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DIRECT SUMS OF RELATIVE (QUASI-) CONTINUOUS MODULES

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Abstract

In [4], relative (quasi-)continuous modules are introduced, and several fundamental results are given. In the present paper, we shall give necessary and sufficient conditions for direct sums of relative (quasi-) continuous modules to be relative (quasi-) continuous modules.

Preliminaries

Throughout this paper, R denotes a ring with identity and all R-modules are unitary right R-modules. For two R-modules X and Y with $X \subseteq Y$, $X \subseteq_e Y$ means that X is an essential submodule of Y.

Let N and M be R-modules. By $\mathcal{A}(N, M)$, we denote the family of all submodule A of M such that $f(\mathrm{X}) \subseteq_e A$ for some $X \subseteq N$ and some f in $Hom_R(X, M)$. It is easy to see that this family $\mathcal{A}(N, M)$ is closed under submodules, essential extensions and isomorphic images.

Definition 1 *For $\mathcal{A}(N, M)$, we consider the following conditions :*

(C_1) *For any $A \in \mathcal{A}(N, M)$, there exists a direct summand $A^* <_\oplus M$ such that $A \subseteq_e A^*$*

(C_2) *For any $A \in \mathcal{A}(N, M)$ and $X <_\oplus M$, $A \simeq X$ implies $A <_\oplus M$*

(C_3) *For any $A \in \mathcal{A}(N, M)$ and $X <_\oplus M$, if $A <_\oplus M$ and $A \cap X = 0$ then $A \oplus X <_\oplus M$*

M is said to be N-continuous if (C_1) and (C_2) hold, and is said to be N-quasi-continuous if (C_1) and (C_3) hold. Furthermore, M is said to be N-CS if

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(C_1) *holds.* We note that these modules are closed under direct summands (cf.[3]).

For R-modules $M = \bigoplus_I M_i$ and X, we use the following conditions :

(A) For every choice of distinct $k_i \in I$ and $m_i \in M_{k_i}$, if the sequence $(0 : m_i)$ is ascending then it becomes stationary.

(B) For any choice of $m_i \in M_{k_i}$ $(i \in \mathbb{N})$ for distinct $k_i \in I$ such that $(0 : x) \subseteq \cap_{i=1}^{\infty}(0 : m_i)$ for some $x \in X$, the ascending sequence $\cap_{i \geq n}(0 : m_i)$ (n$\in \mathbb{N}$) becomes stationary.

(C) For any $x \in X$ and for every choice of distinct $k_i \in I$ and $m_i \in M_{k_i}$, with $(0 : x) \subseteq (0 : m_i)$, if the sequence $(0 : m_i)$ is ascending, then it becomes stationary.

For these conditions, the reader is referred to [2, page4]. We note that (B) implies (C).

Lemma 1 (cf.[1]) *For R-modules X and $\{M_i\}_I$, the following are equivalent:*

(1) $\bigoplus_I M_i$ is X-injective;

(2) (a) each M_i is X-injective

(b) the condition (B) holds for X and $\{M_i\}_I$.

So, in this case (C) holds.

Lemma 2 (cf.[4]) *For an N-(quasi-)continuous module M, the following hold:*

(1) Any direct summand of M is N-(quasi-)continuous.

(2) For any $X <_{\oplus} M$ and $A \in \mathcal{A}(N, M)$ with $X \cap A = 0$, X is A-injective.

(3) For any $A, B \in \mathcal{A}(N, M)$ with $A \cap B = 0$ if $A <_{\oplus} M$ and $A \simeq B$ then $B <_{\oplus} M$.

