inverse Γ-semigroup. Then by (i) S is a regular semigroup and T is a regular Γ-semigroup. Let t_1 be an α-idempotent and t_2 is a β-idempotent in T. Let $e, g \in E(S)$. Since $S \times_\phi T$ is a right inverse Γ-semigroup, $(1, t_1)$ is an α-idempotent and $(1, t_2)$ be a β-idempotent, we have $(1, t_1\alpha t_2\beta t_1) = (1, t_1)\alpha(1, t_2)\beta(1, t_1) = (1, t_2)\beta(1, t_1) = (1, t_2\beta t_1)$ which shows that $t_1\alpha t_2\beta t_1 = t_2\beta t_1$. So T is a right inverse Γ-semigroup. Again, $(ge, t_2^{ge}\beta t_1^e) = (g, t_2^g)\beta(e, t_1^e) = (e, t_1^e)\ \alpha(g, t_2^g)\beta(e, t_1^e) = (ege, t_1^{ege}\alpha t_2^{ge}\beta t_1^e)$ shows $ge = ege$ for $e, g \in E(S)$. Thus S is a right inverse semigroup.

Lemma 3.3 *Let $S \times_\phi T$ be the semidirect product of a monoid S and a Γ-semigroup T corresponding to a given 1-preserving homomorphism $\phi : S \to End(T)$ and let $(s, t) \in S \times_\phi T$, then*

(i) *if* $(s', t') \in V_{\alpha}^{\beta}\Big((s, t)\Big)$ *then* $(s', t') \in V_{\alpha}^{\beta}\Big((s, t^{s's})\Big)$. *In particular if* $s \in E(S)$, *then* $\Big(s, (t')^{s}\beta t^{s's}\alpha t'\Big) \in V_{\alpha}^{\beta}\Big((s, t^{s's})\Big)$.

(ii) *if* t^{s} *is an* α-*idempotent and* $s' \in V(s)$, *then* $(s', t^{ss'}) \in V_{\alpha}^{\alpha}\Big((s, t^{s})\Big)$.

Proof (i) Since $(s', t') \in V_{\alpha}^{\beta}\Big((s, t)\Big)$ we have,

$$(s', t') = (s', t')\beta(s, t)\alpha(s', t') = \Big(s'ss', (t')^{ss'}\beta t^{s'}\alpha t'\Big)$$

and

$$(s, t) = (s, t)\alpha(s', t')\beta(s, t) = \Big(ss's, t^{s's}\alpha(t')^{s}\beta t\Big).$$

This shows that $s' \in V(s)$ and

$$t^{s's}\alpha(t')^{s}\beta t = t \qquad \ldots(1)$$

$$(t')^{ss'}\beta t^{s'}\alpha t' = t' \qquad \ldots(2)$$

From (1) we have, $\Big(t^{s's}\alpha(t')^{s}\beta t\Big)^{s's} = (t)^{s's}$ i.e., $t^{s's}\alpha(t')^{s}\beta t^{s's} = t^{s's}$ and from (2), $\Big((t')^{ss'}\beta t^{s'}\alpha t'\Big)^{s} = (t')^{s}$ i.e., $(t')^{s}\beta t^{s's}\alpha(t')^{s} = (t')^{s}$. Now we have $(s', t')\beta(s, t^{s's})\alpha(s', t') = \Big(s'ss', (t')^{ss'}\beta t^{s'ss'}\alpha t'\Big) = (s', t')$ [by (2)] and hence $(s, t^{s's})\alpha(s', t')\beta(s, t^{s's}) = \Big(ss's, t^{s'ss's}\alpha(t')^{s}\beta t^{s's}\Big) = \Big(s, t^{s's}\alpha(t')^{s}\beta t^{s's}\Big) = (s, t^{s's})$. Thus we have $(s', t') \in V_{\alpha}^{\beta}\Big((s, t^{s's})\Big)$. Again if $s \in E(S)$, $\Big((t')^{s}\beta t^{s's}\alpha t'\Big)^{s} = (t')^{s}\beta t^{s's}\alpha(t')^{s} = (t')^{s}$ and $(s, t^{s's})\alpha\Big(s, (t')^{s}\beta t^{s's}\alpha t'\Big)\beta(s, t^{s's}) = \left(sss, t^{s's}\alpha\Big((t')^{s}\beta t^{s's}\alpha t'\Big)^{s}\beta t^{s's}\right) = \Big(s, t^{s's}\alpha(t')^{s}\beta t^{s's}\Big) = (s, t^{s's})$ and $\Big(s, (t')^{s}\beta t^{s's}\alpha t'\Big)\beta\,(s, t^{s's})\alpha\Big(s, (t')^{s}\beta t^{s's}\alpha t'\Big) = \left(s, \Big((t')^{s}\beta t^{s's}\alpha t'\Big)^{s}\beta t^{s'ss}\alpha(t')^{s}\beta t^{s's}\alpha t'\right)$ $= \Big(s, (t')^{s}\beta t^{s's}\alpha(t')^{s}\ \beta t^{s's}\alpha t'\Big) = \Big(s, (t')^{s}\ \beta t^{s's}\alpha t'\Big)$. Hence $\Big(s, (t')^{s}\beta t^{s's}\alpha t'\Big) \in V_{\alpha}^{\beta}(s, t^{s's})$.