Lemma 3 *Consider two modules $P = \sum \oplus_I T_i \oplus N$ and $Q = \sum \oplus_I W_i \oplus N$ such that $Q \subseteq_e P$. If $\sum \oplus_I T_i$ satisfies (A) and, for any finite subset $F \subseteq I$, if $P = \sum \oplus_F W_i \oplus \bigoplus_{I-F} T_j \oplus N$ then $P = Q$.*

Proof Assume that $P \neq Q$. Since $\sum \oplus_I T_i$ satisfies (A), we can take a finite subset F of I and an element $t \in \sum \oplus_F T_i$ such that $t \notin Q$ and, for any $j \in I-F$ and $s \in T_j$, if $(0 : t) \subsetneq (0 : s)$ then $s \in Q$.

Since $Q \subseteq_e P$, we can take $r \in R$ such that $0 \neq tr \in Q$. So there exists a finite subset $S \subseteq I$ such that $tr \in \bigoplus_G W_i \oplus N$.

We take G as $G \supseteq F$. We express t in $P = \bigoplus_G W_i \oplus \sum \oplus_{I-G} T_j \oplus N$ as $t = w + s + n$, where $w \in \sum \oplus_G W_i$, $s \in \bigoplus_{I-G} T_j$ and $n \in N$.

Since $\bigoplus_{I-G} T_j \ni sr = tr - wr - nr \in \bigoplus_G W_i \oplus N$, we see $sr = 0$;

so $(0 : t) \subsetneq (0 : s)$. This implies $s \in Q$ and hence $t = w + s + n \in Q$, which is a contradiction. Hence $P = Q$.

By a slight modification, we quote [5, Lemma 2.1] as follows :

Lemma 4 *Let* $\{M_\alpha\}_I$ *be a family of* N*-CS-modules and let* $A \in \mathcal{A}(N, P = \bigoplus_I M_\alpha)$*. Then there exists submodules* $T(i) \subseteq_e T(i)^* \subseteq \oplus M_\alpha$*, decompositions* $M_\alpha = T(i)^* \oplus N_i$ *and a submodule* $\bigoplus_I A(i) \subseteq_e A$ *for which the following properties hold :*

(1) $A(i) \subseteq T(i) \oplus \bigoplus_I N_i$

(2) $\sigma(A(i)) = T(i)$ *and* $A(i) \overset{\sigma|A(i)}{\simeq} T(i) \quad (by \sigma|A(i))$ *for each* $i \in I$*, where* σ *is the projection :* $P = \bigoplus_I T(i)^* \oplus \bigoplus_I N_i \to \bigoplus_I T(i)^*$.

So, $T(i), T(i)^* \in \mathcal{A}(N, M_\alpha)$ *and* $A \overset{\sigma|A}{\simeq} \sigma(A) \subseteq_e \bigoplus_I T(i)^*$.

Theorems

We first show the following theorem which is a generalization of [2, Theorem 2.13].

Theorem 1 *Let* $\{M_i\}_I$ *be a family of* R*-modules. Then the following are equivalent:*

(1) $P = \sum \oplus_I M_i$ *is* N*-quasi-continuous;*

(2) (a) Each M_i *is* N*-quasi-continuous;*

(b) $\bigoplus_{I-i} M_j$ *is* A_i*-injective, for any* $i \in I$ *and any* $A_i \in \mathcal{A}(N, M_i)$

(3) (a) Each M_i *is* N*-continuous;*

(b) for any distinct $i, j \in I$

(4) (a) Each M_i *is* N*-quasi-continuous;*

(b) for any distinct $i, j \in I$ *and* $A_i \in \mathcal{A}(N, M)$*,* M_j *is* A_i*-injective*

(c) for any $i \in I$ *and* $A_i \in \mathcal{A}(N, M_i)$*, the condition(B) holds for* $(A_i, \bigoplus_{I-i} M_j)$.

Proof (1)⇒(2) follows from Lemma 2.

(2)⇔(3) follows from Lemma 1.