(ii) $(s, t^{s})\alpha(s', t^{ss'})\alpha(s, t^{s}) = (ss's, t^{ss's}\alpha t^{ss's}\alpha t^{s}) = (s, t^{s})$ since t^{s} is an α-idempotent and $(s', t^{ss'})\alpha(s, t^{s})\alpha(s', t^{ss'}) = (s'ss', t^{ss'ss'}\alpha t^{ss'}\alpha t^{ss'}) = (s', t^{ss'}\alpha t^{ss'}\alpha t^{ss'}) = \Big(s', (t^{s}\alpha t^{s}\alpha t^{s})^{s'}\Big) = (s', t^{ss'})$ i.e., $(s', t^{ss'}) \in V_{\alpha}^{\alpha}(s, t^{s})$.

Lemma 3.4 *Let* S *be a monoid and* T *be a* Γ-*semigroup and* $S \times_{\phi} T$ *be the semidirect product corresponding to a given 1-preserving homomorphism* $\phi : S \rightarrow End(T)$. *Moreover, let* $t \in t^{e}\Gamma T$ *for every* $e \in E(S)$ *and every* $t \in T$. *Then*

(i) (e,t) is an α-idempotent if and only if $e \in E(S)$ and t^e is an α-idempotent.

(ii) if (e,t) is an α-idempotent, then $(e,t^e) \in V_\alpha^\alpha\Big((e,t)\Big)$.

Proof (i) If (e,t) is an α-idempotent, then

$$(e,t) = (e,t)\alpha(e,t) = (e^2, t^e\alpha t) \text{ i.e., } e = e^2 \text{ and } t^e\alpha t = t \tag{3}$$

So, $t^e = (t^e\alpha t)^e = t^e\alpha t^e$ which implies that t^e is an α-idempotent. Conversely, let $e \in E(S)$ and t^e be an α-idempotent. Since $t \in t^e\Gamma T$, $t = t^e\beta t_1$ for some $\beta \in \Gamma$, $t_1 \in T$ and hence $t^e\alpha t = t^e\alpha t^e\beta t_1 = t$. Thus $(e,t)\alpha(e,t) = (e,t^e\alpha t) = (e,t)$ i.e., (e,t) is an α-idempotent.

(ii) If (e,t) is an α-idempotent, from (i) $e \in E(S)$ and t^e is an α-idempotent. Now $(e,t)\alpha(e,t^e)\alpha(e,t) = (e,t^e\alpha t^e\alpha t) = (e,t^e\alpha t) = (e,t)$ [from (3)] and $(e,t^e)\alpha(e,t)\alpha\ (e,t^e) = (e,t^e\alpha t^e\alpha t^e) = (e,t^e)$. Thus $(e,t^e) \in V_\alpha^\alpha\Big((e,t)\Big)$.