(2)⇒(1). First we show that $P = \sum \oplus_I M_i$ is N-CS. Let $X \in \mathcal{A}(N, M)$. By Lemma 4, we have submodules $T(i) \subseteq_e T(i)^* <_\oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\bigoplus_I X_i \subseteq_e X$ such that, for each $i \in I$,

(i) $\sigma(X_i) = T(i)$

(ii) $X_i \simeq T(i)@$ (by $\sigma|X_i$),

where σ is the projection: $P = \bigoplus_I T(i)^* \oplus \bigoplus_I N_i \to \bigoplus_I T(i)^*$. So, we see

(iii) $X \overset{\sigma|X}{\simeq} \sigma(X) \subseteq_e \bigoplus_I T(i)^*$

Since $X \in \mathcal{A}(N,P)$, we see that $X_i \in \mathcal{A}(N,M_i)$, whence $T(i)^* \in \mathcal{A}(N,M_i)$ for each $i \in I$. So, by(b) $\bigoplus_{I-i} N_j$ is $T(i)^*$-injective for each $i \in I$. On the other hand, by(a), N_i is $T(i)^*$-injective. Hence $\bigoplus_I N_i$ is $T(i)^*$-injective for each $i \in I$.

Now, by(iii), the mapping $\varphi : \sigma(X) \to \bigoplus_I N_i$ given by $\varphi(\sigma(x)) = \tau(x)$ is a homomorphisms, where τ is the projection: $P = \bigoplus_I T(i)^* \oplus \bigoplus_I N_i \to \bigoplus_I N_i$.

Since $\bigoplus_I N_i$ is $\bigoplus_I T(i)^*$-injective, φ can be extended to a homomorphism $\varphi^* : \bigoplus_I T(i)^* \to \bigoplus_I N_i$. We put

$X^* = \{x + \varphi^*(x) | x \in \bigoplus_I T(i)^*\}$.

Then $P = X^* \oplus \bigoplus_I N_i$ and moreover we see from $X \overset{\sigma|X}{\simeq} \sigma(X) \subseteq_e \bigoplus_I T(i)^*$ that $X \subseteq_e X^*$. Accordingly, P is N-CS. Here we note that if $X <_\oplus P$, and moreover $\bigoplus_I N_i$ is X-injective.

Next we will show that $P = \bigoplus_I M_i$ satisfies (C_3) for N.

Let $A \in \mathcal{A}(N,P)$ and $X \subseteq P$, and assume that both A and X are direct summands with $A \cap X = 0$; Put $P = X \oplus Q = Y \oplus A$ and let π_Q and π_X be the projections: $P = X \oplus Q \to X$, respectively. Since $X \cap A = 0$, $A \simeq \pi_Q(A)$ by $\pi_Q|A$. Since Q is $N-CS$ and $\pi_Q(A) \in \mathcal{A}(N,P)$, there exists a direct summand $\pi_Q(A)^* <_\oplus Q$ such that $\pi_Q(A) \subseteq_e \pi_Q(A)^*$. Since $\pi_Q(A)^* <_\oplus P$, as we noted above, $P = \pi_Q(A)^* \oplus \bigoplus_I N_i$ for some $N_i <_\oplus M_i$ and $\bigoplus_I N_i$ is $\pi_Q(A)^*$-injective.

Since $X \cap \pi_Q(A)^* = 0$, X is isomorphic to a submodule of $\bigoplus_I N_i$. Hence X is $\pi_Q(A)^*$-injective. Here consider the mapping $\varphi : \pi_Q(A) \to X$ given by $\varphi(\pi_Q(a)) = \pi_X(a)$. Then φ is a homomorphism. So φ can be extended to a homomorphism $\varphi^* : \pi_Q(A)^* \to X$.

Putting $A^* = \{q + \varphi^*(q) | q \in \pi_Q(A)^*\}$, we see that $X \oplus A^* <_\oplus P$ and $A \subseteq_e A^*$. Since $A <_\oplus P$, it follows $A = A^*$ and hence $X \oplus A <_\oplus P$ as required.