Theorem 3.5 *Let S be a monoid and T be a Γ-semigroup. Let $\phi : S \to End(T)$ be a given 1-preserving homomorphism. Then the semidirect product $S \times_\phi T$ is a right (left) orthodox Γ-semigroup if and only if*

(i) *S is an orthodox semigroup and T is a right (left) orthodox Γ-semigroup*

(ii) *for every $e \in E(S)$ and every $t \in T$, $t \in t^e\Gamma T$*

(iii) *if t^e is an α-idempotent, then t^{ge} is an α-idempotent for every $g \in E(S)$, where $e \in E(S), t \in T$.*

Proof Suppose $S \times_\phi T$ is a right orthodox Γ-semigroup. Then by Lemma 3.2 S is an orthodox semigroup and T is a right orthodox Γ-semigroup. For (ii), let $(e,t) \in S \times_\phi T$ with $e \in E(S)$ and let $(e',t') \in V_\alpha^\beta\Big((e,t)\Big)$ for some $\alpha, \beta \in \Gamma$. Then by Lemma 3.3 $(e',t'), \Big(e', (t')^e\beta t^{e'e}\alpha t'\Big) \in V_\alpha^\beta\Big((e,t^{e'e})\Big)$. Thus $V_\alpha^\beta\Big((e,t)\Big) \cap V_\alpha^\beta\Big((e,t^{e'e})\Big) \neq \phi$ and hence we have $V_\alpha^\beta\Big((e,t)\Big) = V_\alpha^\beta\Big((e,t^{e'e})\Big)$ by Theorem 2.6. So $\Big(e, (t')^e\beta t^{e'e}\alpha t'\Big) \in V_\alpha^\beta\Big((e,t)\Big)$. Thus

$$(e,t) = (e,t)\alpha(e,(t')^e\beta t^{e'e}\alpha t'\Big)\beta(e,t) = \Big(e, t^e\alpha(t')^e\beta t^{e'e}\alpha(t')^e\beta t\Big)$$

and hence $t = t^e\alpha(t')^e\beta t^{e'e}\alpha(t')^e\beta t \in t^e\Gamma T$.

For (iii) we shall first show that for an α-idempotent t^e of T if $e \in E(S), t^{e'}$ is an α-idempotent for any $e' \in V(e)$. If $e \in E(S)$ and t^e is an α-idempoetnt, then by Lemma 3.4, (e,t) is an α-idempotent in $S \times_\phi T$ and $(e,t^e) \in V_\alpha^\alpha\Big((e,t)\Big)$. Again since t^e is an α-idempotent (e,t^e) is also an α-idempotent and thus $(e,t^e) \in V_\alpha^\alpha\Big((e,t^e)\Big)$ i.e., $V_\alpha^\alpha\Big((e,t^e)\Big) \cap V_\alpha^\alpha\Big((e,t)\Big) \neq \phi$ and so $V_\alpha^\alpha\Big((e,t^e)\Big) =$

$V_\alpha^\alpha\Big((e,t)\Big)$ and by Lemma 3.3 $(e', t^{ee'}) \in V_\alpha^\alpha\Big((e,t^e)\Big)$ i.e., $(e', t^{ee'}) \in V_\alpha^\alpha\Big((e,t)\Big)$. Thus $(e,t) = (e,t)\alpha(e',t^{ee'})\alpha(e,t) = (ee'e, t^{e'e}\alpha t^{ee'e}\alpha t) = (e, t^{e'e}\alpha t^e \alpha t) = (e, t^{e'e}\alpha t)$ [since $t = t^e\beta u$ for some $\beta \in \Gamma$, $u \in T, t^e\alpha t = t$]. So $t = t^{e'e}\alpha t$ and hence $t^{e'} = \Big(t^{e'e}\alpha t\Big)^{e'} = t^{e'}\alpha t^{e'}$. Thus $t^{e'}$ is an α-idempotent. Let $e, g \in E(S)$ and suppose that t^e ia an α-idempotent for $t \in T$, then $t^{eg}\alpha t^{eg} = (t^e\alpha t^e)^g = t^{eg}$ i.e, t^{eg} is an α-idempotent and we have $eg \in E(S)$ and $ge \in V(eg)$ since S is right orthodox. Then by the above fact t^{ge} is an α-idempotent.