We generalize [2, Theorem 3.16] as follows:

Theorem 2 *Let $\{M_i\}_I$ be a family of R-modules. Then the following are equivalent:*

(1) $P = \bigoplus_I M_i$ is N-continuous;

(2) (a) Each M_i is N-continuous;

(b) $\bigoplus_{I-\{i\}} M_j$ is A_i-injective, for any $i \in I$ and any $A_i \in \mathcal{A}(N,M_i)$

(3) (a) Each M_i is N-continuous;

(b) for any distinct $i,j \in I$ and $A_i \in \mathcal{A}(N,M_i)$, M_j is A_i-injective

(c) for any $i \in I$ and $A_i \in \mathcal{A}(N,M_i)$, the condition(B) holds for

$(A_i, P = \bigoplus_{I-\{i\}} M_j)$.

Proof As in the proof of Theorem 1, we may only show that $P = \bigoplus_I M_i$ satisfies (C_2) for $\mathcal{A} = \mathcal{A}(N, P)$. So, let $A, B \in \mathcal{A}$ such that $A <_\oplus P$ and $A \overset{\tau}{\simeq} B$.

By Lemma 4, there exist submodules $T(i) \subseteq_e T(i)^* <_\oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ and a submodule $\bigoplus_I B(i) \subseteq_e B$ such that, for each $i \in I$, (i) $\sigma(B(i)) = T(i)$

(ii) $B(i) \simeq T(i)$ $\quad$ (*by* $\sigma|B(i)$),

where σ is the projection: $P = \bigoplus_I T(i)^* \oplus \bigoplus_I N_i \to \bigoplus_I N_i$.

Put $A(i) = \tau^{-1}(B(i))$ for each $i \in I$. Since A is CS, for each $i \in I$, there exists direct summand $A(i)^* <_\oplus A$ such that $A(i) \subseteq_e A(i)^*$.

Let fix $i_0 \in I$. By the proof of Theorem 1, there exist direct summands $K_j <_\oplus M_j$ such that $A(i_0)^*$ is isomorphic to $\bigoplus_I K_j$; say $A(i_0)^* \overset{\varphi}{\simeq} \bigoplus_I k_j$.

Now put $B(i_0)^* = \tau(A(i_0^*))$. Then $B(i_0) \subseteq_e B(i_0)^*$ and $B(i_0)^* \overset{\sigma|B(i_0)^*}{\simeq} \sigma(B(i_0^*)) \subseteq_e T(i_0)^*$. Since $\bigoplus_{I-i_0} M_j \oplus_> \bigoplus_{I-i_0} k_j \simeq \tau\varphi(\bigoplus_{I-i_0} K_j)$ (by $\tau\varphi| \bigoplus_{I-i_0} K_j$) and $\tau\varphi(\bigoplus_{I-i_0} K_j) \subseteq T(i_0^*) <_\oplus M_{i_0}$, we see from (b) that $\tau\varphi(\bigoplus_{I-i_0} K_j) <_\oplus T(i_0)^*$.

On the other hand, $\tau\varphi(K_{i_0}) <_\oplus T(I_0)^*$ by (a). As a result we see that $T(i_0) \subseteq_e \tau\varphi(\bigoplus_I K_i) <_\oplus T(I_0)^*$; whence $\tau\varphi(\bigoplus_I K_i) = \sigma(B(i_0)^*) = T(i_0)^*$. Thus we have $P = B(i_0)^* \oplus \bigoplus_{I-i_0} T(i)^* \oplus \bigoplus_I N_i$

Inductively, we see that, for any finite subset F of I, $P = \bigoplus_F B(i)^* \oplus \bigoplus_{I-F} T(i)^* \oplus \bigoplus_I N_i$.

Here using Lemma 3 we get $P = \bigoplus_I B(i)^* \oplus \bigoplus_I N_i$ and hence $B = \bigoplus_I B(i)^* <_\oplus P$.

This completes the proof.

Remarks

Remark 1 In Theorem 1, $\bigoplus_F M_i$ is N-quasi-continuous for any finite subset F of I if and only if (a), (b) hold.

Remark 2 In Theorem 2, $\bigoplus_F M_i$ is N-continuous if and only if (a), (b) hold (See [4]).

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