Conversely, suppose that S and T satisfy (i), (ii), (iii). Let $(s,t) \in S \times_\phi T$ be given. Since S is regular, there exists $s' \in S$ such that $s = ss's$ and $s' = s'ss'$. Then $Ss' = Ss$. We take $e = s's$, then $e \in E(S)$. By (ii) $t \in t^e\Gamma T$ which implies $t = t^e\beta u$ for some $\beta \in \Gamma$, $u \in T$. Let $t' = v^{s'}$ where $v \in V_\gamma^\delta(t)$ where $\gamma, \delta \in \Gamma$. Now $t^{s's}\gamma(t')^s\delta t = t^{s's}\gamma v^{s's}\delta t^e\beta u = (t\gamma v\delta t)^e\beta u = t$ i.e, $(s,t) = (ss's, t^{s's}\gamma(t')^s\delta t) = (s,t)\gamma(s',t')\delta(s,t)$. Again $(t')^{ss'}\delta t^{s'}\gamma t' = (v^{s'})^{ss'}\delta t^{s'}\gamma v^{s'} = v^{s'}\delta t^{s'}\gamma v^{s'} = v^{s'} = t'$ i.e., $(s',t') = (s'ss', (t')^{ss'}\delta t^{s'}\gamma t') = (s',t')\delta(s,t)\gamma(s',t')$. Thus we have $(s',t') \in V_\gamma^\delta(s,t)$ which yields $S \times_\phi T$ is a regular Γ-semigroup.

Now let (e,t) be an α-idempotent and (g,u) be a β-idempotent. Then by Lemma 3.4 $e, g \in E(S)$, t^e is an α-idempotent and u^g is a β-idempotent. By (iii) t^{ge} is an α-idempotent, u^{eg} is a β-idempotent and $t^{geg}\alpha t^{geg} = (t^{ge}\alpha t^{ge})^g = t^{geg}$ i.e., t^{geg} is an α-idempotent. By our assumption $e, g \in E(S)$ and $(t^g\alpha u)^{eg} = t^{geg}\alpha u^{eg}$ is a β-idempotent. Thus by Lemma 3.4 $(e,t)\alpha(g,u) = (eg, t^g\alpha u)$ is a β-idempotent which shows that $S \times_\phi T$ is a right orthodox Γ-semigroup.

Theorem 3.6 *Let S be a monoid, T be a Γ-semigroup and $\phi : S \to End(T)$ be a given 1-preserving homomorphism. Then the semidirect product $S \times_\phi T$ is a right inverse Γ-semigroup if and only if*

(i) *S is a right inverse semigroup and T is a right inverse Γ-semigroup*
(ii) *for every $e \in E(S)$ and every $t \in T$, $t \in t^e\Gamma T$.*

Proof Let $S \times_\phi T$ be a right inverse Γ-semigroup. Then by Lemma 3.2 S is a right inverse semigroup and T is a right inverse Γ-semigroup. Again since every right inverse Γ-semigroup is a right orthodox Γ-semigroup from the above theorem, condition (ii) holds.

Conversely, suppose that S and T satisfy (i) and (ii). Then $S \times_\phi T$ is a regular Γ-semigroup by Lemma 3.2. Let (e,t) be an α-idempotent and (g,u) be a β-idempotent in $S \times_\phi T$. Then by Lemma 3.4 $e, g \in E(S)$, t^e is an α-idempotent, u^g is a β-idempotent. From (ii) $t = t^e\gamma v$ for some $\gamma \in \Gamma$, $v \in T$ and thus $t^e\alpha t = t$ and similarly $u^g\beta u = u$. So $u^{ge} = (u^g\beta u)^{ge} = u^{ge}\beta u^{ge}$ and $t^{ge} = (t^e\alpha t)^{ge} = t^{ege}\alpha t^{ge} = t^{ge}\alpha t^{ge}$ since S is a right inverse semigroup. Now by (ii) we have $u^e\beta t = (u^e\beta t)^{ge}\delta v_1$ for some $\delta \in \Gamma$, $v_1 \in T$ and hence $u^e\beta t = u^{ege}\beta t^{ge}\delta v_1 = u^{ge}\beta t^{ge}\delta v_1$. Thus we have $(e,t)\alpha(g,u)\beta(e,t) = (ege, t^{ge}\alpha u^e\beta t)$ $= (ge, t^{ge}\alpha u^{ge}\beta t^{ge}\delta v_1) = (ge, u^{ge}\beta t^{ge}\delta v_1) = (ge, u^e\beta t) = (g,u)\beta(e,t)$ which implies $S \times_\phi T$ is a right inverse Γ-semigroup.

Theorem 3.7 *Let S be a monoid, T be a Γ-semigroup and $\phi : S \to End(T)$ be a given 1-preserving homomorphism. Then the semidirect product $S \times_\phi T$ is a left inverse Γ-semigroup if and only if*

(i) *S is a left inverse semigroup and T is a left inverse Γ-semigroup*

(ii) *for every $e \in E(S)$ and every $t \in T$, $t = t^e$.*

Proof Let $S \times_\phi T$ be a left inverse Γ-semigroup. Then by Lemma 3.1 S is a left inverse semigroup and T is a left inverse Γ-semigroup. For (ii) let (e, u) be an α-idempotent in $S \times_\phi T$. Then $(e, u) = (e, u)\alpha(e, u) = (e, u^e\alpha u)$ i.e., $u^e\alpha u = u$. Again $(e, u^e)\alpha(e, u^e) = (e, u^{ee}\alpha u^e) = (e, u^e)$ which yields (e, u^e) is an α-idempotent and we have $(e, u^e)\alpha(e, u) = (e, u^e\alpha u) = (e, u)$. Since $S \times_\phi T$ is a left inverse Γ-semigroup, $(e, u) = (e, u^e)\alpha(e, u) = (e, u^e)\alpha(e, u)\alpha(e, u^e)$ $= (e, u^{eee}\alpha u^{ee}\alpha u^e) = (e, u^e\alpha u)^{ee}\alpha u^e) = (e, u^{ee}\alpha u^e) = (e, u^e)$ i.e., $u = u^e$. Thus if (e, u) is an α-idempotent then $u = u^e$. Now $(e, t) \in S \times_\phi T$ with $e \in E(S)$ and let $(e', t') \in V_\gamma^\delta\Big((e, t)\Big)$ for some $\gamma, \delta \in \Gamma$. Then we get $e' \in V(e)$, $t^{e'e}\gamma(t')^e\delta t = t$ i.e., $t^{e'e}\gamma(t')^{ee'e}\delta t^{e'e} = t^{e'e}$ which implies $t^{e'e}\gamma(t')^e\delta t^{e'e} = t^{e'e}$. Since $\Big(e'e, (t')^e\delta t\Big) = (e', t')\delta(e, t)$ and $S \times_\phi T$ is left orthodox (since it is left inverse), $\Big(e'e, (t')^e\delta t\Big)$ is a γ-idempotent and hence $(t')^e\delta t = \Big((t')^e\delta t\Big)^{e'e} =$ $(t')^e\delta t^{e'e}$. Thus $t^{e'e} = t^{e'e}\gamma(t')^e\delta t^{e'e} = t^{e'e}\gamma(t')^e\delta t = t$ and hence $t^e = (t^{e'e})^e =$ $t^{e'e} = t$.

Conversely suppose that S and T satisfy (i) and (ii). Then $S \times_\phi T$ is regular. Now let (e, t) be an α-idempotent and (g, u) be a β-idempotent. Then $e^2 = e$ and $t = t^e\alpha t = t\alpha t$ [by (ii)] and similarly $g^2 = g$ and $u\beta u = u$ i.e., $e, g \in E(S)$ and t is an α-idempotent, u is a β-idempotent. Thus we have $(e, t)\beta(g, u)\alpha(e, t) = (ege, t^{ge}\beta u^e\alpha t) = (ege, t\beta u\alpha t)$ [by (ii)] $=(eg, t\beta u) =$ $(eg, t^g\beta u) = (e, t)\beta(g, u)$. Thus $S \times_\phi T$ is a left inverse Γ-semigroup.

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